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Abstract

In this paper, a Laplace Homotopy perturbation method is employed for solving one-dimensional non-homogeneous partial differential equation with a variable coefficient. This method is a combination of the Laplace transform and Homotopy Perturbation Method (LHPM). LHPM presents an accurate methodology to solve non-homogeneous partial differential equation with variable coefficient. The aim of using the Laplace transform is to overcome the deficiency that mainly caused by unsatisfied conditions in other semi-analytical methods such as HPM, VIM, and ADM. The approximate solutions obtained by means of LHPM in a wide range of problem’s domain were compared with those results obtained from the actual solutions, Homotopy perturbation method (HPM) and the finite element method. The comparison shows a precise agreement between the results, and introduces this new method as an applicable one which it needs less computations and is much easier and more convenient than others, so it can be widely used in engineering too.

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1. Introduction

Differential equation, integral equation or combinations of them, Integro-differential equation, are obtained in modeling of real-life engineering phenomena that are inherently nonlinear with variable coefficient. Most of these types of equations do not have analytical solution. Therefore, those problems should be solved by using numerical or semi-analytical techniques. In numeric methods, computer codes and more powerful processor are required to achieve accurate results. Acceptable results are obtained via semi-analytical methods which are more convenient than numerical methods. The main advantage of semi-analytical methods, compared with other methods, is based on the fact that they can be conveniently applied to solve various complicated problems. Several numerical and analytical methods including non-polynomial cubic spline methods, finite difference, Laplace decomposition method, Homotopy perturbation transform method, Variational iteration method, Adomian decomposition method and the homogeneous Adomian decomposition method have been developed for solving linear or nonlinear non-homogeneous partial differential equations, see [1-7]. HPM, ADM, and VIM methods can be used to solve the non-homogeneous variable coefficient partial differential equations with accurate approximation, but this approximation is acceptable only for a small range [7], because, boundary conditions in one dimension are satisfied via those methods. Therefore, unsatisfied conditions play no rolls in final results. Consequently, this shows that most of these semi-analytical methods encounter the inbuilt deficiencies and involve huge computational work. One of those semi-analytical solution methods is the Homotopy Perturbation Method (HPM). He [8-12] developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems by merging two techniques, the standard homotopy and the perturbation. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods and has been modified later by some scientists to obtain more accurate results, rapid convergence, and to reduce amount of
computation [13-16]. The homotopy perturbation method (HPM) has been applied to a wide class of functional equations; see [17-32] and the references therein. The partial differential equations can be calculated with the help of HPM and other semi-analytical methods in small range with high iteration which observed in [33-37].

The basic motivation of this paper is to propose a new modification of HPM to overcome this deficiency. By using this new method, combination of Laplace transform and Homotopy Perturbation Method (LHPM), all conditions can be satisfied. Also, very accurate results are obtained in wide range via one or two iteration steps. In this work, the proposed Laplace transform Homotopy perturbation method (LHPM) is used to handle three time dependent non-homogeneous partial differential equations. Furthermore, Comparisons are made between the present method and the HPM, in order to verify the efficiency of the present method.

2. Homotopy Perturbation Method (HPM)

To explain the homotopy perturbation method, we consider a general equation of the type

\[ L(u) = 0, \]

where \( L \) is any integral or differential operator. We define a convex homotopy \( H(u, p) \) by

\[ H(u, p) = (1 - p)F(u) + pL(u), \]

where \( F(u) \) is a functional operator with known solutions \( v_0 \), which can be obtained easily. It is clear that, for \( H(u, p) = 0 \), we have

\[ H(u,0) = F(u), \quad H(u,1) = L(u). \]

This shows that \( H(u, p) \) continuously traces an implicitly defined curve from a starting point \( H(v_0,0) \) to a solution function \( H(f, 1) \). The embedding parameter monotonically increases from zero to unit as the trivial problem \( F(u) = 0 \) is continuously deforms the original problem \( L(u) = 0 \). The embedding parameter \( p \in [0,1] \) can be considered as an
expanding parameter [8-12]. The homotopy perturbation method (HPM) uses the homotopy parameter \( p \) as an expanding parameter [8-12] to obtain

\[
u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots\]

(3)

if \( p \to 1 \), then (3) corresponds to (2) and becomes the approximate solution of the form,

\[
f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i.
\]

(4)

It is well known that series (4) is convergent for most of the cases and also the rate of convergence is dependent on \( L(u) \); see [8-12]. We assume that (4) has a unique solution.

