Analytical Solution of Linear and Non-Linear Space-Time Fractional Reaction-Diffusion Equations

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Abstract

In this study, we present the homotopy perturbation method (HPM) for finding the analytical solution of linear and non-linear space-time fractional reaction-diffusion equations (STFRDE) on a finite domain. These equations are obtained from standard reaction-diffusion equations by replacing a second-order space derivative by a fractional derivative of order and a first-order time derivative by a fractional derivative of order. Some examples are given. Numerical results show that the homotopy perturbation method is easy to implement and accurate when applied to linear and non-linear space-time fractional reaction-diffusion equations.

KEYWORDS: homotopy perturbation method, fractional reaction-diffusion equation, space-time fractional derivative
1. Introduction

In recent years, it has been found that derivatives of non-integer order are very effective for the description of many physical phenomena such as rheology, damping laws, diffusion process. These findings invoked the growing interest of studies of the fractal calculus in various fields such as physics, chemistry and engineering [1-5].

Fractional differential equations have been caught much attention recently due to exact description of nonlinear phenomena, especially in fluid mechanics, e.g. nano-hydrodynamics, where continuum assumption does not well, and fractional model can be considered to be a best candidate. Hence, great attention has been given to finding solutions of fractional differential equations. The solution of a fractional differential equation is much involved. In general, there exists no method that yields an exact solution for a fractional differential equation. Only approximate solutions can be derived using the linearization or perturbation methods. No analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method was first proposed to solve fractional differential equations with greatest success, see ref.[3]. Many authors found VIM as an effective way to solving fractional equations, both linear and nonlinear [6,7]. Momani [8] Ganji [9] and Yıldırım [10,11] applied the homotopy perturbation method (HPM) to fractional differential equations and revealed that HPM is an alternative analytical method for solving fractional differential equations. Momani [12] and Odibat [13] compared solution procedure between VIM and HPM. Although the fractional calculus was invented by Newton and Leibnitz over three centuries ago, it only became a hot topic recently owing to the development of the computer and its exact description of many real-life problems. A physical interpretation of the fractional calculus was given in [45]. According to fractal spacetime theory (El Naschie's e-infinity theory), time and space do be discontinuous according, and the fractional model is the best candidate to describe such problems [45].

Khan et al [46] presented the approximate solutions of the time fractional chemical engineering equations by means of the variational iteration method (VIM) and homotopy perturbation method (HPM).

Das et al [47] suggested a fractional Lotka-Volterra model using the homotopy perturbation method. The effect of the fractional order on populations of the predator and the prey was discussed. Also Das [48] used Variational Iteration Method and Modified Decomposition Method for solving Fractional Vibration Equation. But we used the homotopy perturbation method for solving linear and non-linear space–time fractional reaction–diffusion equations (STFRDE) on a finite domain. These problems are more difficult than other...
problems. Instead of fractional equations, the fractal equations might be more suitable for these type problems [49].

Variational iteration method and homotopy perturbation method are the two main tools for fractional differential equations [50-54]. How to choose the initial solution is important for the present paper, Hesameddini [55,56] presented an optimal choice of initial solutions in the homotopy perturbation method.

Reaction–diffusion equations are commonly used to model the growth and spreading of biological species. A fractional reaction–diffusion equation (FRDE) can be derived from a continuous-time random walk model when the transport is dispersive [14] or a continuous-time random walk model with temporal memory and sources [15]. The topic has received a great deal of attention recently, for example, in systems biology [16], chemistry and biochemistry applications [17]. Continuous-time random walks, where each random particle jump occurs after a random waiting time, can be used to derive anomalous diffusion. Very large particle jumps are associated with fractional derivatives in space [18], while very long waiting times lead to fractional derivatives in time [19]. In the continuous-time random walk, the size of particle jumps can depend on the waiting time between jumps. For these models, the limiting particle distribution is governed by a fractional differential equation involving coupled space–time fractional derivative operators [20].

