Analytical Solution of Second-Order Hyperbolic Telegraph Equation by Variational Iteration and Homotopy Perturbation Methods

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Analytical Solution of Second-Order Hyperbolic Telegraph Equation by Variational Iteration and Homotopy Perturbation Methods

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Abstract. In this research, two analytical methods, namely homotopy perturbation method and variational iteration method are introduced to obtain solutions of the initial value problem of hyperbolic type which is called telegraph equation. Some illustrative examples are presented to show the efficiency of the methods.

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1. Introduction

Recently, it is found that telegraph equation is more suitable than ordinary diffusion equation in modelling reaction diffusion for such branches of sciences. The hyperbolic partial differential equations model the vibrations of structures (e.g., buildings, beams, and machines) and they are the basis for fundamental equations of atomic physics. The telegraph equation is important for modeling several relevant problems such as signal analysis [1], wave propagation [2], random walk theory [3], etc. Partial differential equations are often very complicated to be solved exactly and even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or it might be difficult to interpret the outcome. Recently, some promising approximate analytical solutions are proposed, such as Exp-function method [4,5], Adomian

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decomposition method (ADM) \[6–10\], variational iteration method (VIM) \[11–17\] and homotopy perturbation method (HPM) \[18–22, 25\]. Other methods are reviewed in Refs. \[23, 24\]. VIM is to construct correction functional using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants. The method is the most effective and convenient one for both linear and nonlinear equations. The method also has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. VIM was first proposed by He \[11\] and was successfully applied to various engineering problems \[12–17\].

HPM is also a straightforward and convenient method for both linear and nonlinear equations. This method does not depend on a small parameter. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter \(p \in [0, 1]\), which is considered as a “small parameter” \[18\]. These methods are useful for obtaining both a closed form and explicit solutions and numerical approximations of linear or nonlinear differential equations and it is also quite straightforward to write computer codes. The interested reader can see \[26–29\] for the last development of the VIM and the HPM.

In this paper, we propose the application of HPM and VIM to solve the following one-dimensional second-order hyperbolic telegraph equation:

\[
\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u = \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad x \in \Omega = [a, b], \quad 0 < t \leq T,
\]

with the following initial conditions

\[
\begin{align*}
  u(x, 0) &= g_1(x), \quad x \in \Omega, \\
  u_t(x, 0) &= g_2(x), \quad x \in \Omega.
\end{align*}
\]

where \(\alpha\) and \(\beta\) are known constant coefficients, and \(f\) is a known function, and the function \(u\) is unknown. Equation (1.1), referred to the second-order telegraph equation with constant coefficients, models mixture between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation \[30\]. The reminder of this paper is organized as follows.

In Sect. 2, we describe the HPM and apply this technique to solve Eq. (1.1). In Sect. 3, we describe the procedure of VIM for solving Eq. (1.1). To show the efficiency of these methods, we give some examples in Sect. 4. The paper is concluded in Sect. 5. (The case I for the HPM is much attractive, see Eq. (10), how to construct homotopy equation is a hot topic recently. Hesameddini and Latifizadeh \[31, 32\] suggested some effective ways in his recent publications. 4 cases are discussed for the VIM, the case 2 of the VIM is very much suitable for inverse problems \[33\].)
2. Homotopy Perturbation Method

To illustrate the basic ideas of the HPM, we consider the following nonlinear differential equation:

\[ A(u) - f(r) = 0, \quad r \in \Omega, \]  

(2.1)

with the boundary conditions

\[ B \left( u, \frac{\partial u}{\partial n} \right) = 0, \quad r \in \Gamma, \]  

(2.2)

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f(r) \) is a known analytical function and \( \Gamma \) is the boundary of the domain \( \Omega \). Generally speaking, the operator \( A \) can be divided into two parts which are \( L \) and \( N \) where \( L \) is linear, but \( N \) is nonlinear. Therefore Eq. (2.1) can be rewritten as:

\[ L(u) + N(u) - f(r) = 0. \]  

(2.3)