The comparisons of like powers of \( p \) give solutions of various orders.

3. Basic idea of LHPM

To illustrate the basic idea of this method, we consider the general form of one-dimensional non-homogeneous partial differential equations with a variable coefficient of the forms [7]:

\[
\frac{\partial^2 u}{\partial t^2} = \mu(x) \frac{\partial^2 u}{\partial x^2} + \varphi(x,t)
\]

(5)

and

\[
\frac{\partial^2 u}{\partial t^2} = \mu(x) \frac{\partial^2 u}{\partial x^2} + \varphi(x,t)
\]

(6)

which subject to the boundary conditions

\[
u(0,t) = g_0(t) , \quad u(1,t) = g_1(t)
\]

(7)

And the initial condition

\[
u(x,0) = f(x)
\]

(8)
The methodology consists of applying Laplace transform on both sides of Eq. (5), Eq. (7) and in view of the initial condition, we get

\[ \frac{d^2\bar{u}}{dx^2} - \frac{s\bar{u}(x,s)}{\mu(x)} + \frac{\bar{\phi}(x,s) + f(x)}{\mu(x)} = 0 \]  

(9)

\[ \bar{u}(0,s) = g_0(s), \quad \bar{u}(1,s) = g_1(s) \]  

(10)

which is second-order boundary value problem. According to HPM, we construct a homotopy in the form

\[ H(v,p) = (1-p)\left[ \frac{d^2v}{dx^2} - \frac{d^2\bar{u}_0}{dx^2} \right] + p\left[ \frac{d^2v}{dx^2} - \frac{sv}{\mu(x)} + \frac{\bar{\phi}(x,s) + f(x)}{\mu(x)} \right] = 0 \]

Where \( \bar{u}_0 \) is the arbitrary function that satisfy boundary conditions Eq. (10), therefore

\[ v(x,s) = \sum_{i=0}^{\infty} p^i v_i(x,s) = v_0(x,s) + p^1v_1(x,s) + p^2v_2(x,s) + \ldots \]  

(11)

Taking the inverse Laplace transform from both sides of Eq. (11), one obtains

\[ v(x,t) = \sum_{i=0}^{\infty} p^i v_i(x,t) = v_0(x,t) + p^1v_1(x,t) + p^2v_2(x,t) + \ldots \]  

(12)

Setting \( p = 1 \) results in the approximate solution of Eq. (5)

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots, \]  

(13)

4. Case study

In this section, we demonstrate the effectiveness of the LHFM with several illustrative examples. Let us define \( S_n(x,t) = \sum_{i=0}^{n} v_i(x,t) \) that is the \( n \) th partial sum of the infinite series of approximate solution [7], so the relative errors \( S_n(\%) \) is calculated as

\[ RE(\%) = \left[ \frac{|u(x,t) - S_n(x,t)|}{u(x,t)} \right] \times 100 \]  

(14)
In order to investigate the validity of the LHPM, the relative errors $S_n(\%)$, $n = 1$ or $2$, for some position value of $x$, $0 \leq x \leq 1$, obtained in wide range of $t$ by the LHPM are compared with the exact solutions.

**Example-1**

Let us consider the one dimensional non-homogeneous problem

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + u - e^{-t} (1 + 2t) = 0$$  \hspace{1cm} (15)

Subject to boundary conditions

$$u(0,t) = t, \quad u_t(0,t) = e^{-t} - t$$  \hspace{1cm} (16)

and the initial condition $u(x,0) = x$ that is easily seen to have the exact solution

$$u(x,t) = te^{-x} + xe^{-t}.$$  

By applying the aforesaid method subject to the initial condition, i.e. $u(x,0) = x$, we have

$$\frac{d^2 u}{dx^2} + (s+1)u - e^{-x}\left(\frac{1}{s} + \frac{2}{s^2}\right) = 0$$  \hspace{1cm} (17)

$$u(0,s) = \frac{1}{s^2}, \quad u_t(0,s) = -\frac{1}{s^2} + \frac{1}{1+s}$$  \hspace{1cm} (18)