In this paper, we first consider the linear space-time fractional reaction-diffusion equation (LSTFRDE) of the form:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^\beta u(x,t)}{\partial x^\beta} = b(x)u(x,t) + f(x,t), \quad 0 < x < L, \quad t > 0, \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2
\]  

(1)

where \( \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \) is the Caputo time-fractional derivative of order \( 0 < \alpha \leq 1 \), \( \frac{\partial^\beta u(x,t)}{\partial x^\beta} \) is the Caputo space-fractional derivative of order \( 1 < \beta \leq 2 \) and \( 0 < b(x) \leq b_{\text{max}} \) and \( 0 < c(x) \leq c_{\text{max}} \) are continuous for \( 0 < x < L \).

We assume Dirichlet boundary conditions and an initial condition for this problem:

\[
u(x,0) = p(x) \quad (2)
\]
\[
u(0,t) = q_1(t) \quad (3)
\]
\[
u_t(0,t) = q_2(t) \quad (4)
\]
when $\alpha = 1$, $\beta = 2$, $b(x) = 1$, $c(x) = 0$ and $f(x,t) = 0$, we recover in the limit the well-known diffusion equation (Markovian process)

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

In the case $0 < \alpha < 1$ we have to consider all previous time levels (non-Markovian process). Then we consider the non-linear space-time fractional reaction-diffusion equation (NSTFRDE) of the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b \frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(u(x,t)) + g(x,t), \quad 0 < x < L, \quad t > 0, \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2$$ (5)

where $b > 0$ is a constant; $f(u)$ is a non-linear reaction term that models population growth. For example, a typical choice is Fisher’s equation $f(u) = ru(1 - u / K)$, where $r$ is the intrinsic growth rate and $K$ the carrying capacity. We assume that this NSTFRDE has a unique and sufficiently smooth solution under suitable initial conditions.

There are several definitions of a fractional derivative of order $\alpha > 0$. [4,21-25]. The two most commonly used definitions are the Riemann-Liouville and Caputo. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. The Riemann-Liouville fractional integration of order $\alpha$ is defined as

$$J_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0,$$

and the Caputo fractional derivatives of order $\alpha$ is defined as

$$D_0^\alpha f(x) = J_0^{\alpha-m} D^m f(x).$$

The Caputo fractional derivatives are considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the nonlinear fractional reaction-diffusion equation, and the fractional derivatives are taken in Caputo sense as follows.

**Definition 1.1.** For $m$ to be the smallest integer that exceeds $ \alpha $, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as
\[ D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial t^m} d\tau, & \text{for } m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N} \end{cases} \]

For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references. Recently Yu et al. [26] used Adomian decomposition method for solving the governing equation. In this paper, we will use Homotopy perturbation method for solving fractional reaction-diffusion equations.

The homotopy perturbation method (HPM) was first proposed by the Chinese mathematician Ji-Huan He [27-30]. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter \( p \in [0,1] \) which is considered as a “small parameter”. Considerable research works have been conducted recently in applying this method to a class of linear and non-linear equations [31-41]. The interested reader can see the Refs. [42-44] for last development of HPM.

2. HPM solutions of linear space-time fractional reaction-diffusion equation (LSTFDRE)

The LSTFDRE (1) can be written in terms of operator form as

\[ D_t^\alpha u(x,t) = b(x)D_x^\beta u(x,t) - c(x)u(x,t) + f(x,t), \quad 0 < x < L, \quad t > 0, \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2 \quad (6) \]

\[ u(x,0) = p(x) \quad (7) \]

\[ u(0,t) = q_1(t) \quad (8) \]

\[ u_x(0,t) = q_2(t) \quad (9) \]

In the HPM method we only use one of the initial and boundary conditions, depending on which operator is used for (6). If we use the inverse operator \( J_t^\alpha \) of \( D_t^\alpha \), then only the initial condition is used; if we used the inverse operator \( J_x^\beta \) of \( D_x^\beta \), then only use the boundary condition. In this study we will analyse the first situation. To solve Eqs. (6 - 9) by homotopy perturbation method, we construct the following homotopy

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = p \left[ b(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} - c(x)u(x,t) \right] + f(x,t) \quad (10) \]
In view of the HPM, we use the homotopy parameter \( p \) to expand solution

\[
u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots
\]

(11)

Substituting (11) into (10) and equating the coefficients of like powers of \( p \), we get the following set of fractional differential equations

\[
p^0 : \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = f(x,t),
\]

(12)

\[
p^1 : \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = b(x)\frac{\partial^\beta u_0(x,t)}{\partial x^\beta} - c(x)u_0(x,t),
\]