By the homotopy perturbation technique, we construct a homotopy \( v(r, p) : \Omega \times [0, 1] \rightarrow R \) which satisfies:

\[ H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega, \]  

(2.4)

where \( p \in [0, 1] \) is an embedding parameter and \( u_0 \) is an initial approximation of Eq. (1.1). Obviously, from these we have:

\[ H(v, 0) = L(v) - L(u_0), \quad H(v, 1) = A(v) - f(r). \]  

(2.5)

The changing process of \( p \) from zero to one is just that of \( v(r, p) \) from \( u_0(r) \) to \( u(r) \). In topology, this is called deformation, and \( L(v) - L(u_0) \) and \( A(v) - f(r) \) are called homotopy. According to the HPM, we can first use the embedding parameter \( p \) as a “small parameter”, and assume that the solution of (2.4) can be written as a power series in \( p \):

\[ v = v_0 + pv_1 + p^2v_2 + \cdots. \]  

(2.6)

Setting \( p = 1 \), results in the exact solution of (3):

\[ v = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \]  

(2.7)

In order to solve the Eq. (1.1) using HPM, we construct the following homotopies:

Case 1.

\[ H(v, p) = (1 - p) \left( \frac{\partial^2}{\partial t^2} v(x, t) - f(x, t) \right) + p \left( \frac{\partial^2}{\partial t^2} v(x, t) + \alpha \frac{\partial}{\partial t} v(x, t) + \beta v(x, t) - \frac{\partial^2 v(x, t)}{\partial x^2} - f(x, t) \right) = 0. \]  

(2.8)
Substituting (2.6) into (2.8) and equating the coefficients of similar powers of $p$, yields

\[ p^0 : \frac{\partial^2 v_0(x,t)}{\partial t^2} - f(x,t) = 0, \quad v_0(x,0) = g_1(x), \quad \frac{\partial}{\partial t}v_0(x,0) = g_2(x). \quad (2.9) \]

\[ p^n : \frac{\partial^2 v_n(x,t)}{\partial t^2} + \alpha \frac{\partial v_{n-1}(x,t)}{\partial t} + \beta v_{n-1}(x,t) - \frac{\partial^2 v_{n-1}(x,t)}{\partial x^2} = 0, \quad v_{n-1}(x,0) = 0, \quad \frac{\partial}{\partial t}v_{n-1}(x,0) = 0, \quad (n \geq 1). \quad (2.10) \]

**Case 2.**

\[ H(v,p) = (1-p) \left( \frac{\partial^2 v(x,t)}{\partial t^2} \right) + p \left( \frac{\partial^2 v(x,t)}{\partial t^2} + \alpha \frac{\partial v(x,t)}{\partial t} + \beta v(x,t) - \frac{\partial^2 v(x,t)}{\partial x^2} - f(x,t) \right) = 0. \quad (2.11) \]

Substituting (2.6) in (2.11), yields

\[ p^0 : \frac{\partial^2 v_0(x,t)}{\partial t^2} = 0, \quad v_0(x,0) = g_1(x), \quad \frac{\partial}{\partial t}v_0(x,0) = g_2(x). \quad (2.12) \]

\[ p^1 : \frac{\partial^2 v_1(x,t)}{\partial t^2} + \alpha \frac{\partial v_0(x,t)}{\partial t} + \beta v_0(x,t) - \frac{\partial^2 v_0(x,t)}{\partial x^2} = 0, \quad v_1(x,0) = 0, \quad \frac{\partial}{\partial t}v_1(x,0) = 0. \quad (2.13) \]

\[ p^n : \frac{\partial^2 v_n(x,t)}{\partial t^2} + \alpha \frac{\partial v_{n-1}(x,t)}{\partial t} + \beta v_{n-1}(x,t) - \frac{\partial^2 v_{n-1}(x,t)}{\partial x^2} = 0, \quad v_{n-1}(x,0) = 0, \quad \frac{\partial}{\partial t}v_{n-1}(x,0) = 0, \quad (n \geq 2). \quad (2.14) \]

We note for $f = 0$ cases 1 and 2 are similar.