To solve Eq. (17) by means of HPM, a homotopy equation can be readily constructed as follows

$$H(v, p) = (1-p)\left[\frac{d^2 v}{dx^2} - \frac{d^2 u}{dx^2}\right] + p\left[\frac{d^2 v}{dx^2} + (s+1)v - x - e^{-x}\left(\frac{1}{s} + \frac{2}{s^2}\right)\right] = 0, \quad p \in [0,1]$$  \hspace{1cm} (19)

we assume that the solution to Eq. (19) may be written as a power series in $p$:

$$v(x,s) = \sum_{i=0}^{\infty} p^i v_i(x,s) = v_0(x,s) + p v_1(x,s) + p^2 v_2(x,s) + \ldots$$  \hspace{1cm} (20)
Substituting Eq. (20) into Eq. (19) and equating the terms with the identical powers of \( p \), we have

\[
p^0 \frac{d^2 v_0}{dx^2} - \frac{d^2 \bar{p}_0}{dx^2} = 0, \quad v_0(0, s) = \frac{1}{s^2}, \quad v_{0x}(0, s) = -\frac{1}{s^2} + \frac{1}{1+s}
\]

\[
p^1 \frac{d^2 v_i}{dx^2} + (s+1)v_0 - x - e^{-x}\left(\frac{1}{s} + \frac{2}{s^2}\right) = 0, \quad v_i(0, s) = 0, \quad v_{is}(0, s) = 0
\]

(21)

\[
p^i \frac{d^2 v_i}{dx^2} + (s+1)v_{i-1} = 0, \quad v_i(0, s) = 0, \quad v_{is}(0, s) = 0, \quad i = 2, 3, 4, ...
\]

The initial approximation \( v_0(x, s) \) or \( \bar{p}_0(x, s) \) can be freely chosen, here we set

\[
v_0(x, s) = \bar{p}_0(x, s) = \frac{1-x}{s^2} + \frac{x}{1+s}
\]

(22)

which satisfies boundary conditions, Eq. (18).

By substitution of Eq. (22) into Eq. (21), the first component of the Homotopy Perturbation solution for Eq. (17) are derived as follows

\[
v_1(x, s) = \frac{1}{s}\left(-\frac{1}{6}x^3 + \frac{1}{2}x^2 + 2x - 2 + 2e^{-x}\right) + \frac{1}{s}\left(-\frac{1}{6}x^3 - \frac{1}{2}x^2 + x - 1 + e^{-x}\right)
\]

Consequently

\[
S_1(x, s) = v_0 + v_1 = \frac{x}{1+s} + \frac{1}{s}\left(-\frac{1}{6}x^3 + \frac{1}{2}x^2 + x - 1 + 2e^{-x}\right) + \frac{1}{s}\left(-\frac{1}{6}x^3 - \frac{1}{2}x^2 + x - 1 + e^{-x}\right)
\]

In the same manner, the rest of components were obtained using the Maple package.

By taking the inverse Laplace of components yields

\[
S_0(x, t) = xe^{-t} + t(1-x)
\]

\[
S_1(x, t) = xe^{-t} + t\left(\frac{1}{6}x^3 - \frac{1}{2}x^2 + x - 1 + 2e^{-x}\right) + \frac{1}{s}\left(-\frac{1}{6}x^3 - \frac{1}{2}x^2 + x - 1 + e^{-x}\right)
\]

\[
S_2(x, t) = xe^{-t} + t\left(-\frac{x^5}{120} + \frac{x^4}{24} - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + 1\right) + 2\left(-\frac{x^5}{120} + \frac{x^4}{24} - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + 1\right) - 2e^{-x}
\]

The REs \( S_n \% \) in LHPM for \( n = 1, 2 \) are compared with the HPM REs \( S_5 \) and \( S_8 \) at some position value \( x \) are depicted in Fig. 1(a-d), which show that, unlike the LHPM, the
amount of the HPM REs \( S_5 \) and \( S_8 \) are dramatically increased as the time value \( t \) increasing. So HPM solution validity range restricted to short region \((t < 1.5)\). But the validity range can been increased with increasing the terms of solution to more than 8 terms.