(13)

\[
p^2 : \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = b(x)\frac{\partial^\beta u_1(x,t)}{\partial x^\beta} - c(x)u_1(x,t),
\]

(14)

\[
p^3 : \frac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} = b(x)\frac{\partial^\beta u_2(x,t)}{\partial x^\beta} - c(x)u_2(x,t),
\]

(15)

\[
:\vdots
\]

Solving the above equations and using the initial conditions yield

\[
u_0(x,t) = p(x) + J_i^{\alpha} f(x,t)
\]

(16)

\[
u_1(x,t) = \left[ b(x)D_x^\beta - c(x) \right] p(x) \frac{t^\alpha}{\Gamma(\alpha + 1)} + J_i^{2\alpha} f(x,t)
\]

(17)

\[
u_2(x,t) = \left[ b(x)D_x^\beta - c(x) \right]^2 p(x) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + J_i^{3\alpha} f(x,t)
\]

(18)

\[
u_3(x,t) = \left[ b(x)D_x^\beta - c(x) \right]^3 p(x) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + J_i^{4\alpha} f(x,t)
\]

(19)

\[
:\vdots
\]

\[
u_n(x,t) = \left[ b(x)D_x^\beta - c(x) \right]^n p(x) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} + J_i^{(n+1)\alpha} f(x,t)
\]

(20)

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Thus, we have the solution in series form is given by

\[
u(x,t) = p(x) + J_i^{\alpha} f(x,t) + \left[ b(x)D_x^\beta - c(x) \right] \left[ p(x) \frac{t^{\alpha}}{\Gamma(\alpha + 1)} + J_i^{2\alpha} f(x,t) \right] + \ldots
\]

(21)
3. HPM solutions of non-linear space-time fractional reaction-diffusion equation (NLSTFDRE)

The NLSTFDRE (5) can be written in terms of operator form as

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b \frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(u(x,t)) + g(x,t), \quad 0 < x < L, \quad t > 0, \quad 0 < \alpha \leq 1, \quad 1 < \beta \leq 2 \]  \hspace{1cm} (22)

\[ u(x,0) = p(x) \]  \hspace{1cm} (23)

To solve Eqs. (22-23) by homotopy perturbation method, we construct the following homotopy:

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = p \left[ b \frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(u(x,t)) \right] + g(x,t) \]  \hspace{1cm} (24)

In view of the HPM, we use the homotopy parameter \( p \) to expand solution

\[ u = u_0 + pu_1 + p^2u_2 + p^3u_3 + ... \]  \hspace{1cm} (25)

Substituting (25) into (24) and equating the coefficients of like powers of \( p \), we get the following set of fractional differential equations

\[ p^0: \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = g(x,t), \]  \hspace{1cm} (26)

\[ p^1: \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = b \frac{\partial^\beta u_0(x,t)}{\partial x^\beta} + A_0 \]  \hspace{1cm} (27)

\[ p^2: \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = b \frac{\partial^\beta u_1(x,t)}{\partial x^\beta} + A_1 \]  \hspace{1cm} (28)

\[ p^3: \frac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} = b \frac{\partial^\beta u_2(x,t)}{\partial x^\beta} + A_2 \]  \hspace{1cm} (29)

\[ \vdots \]

where

\[ f(u(x,t)) = \sum_{n=0}^{\infty} p^{n+1} A_n \]  \hspace{1cm} (30)

Solving the above equations and using the initial conditions yield

\[ u_0(x,t) = p(x) + J^\alpha_t g(x,t) \]  \hspace{1cm} (31)

\[ u_t(x,t) = J^\alpha_t \left[ bD^\beta_x u_0(x,t) \right] + J^\alpha_t \left[ A_0 \right] \]  \hspace{1cm} (32)
and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Thus, we have the solution in series form is given by

\[
u(x,t) = p(x) + J_t^\alpha g(x,t) + J_t^\alpha \left[ bD_t^\beta u_0(x,t) \right] + J_t^\alpha \left[ A_0 \right] + \ldots + J_t^\alpha \left[ bD_t^\beta u_n(x,t) \right] + J_t^\alpha \left[ A_n \right]...
\]

(36)