### 3. Variational Iteration Method

To illustrate the basic concepts of the VIM, we consider the following differential equation:

\[ L(u) + N(u) = g(x,t), \quad (3.1) \]

where $L$ is a linear operator, $N$ is a nonlinear operator and $g(x,t)$ is an inhomogeneous term. Then we can construct a correction functional as follows:

\[ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \{ Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau) \} \, d\tau, \quad (3.2) \]
where $\lambda$ is a general Lagrange multiplier, which can be identified optimally via variational theory. The second term on the right hand side is called the correction and is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$.

By determining $\lambda$, the approximations $u_n(x, t), \ n \geq 0$ obtain immediately. Consequently, the exact solution may be obtained by

$$u(x, t) = \lim_{n \to \infty} u_n(x, t). \quad (3.3)$$

We consider the Eq. (1.1) subject to the initial condition (1.2). According to the VIM, we can construct the following correction functional:

**Case 1.**

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} + \beta \tilde{u}_n(x, \tau) - \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau, \quad (3.4)$$

Making the above correction functional stationary, and noticing that $\delta \tilde{u}_n = 0$, we have

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} + \beta \tilde{u}_n(x, \tau) - \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau$$

$$= \delta u_n(x, t) + \lambda \delta \frac{\partial u_n(x, \tau)}{\partial \tau} - \lambda' \delta u_n(x, \tau) + \int_0^t \lambda'' \delta u_n(x, \tau) d\tau,$$

which yields the following stationary conditions

$$\delta u_n : 1 - \lambda'(t) = 0,$$

$$\delta \frac{\partial u_n}{\partial t} : \lambda(t) = 0,$$

$$\delta u_n : \lambda(\tau) = 0,$$ \quad (3.6)

Therefore, the general Lagrange multiplier can be readily identified as

$$\lambda(\tau) = \tau - t.$$ \quad (3.7)
Substituting this value of the Lagrangian multiplier into functional (3.4) gives

\[
u_{n+1}(x, t) = u_n(x, t) + \int_0^t (\tau - t) \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial u_n(x, \tau)}{\partial \tau} + \beta u_n(x, \tau) - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau,
\]

(3.8)

Case 2.

\[
u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left\{ \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial \tau^2} + \beta \tilde{u}_n(x, \tau) + \alpha \frac{\partial \tilde{u}_n(x, \tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau,
\]

(3.9)

By a similar calculation, the following stationary conditions obtain:

\[
\begin{align*}
\delta u_n &: 1 - \lambda'(t) = 0, \\
\delta \frac{\partial u_n}{\partial t} &: \lambda(t) = 0, \\
\delta u_n &: \lambda''(\tau) + \beta \lambda(\tau) = 0,
\end{align*}
\]

(3.10)

Therefore \( \lambda \) can be identified as if \( \beta > 0 \)

\[
\lambda(\tau) = \frac{1}{\sqrt{\beta}} \sin \sqrt{\beta}(\tau - t)),
\]

(3.11)

and if \( \beta < 0 \)

\[
\lambda(\tau) = \frac{1}{2\sqrt{-\beta}}(\exp(\sqrt{-\beta}(\tau - t)) - \exp(\sqrt{-\beta}(t - \tau))).
\]

(3.12)

Thus, we obtain the following iteration formulas: if \( \beta > 0 \)

\[
u_{n+1}(x, t) = u_n(x, t) + \int_0^t \frac{1}{\sqrt{\beta}} \sin \sqrt{\beta}(\tau - t)) \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial u_n(x, \tau)}{\partial \tau} + \beta u_n(x, \tau) - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau,
\]

(3.13)

and if \( \beta < 0 \)

\[
u_{n+1}(x, t) = u_n(x, t) + \int_0^t \frac{1}{2\sqrt{-\beta}} \left( \exp(\sqrt{-\beta}(\tau - t)) - \exp(\sqrt{-\beta}(t - \tau)) \right) \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial u_n(x, \tau)}{\partial \tau} + \beta u_n(x, \tau) - \frac{\partial^2 u_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau,
\]

(3.14)
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Case 3.