On the other hand, results of LHPM are in good agreement with those obtained by exact solution. LHPM solution is almost valid for large wide range of times that shows present method can solve non-homogeneous parabolic equation with high degree of accuracy by the two first terms only. Taylor expansion of \( e^{-n} \) is given by

\[
e^{-n} \approx \sum_{i=0}^{n} \frac{(-1)^i}{i!} n^i = 1 - n + \frac{n^2}{2} - \frac{n^3}{6} + \ldots
\]

Consequently \( S_1(x,t) \) and \( S_2(x,t) \) can be reduced to \( ts e^{-x} + xe^{-t} \) which is the same as the exact solution. This exact solution can not been achieved using HPM, because all the conditions are not taken into account. In other word, the same results are obtained in HPM when boundary condition changed to \( u(0,t) = g_2(t), \ u(1,t) = g_3(t) \).

**Example-2**

Let us consider the problem

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u = 0 \tag{23}
\]

With boundary conditions

\[
u(0,t) = \cosh(t), \quad u(1,t) = 1 \tag{24}
\]

And the initial conditions

\[
u(x,0) = \sin(x) + 1, \quad u_t(x,0) = 0 \tag{25}
\]

That is easily seen to have the exact solution \( u(x,t) = \sin(x) + \cosh(t) \). Operating the Laplace transforms on both sides of Eqs. (23)-(24) subject to the initial condition, we arrive at
\[
\frac{d^2 \bar{u}}{dx^2} + (1 - s^2) \bar{u} + s(1 + \sin(x)) = 0 \quad (26)
\]

\[
\bar{u}(0, s) = \frac{s}{s^2 - 1} \quad , \quad \bar{u}_x(0, s) = \frac{1}{s} \quad (27)
\]

In view of Eq. (2), the homotopy for Eq. (25) can be constructed as

\[
H(v, p) = (1 - p) \left[ \frac{d^2 v}{dx^2} - \frac{d^2 \bar{u}_0}{dx^2} \right] + p \left[ \frac{d^2 v}{dx^2} + (1 - s^2) v + s(1 + \sin(x)) \right] = 0 \quad (28)
\]

With initial approximation \( u_0(x, s) = \frac{x}{s} + \frac{s}{s^2 - 1} \), which was chosen in such way that satisfies boundary conditions Eq. (27), same as example 1 we assume the solution of Eq. (26) has a form:

\[
v(x, s) = \sum_{i=0}^{n} p^i v_i(x, s) = v_0(x, s) + p v_1(x, s) + p^2 v_2(x, s) + \ldots \quad (29)
\]

Substituting Eq. (28) in to Eq. (27) and rearranging the resultant equation based on powers of \( p \)-terms, we obtain the following sets of linear differential equations

\[
p^0 : \frac{d^2 v_0}{dx^2} - \frac{d^2 \bar{u}_0}{dx^2} = 0 \quad , \quad v_0(0, s) = \frac{s}{s^2 - 1} \quad , \quad v_0_x(0, s) = \frac{1}{s}
\]

\[
p^1 : \frac{d^2 v_1}{dx^2} + (1 - s^2) v_0 + s(1 + \sin(x)) = 0 \quad , \quad v_1(0, s) = 0 \quad , \quad v_1_x(0, s) = 0 \quad (30)
\]

\[
p^i : \frac{d^2 v_i}{dx^2} + (1 - s^2) v_{i-1} = 0 \quad , \quad v_i(0, s) = 0 \quad , \quad v_{ix}(0, s) = 0 \quad , \quad i = 2, 3, 4, \ldots
\]

Consequently, the first few components of the homotopy perturbation solution for Eq. (26) are derived as follows
Taking the inverse Laplace transform of components notice that

\[
\mathcal{L}^{-1}\{f(x)s^n\} = f(x)\frac{d^n}{dt^n}\delta(t),
\]

in which \(\delta(t)\) is the Dirac delta function that is zero everywhere except at the origin so, those terms vanished, also each \(S^n\) coefficients are approximately equal to zero because of \(\sin(x)\) Taylor expansion, therefore Eq. (31) reduced to

\[
v_0(x,t) = x + \cosh(t) \\
v_1(x,s) = -\frac{1}{6}x^3 \\
v_2(x,s) = \frac{1}{120}x^5
\]

In the same manner, the rest of components can be obtained using the Maple package, therefore according to Eq. (13) and the assumption \(p = 1\), we get:

\[
u(x,t) = \cosh(t) + x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots \approx \cosh(t) + \sin(x)
\]

which is the same as the exact solution.