4. Numerical examples

4.1 Example 1

Consider the following linear space – time fractional diffusion – reaction equation with boundary and initial conditions: [26]

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = b(x) \frac{\partial^\beta u(x,t)}{\partial x^\beta} - c(x)u(x,t) + f(x,t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1
\]

(37)

\[
u(0,t) = u_x(0,t) = 0
\]

(38)

\[
u(x,0) = p(x) = x^2 - x^3
\]

(39)

where the source function

\[
f(x,t) = 3(4t^2 + 1)x^3 + \frac{32}{3\sqrt{\pi}}t^{1.5}(x^2 - x^3)
\]

the coefficients of the diffusion and reaction terms are \( b(x) = \Gamma(1.2)x^{1.8}, \quad c(x) = 2 \).

When \( \alpha = 0.5, \quad \beta = 1.8 \), the exact solution of this problem is \( (4t^2 + 1)(x^2 - x^3) \), which can be verified by direct fractional differentiation of the given solution, and substituting in the fractional differential equation. The initial and boundary conditions are clearly satisfied. When \( \alpha = 0.5, \quad \beta = 1.8 \), according to homotopy perturbation procedures Eqs. (10)-(21), we now successively obtain
\[ u_0(x,t) = (4t^2 + 1)(x^2 - x^3) + 6x^3 \left( \frac{32t^{2.5}}{15\sqrt{\pi}} + \frac{t^{0.5}}{\sqrt{\pi}} \right) \]

\[ u_1(x,t) = -3x^3 \left( \frac{64t^{1.5}}{15\sqrt{\pi}} + \frac{2t^{0.5}}{\sqrt{\pi}} - 4t^3 - 3t \right) \]

\[ u_2(x,t) = -9x^3 \left( \frac{4t^3}{3} - \frac{128t^{1.5}}{35\sqrt{\pi}} + t - \frac{4t^{1.5}}{\sqrt{\pi}} \right) \]

\[ u_3(x,t) = -27x^3 \left( \frac{4t^{1.5}}{3\sqrt{\pi}} + \frac{128t^{3.5}}{105\sqrt{\pi}} - t^4 - \frac{3t^2}{2} \right) \]

\[ \vdots \]

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Thus, we have the solution in series form is given by

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \ldots \]  

(40)

We define the error between the solution of Equ. (9) and the exact solution as \( R_n \), where \( n \) is the number of series. Then, using mathematical induction we obtain

\[ R_n = 3^{n+1}x^3 \left( J_{(n+1)/2} (4t^2 + 1) \right) \]

\[ = 3^{n+1}x^3 \left( \frac{t^{(n+1)/2}}{\Gamma \left( \frac{n+3}{2} \right)} + \frac{8t^{(n+5)/2}}{\Gamma \left( \frac{n+7}{2} \right)} \right) \]  

(41)

Therefore,

\[ \phi_n(x,t) = \sum_{k=0}^{n-1} u_k(x,t) \]

\[ = (4t^2 + 1)(x^2 - x^3) + 3^n x^3 \left( \frac{t^{n/2}}{\Gamma \left( \frac{n+2}{2} \right)} + \frac{8t^{(n+4)/2}}{\Gamma \left( \frac{n+6}{2} \right)} \right) \]  

(42)

\[ u(x,t) = \lim_{n \to \infty} \phi_n(x,t) = (4t^2 + 1)(x^2 - x^3). \]
When $\alpha = 0.5$, $\beta = 1.8$, Figs. (1-2-3a-3c-3e) show the different approximate solutions obtained by applying the HPM with different numbers of terms in truncated series and the exact solution of STFDRE. Also Figs.(3b-3d) show the absolute error between approximate solutions and the exact solution.