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial u_n(x, \tau)}{\partial \tau} + \beta \tilde{u}_n(x, \tau) - \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau, \]  

(3.15)

In this case the following stationary conditions result in

\[ \delta u_n : 1 + \alpha \lambda(t) - \lambda'(t) = 0, \]
\[ \delta \frac{\partial u_n}{\partial t} : \lambda(t) = 0, \]
\[ \delta u_n : \lambda''(\tau) - \alpha \lambda'(\tau) = 0, \]  

(3.16)

and therefore we get

\[ \lambda(\tau) = \frac{1}{\alpha} \left( \exp(\alpha(\tau - t)) - 1 \right), \]  

(3.17)

also the iteration formula is given as

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \frac{1}{\alpha} \left( \exp(\alpha(\tau - t)) - 1 \right) \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial u_n(x, \tau)}{\partial \tau} + \beta u_n(x, \tau) - \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau, \]  

(3.18)

Case 4. For the last case we set

\[ u_{n+1}(x, t) = u_n(x, t) + \int_0^t \left\{ \frac{\partial^2 u_n(x, \tau)}{\partial \tau^2} + \alpha \frac{\partial u_n(x, \tau)}{\partial \tau} + \beta u_n(x, \tau) - \frac{\partial^2 \tilde{u}_n(x, \tau)}{\partial x^2} - f(x, \tau) \right\} d\tau, \]  

(3.19)

And then the following stationary conditions result in

\[ \delta u_n : 1 + \alpha \lambda(t) - \lambda'(t) = 0, \]
\[ \delta \frac{\partial u_n}{\partial t} : \lambda(t) = 0, \]
\[ \delta u_n : \lambda''(\tau) - \alpha \lambda'(\tau) + \beta \lambda(\tau) = 0, \]  

(3.20)

Therefore \( \lambda \) can be identified as:

\[ \text{if } \alpha^2 - 4\beta > 0 \]
\[ \lambda(\tau) = \frac{1}{\sqrt{\alpha^2 - 4\beta}} \left( \exp \left( \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2} (\tau - t) \right) - \exp \left( \frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2} (t - \tau) \right) \right), \]  

(3.21)
and if $\alpha^2 - 4\beta < 0$

$$\lambda(\tau) = \frac{2}{\sqrt{4\beta - \alpha^2}} \exp\left(\frac{\alpha}{2}(\tau - t)\right) \sin\left(\frac{\sqrt{4\beta - \alpha^2}}{2}(\tau - t)\right), \quad (3.22)$$

else

$$\lambda(\tau) = (\tau - t) \exp\left(\frac{\alpha}{2}(\tau - t)\right). \quad (3.23)$$

Now we begin with an initial approximation $u_0(x, t) = g_1(x) + tg_2(x)$ by the above iteration formula, we can obtain $u_n(x, t)$ for $n \geq 1$.

4. Test Examples

In order to assess the advantages and accuracy of HPM and VIM, we consider the following examples. Results are computed using the program written by Maple 11.

**Example 1.** As the first example, consider the hyperbolic telegraph Eq. (1.1) in the interval $0 \leq x \leq \pi$ [34]. The initial conditions are given by

$$u(x, 0) = \sin(x), \quad 0 \leq x \leq \pi,$$

$$u_t(x, 0) = -\sin(x), \quad 0 \leq x \leq \pi.$$ 

In this example we set $\alpha = 4$, $\beta = 2$ and $f(x, t) = 0$

A: Implementation of HPM

By using HPM (in the cases 1 and 2) for this example, we obtain

$$v_0(x, t) = (1 - t) \sin(x),$$

$$v_1(x, t) = \left(\frac{1}{2}t^2 - \frac{1}{2}t^3\right) \sin(x),$$

$$v_2(x, t) = \left(-\frac{2}{3}t^3 - \frac{5}{8}t^4 - \frac{3}{40}t^5\right) \sin(x),$$

$$v_3(x, t) = \left(\frac{2}{3}t^4 + \frac{3}{5}t^5 + \frac{9}{80}t^6 + \frac{3}{560}t^7\right) \sin(x),$$

$$\cdots$$

The exact solution of Example 1 when $p \to 1$ is as

$$u(x, t) = v_0(x, t) + v_1(x, t) + v_2(x, t) + v_3(x, t) + \cdots$$

$$= \left(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 - \cdots\right) \sin(x) = \exp(-t) \sin(x).$$