Fig. 2(a-d) shows the LHPM relative errors \(S_n(\%)\) for \(n = 1, 2\) and the HPM relative errors \(S_3\) and \(S_4\) at some position value \(x\). The amount of the LHPM relative errors are decreased as the time value \(t\) increasing, which has contrary behavior with HPM solution. The LHPM solution is in good agreement with exact solution and almost valid for large wide range of times that shows that present method can solve second order time-dependent partial differential equation with high degree of accuracy by the two first terms.
only. When boundary conditions change, the HPM solutions give the same results. This is the deficiency of HPM and other semi-analytical methods. In contrast, the results of LHPM which satisfies all conditions have good accuracy that has been studied in next example.

Example-3

The LHPM results are compared with other semi-analytical and numerical methods in this example.

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-x} \left( \cos(t) - \sin(t) \right) \]

With initial condition \( u(x,0) = x \), and two sets of different boundary conditions

**case A:** \( u(0,t) = \sin(t) \), \( u(1,t) = 1 + \frac{\sin(t)}{t} \)

**case B:** \( u(0,t) = \cos(t) \), \( u(1,t) = \sin(x) \)  \( (33) \)

As can be seen in figures 3 and 4 results obtained from LHPM by the first iteration \( S_1 \), in a wide range for both case A and B are in excellent conformance with numerical and exact solution. Numerical results were obtained using the COMSOL 3.4 which uses finite element method.

Fig. 5 illustrates HPM results [35] that as can be seen from figure its validity range restricted to short region \( t < 1.5 \). The other semi-analytical results for case A can be found in literatures [7, 35, and 36]. Also HPM and other semi-analytical method results for case B are as same as dashed line in Fig. 5 that are not in good conformance with those obtained numerically in Fig. 4 at all.

5. Conclusion
The paper suggests an effective modification of the homotopy perturbation method called the Laplace transform Homotopy perturbation method (LHPM) that combines the Laplace transform and the He’s Homotopy perturbation method and studied its validity in a wide range with three examples. The new approach is implemented to solve non-homogenous partial differential equation with variable coefficient and compared with exact solution, numerical solution, and standards HPM. The negligible relative errors have been observed even with just two first terms of the LHPM solution, which indicate that LHPM needs much less computational work compared with standard HPM methods. The main advantage of this method is to overcome the deficiency that is mainly caused by unsatisfied conditions. Thus, it can be concluded that the LHPM methodology is very powerful and efficient in finding approximate solutions as well as numerical solutions. This paper can be used as a standard paradigm for other applications.

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Reference

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Fig. 1: Relative error (%) of LHPM results, $S_1$ and $S_2$ and HPM results, $S_5$ and $S_8$ at different position value $x = 0.1, 0.3, ..., $, when the exact solution is $u(x, t) = te^{-x} + xe^{-2t}$.

Fig. 2: Relative error (%) of LHPM results, $S_1$ and $S_2$ and HPM results, $S_5$ and $S_8$ at different position value $x = 0.1, 0.3, ..., $, when the exact solution is $u(x, t) = \sin(x) + \cosh(t)$.

Fig. 3: Results of first order approximation, $S_1$ obtained from LHPM (hollow symbols) and exact solution (filled symbols) for case A at different position value $x = 0.1, 0.5, 0.9$ when its exact solution is $u(x, t) = x + e^{-t} \sin(t)$.

Fig. 4: Results of first iteration $S_1$ obtained from LHPM (hollow symbols) and numerical solution (filled symbols) for case B at different position value $x = 0.1, 0.5, 0.9$.

Fig. 5: comparison between HPM results and exact solution of case A at different position value $x = 0.1, 0.5, 0.9$ when its exact solution is $u(x, t) = x + e^{-t} \sin(t)$.
Fig. 1
Fig. 2
Fig. 3
Fig. 4

The figure illustrates the relationship between $u(x, t)$ and $S_1(x, t)$ for different values of $x$: 0.1, 0.5, and 0.9. The data points and trend lines show how these variables change over time. The x-axis represents time, while the y-axis represents the values of $u(x, t)$ and $S_1(x, t)$. The trends for each value of $x$ are clearly distinguishable, with the highest peak at $x=0.1$ and the lowest at $x=0.9$. The figure provides a visual representation of the dynamics of these variables under varying conditions.
Fig. 5

$u(x,t)$

Time

$x=0.1$
$x=0.1$
$x=0.5$
$x=0.5$
$x=0.9$
$x=0.9$