**Fig 1.** Comparison of the exact solution and approximate solution at time $t=0.4$ for $\alpha = 0.5$ and $\beta = 1.8$. 
Fig 2. Comparison of the exact solution and approximate solution at time $t=0.4$ for $\alpha = 0.5$ and $\beta = 1.8$.  

\[
\alpha = \frac{1}{2}, \quad \beta = 1.8
\]

the number of terms in truncated series is 25
Fig 3. $\alpha = 0.5 \beta = 1.8$; (a) Approximate Solution (the number of terms in truncated series is 50); (b) Absolute Error; (c) Approximate Solution (the number of terms in truncated series is 4); (d) Absolute Error; (e) Exact Solution.
4.2 Example 2

Consider the following non-linear space–time fractional diffusion–reaction equation: [26]

\[
\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{\partial^\beta}{\partial x^\beta} u(x,t) + f(u(x,t)) + g(x,t), \quad 0 \leq x \leq 1, \quad t > 0
\]

(43)

\[
u(x,0) = 0, \quad 0 \leq x \leq 1
\]

(44)

where the non-linear reaction term is Fisher’s growth equation:

\[
f(u(x,t)) = 0.25u(x,t)(1-u(x,t))
\]

and

\[
g(x,t) = -0.0104649t^{0.9} + 0.00961766x^{1.1} - 0.0025t^{0.9}x^{1.1} + 0.000025t^{1.8}x^{2.2}.
\]

When \( \alpha = 0.9, \ \beta = 1.1 \), the exact solution of this problem is \( 0.01x^{1.1}t^{0.9} \), which can be verified by direct fractional differentiation of the given solution, and substituting in the fractional differential equation. The initial and boundary conditions are clearly satisfied. Using Eqs. (24) and (30), the first few polynomials \( A_n \) that represent the non-linear term \( 0.25u(x,t)(1-u(x,t)) \) are obtained as

\[
A_0 = 0.25u_0(1-u_0)
\]

\[
A_1 = u_1(0.25 - 0.5u_0)
\]

\[
A_2 = u_2(0.25 - 0.5u_0) + \frac{u_1^2}{2}(-0.5)
\]

\[
A_3 = u_3(0.25 - 0.5u_0) + u_1u_2(-0.5)
\]
When $\alpha = 0.9$, $\beta = 1.1$, using Eqs.(45), according to homotopy perturbation procedures Eqs. (24) - (36) we now successively obtain

$$u_0(x,t) = 0.01x^{1.1}t^{0.9} - 0.0143419x^{1.1}t^{1.8} - 0.0600346t^{1.8} + 0.0000100493x^{2.2}t^{2.7}$$
$$u_1(x,t) = 0.00600346t^{1.8} - 0.00120662x^{1.1}t^{2.7} - 2.30348 \times 10^{-6}t^{4.5} + (0.00143419t^{1.8} - 0.000144127t^{2.7} + 0.0000166107t^{3.6} - 1.10058 \times 10^{-6}t^{4.5})x^{1.1} + (-0.0000100493t^{2.7} + 3.01807 \times 10^{-6}t^{3.6} - 1.31461 \times 10^{-7}t^{4.5} + 6.55614 \times 10^{-9}t^{5.4})x^{2.2} + (-1.28454 \times 10^{-8}t^{4.5} + 1.56623 \times 10^{-9}t^{5.4})x^{3.3} + (-4.78235 \times 10^{-12}t^{6.3})x^{4.4}$$

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Thus, we have the solution in series form is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$  \hspace{1cm} (46)$$

When $\alpha = 0.9$ $\beta = 1.1$, Figs. (4-5) show the different approximate solutions obtained by applying the HPM with different numbers of terms in truncated series compared with the exact solution of NSTFDRE at time $t=0.4$ and $4$, respectively. From Figs. (4-5), it can be seen that the approximate solutions are in excellent agreement with the exact solution.
Fig 4. Comparison of the exact solution and the approximate solution at time $t=0.4$ for $\alpha = 0.5$ and $\beta = 1.8$.

Fig 5. Comparison of the exact solution and the approximate solution at time $t=4$ for $\alpha = 0.5$ and $\beta = 1.8$. The number of terms in the truncated series is 2, 3, and 5 for the exact solution.
4.3 Example 3

Consider the following non-linear space-time fractional diffusion-reaction equation: [26]

\[
\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = b \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + f(u(x,t)) + g(x,t), \quad 0 \leq x \leq 1, \quad t > 0
\]  

\[
u(x,0) = 0, \quad 0 \leq x \leq 1
\]  

where the non-linear reaction term

\[
f(u(x,t)) = u^3(x,t) - u^2(x,t)
\]

and

\[
g(x,t) = -0.00111018t^{0.8} + 0.000931384x^{1.2} + 1 \times 10^{-6}t^{1.6}x^{2.4} - 1 \times 10^{-9}t^{2.4}x^{3.6}.
\]

When \( \alpha = 0.8, \beta=1.2 \), the exact solution of this problem is \( 0.001x^{1.2}t^{0.8} \), which can be verified by direct fractional differentiation of the given solution, and substituting in the fractional differential equation. The initial and boundary conditions are clearly satisfied. Using Eqs.(24) and (30), the first few polynomials \( A_n \) that represent the non-linear term \( u^3(x,t) - u^2(x,t) \) are obtained as

\[
\begin{align*}
A_0 &= u_0^3 - u_0^2, \\
A_1 &= 3u_0^2u_1 - 2u_0u_2, \\
A_2 &= 3u_0u_1^2 + 3u_0^2u_2 - 2u_0u_2 - u_1^2, \\
A_3 &= u_1^3 + 6u_0u_1u_2 + 3u_0^2u_3 - 2u_1u_2 - 2u_0u_3.
\end{align*}
\]

When \( \alpha = 0.8, \beta=1.2 \), using Eqs.(49), according to homotopy perturbation procedures Eqs. (24) - (36) we now successively obtain
\[ u_0(x,t) = -0.000717812t^{1.6} + 0.001t^{0.8}x^{1.2} + 4.79546 \times 10^{-7}t^{2.4}x^{2.4} - 3.8434 \times 10^{-10}t^{3.2}x^{3.6} \]

\[ u_i(x,t) = 0.000717812t^{1.6} - 1.66528 \times 10^{-7}t^{4.0} - 9.18695 \times 10^{-11}t^{5.6} + (1.05046 \times 10^{-6}t^{2.2} + 4.33281 \times 10^{-10}t^{4.8})x^{1.2} \]

\[ +(-4.79546 \times 10^{-7}t^{2.4} - 1.25353 \times 10^{-9}t^{4.0} + 1.92973 \times 10^{-10}t^{4.8} + 1.65775 \times 10^{-13}t^{6.4})x^{2.4} \]

\[ +(3.8434 \times 10^{-10}t^{3.2} - 3.09974 \times 10^{-10}t^{4.0} - 6.50073 \times 10^{-13}t^{5.6} - 1.21088 \times 10^{-16}t^{7.2})x^{3.6} \]

\[ +(6.18717 \times 10^{-11}t^{4.8} - 5.71216 \times 10^{-14}t^{5.6} + 3.70189 \times 10^{-16}t^{6.4} - 1.00934 \times 10^{-16}t^{7.2})x^{4.8} \]

\[ +(-2.86403 \times 10^{-16}t^{5.6} + 2.36723 \times 10^{-16}t^{6.4} + 1.48881 \times 10^{-19}t^{8.0})x^{6.0} + (-2.55501 \times 10^{-19}t^{7.2} \]

\[ + 2.06834 \times 10^{-20}t^{8.0} - 5.53331 \times 10^{-23}t^{8.8})x^{7.2} + (8.31161 \times 10^{-21}t^{8.0} - 4.61232 \times 10^{-23}t^{8.8})x^{8.4} \]

\[ +(3.45072 \times 10^{-26}t^{9.6})x^{9.6} - (8.65263 \times 10^{-30}t^{10.4})x^{10.8} \]

and so on; in this manner, the rest of the components of the homotopy perturbation series can be obtained. Thus, we have the solution in series form is given by

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots \] (50)

When \( \alpha = 0.8 \) \( \beta = 1.2 \), Figs. (6-7) show the different approximate solutions obtained by applying the HPM with different numbers of terms in truncated series compared with the exact solution of NSTF DRE at time \( t=0.5 \) and \( 4 \), respectively. From Figs. (6-7), it can be seen that the approximate solutions are in excellent agreement with the exact solution.

Fig 6. Comparison of the exact solution and the approximate solution at time \( t=0.5 \) for \( \alpha = 0.8 \) and \( \beta = 1.2 \)
5. Conclusion

In this study, we used the homotopy perturbation method for the linear and nonlinear space–time fractional diffusion–reaction equations. The HPM is clearly a very efficient technique for finding the solutions of the proposed equations. Numerical results show that the approximate solutions obtained by applying the HPM are in excellent agreement with the exact solution. The mathematical technique employed in this paper is significant in studying some other problems of physics and engineering.

References


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