B: Implementation of VIM

By $u_0(x, t) = (1 - t) \sin(x)$ we have
Case 1.

\[ u_1(x, t) = \left(1 - t + \frac{1}{2}t^2 - \frac{1}{2}t^3\right)\sin(x), \]

\[ u_2(x, t) = \left(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 - \frac{5}{8}t^4 - \frac{3}{40}t^5\right)\sin(x), \]

\[ u_2(x, t) = \left(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{147}{280}t^5 + \frac{63}{560}t^6 + \frac{3}{560}t^7\right)\sin(x), \]

Therefore, the exact solution of \( u(x, t) \) in a closed form is

\[ u(x, t) = \exp(-t)\sin(x). \]

Figures 1, 2 and 3 express the absolute errors arising from the VIM in cases 2–4.

Example 2. In the following problem, we consider the hyperbolic telegraph Eq. (1.1) with \( \alpha = 1, \beta = 1 \) and \( f(x, t) = x^2 + t - 1 \) in the interval \( 0 \leq x \leq 1 \) [35]. The initial conditions are given by

\[ u(x, 0) = x^2, \quad 0 \leq x \leq 1, \]

\[ u_t(x, 0) = 1, \quad 0 \leq x \leq 1. \]

A. Implementation of HPM
Figure 2. Plot of absolute error function for Example 1 by VIM (case 3) with $n = 10$

Figure 3. Plot of absolute error function for Example 1 by VIM (case 4) with $n = 8$

By using HPM for this example, we obtain

Case 1. We can identify $v_n$ for $n = 0, 1, 2, \ldots$ and therefore we obtain the $n$th approximation of the exact solution as. Figure 4 represents the absolute errors for this equation obtained by HPM.

Case 2.

\begin{align*}
    v_0(x, t) &= x^2 + t, \\
    v_1(x, t) &= v_2(x, t) = v_3(x, t) = \cdots = 0,
\end{align*}
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Therefore, we have

\[ u(x, t) = v_0(x, t) + v_1(x, t) + v_2(x, t) + \cdots = x^2 + t + 0 + 0 + \cdots = x^2 + t, \]

which is the exact solution.

B: Implementation of VIM

Starting with \( u_0(x, t) = x^2 + t \) in cases 1–4, we have

\[
\begin{align*}
  u_1(x, t) &= x^2 + t \\
  u_2(x, t) &= x^2 + t \\
  u_3(x, t) &= x^2 + t \\
  &\vdots \\
  &\vdots 
\end{align*}
\]

Therefore, we have

\[ u(x, t) = \lim_{n \to \infty} u_n(x, t) = \lim_{n \to \infty} x^2 + t = x^2 + t, \]

which is the exact solution.

Example 3. In our third example, we consider the hyperbolic telegraph Eq. (1.1) with \( \alpha = 1 \) and \( \beta = 1 \) in the interval \( 0 \leq x \leq 1 \) [30]. The initial conditions are taken as

\[
\begin{align*}
  u(x, 0) &= 0, \quad 0 \leq x \leq 1, \\
  u_t(x, 0) &= 0, \quad 0 \leq x \leq 1, 
\end{align*}
\]
and \( f(x, t) = (2 - 2t + t^2)(x - x^2)\exp(-t) + 2t^2\exp(-t) \). The exact solution of this problem is \( u(x, t) = (x - x^2)\exp(-t)t^2 \).

A. Implementation of HPM

Similar to Example 2 we obtain the \( n \)th approximation of the exact solution as \( u_n = v_0 + v_1 + \cdots + v_n \). Absolute error functions for this equation obtained by HPM are plotted in Figs. 5 and 6.
Figure 7. Plot of absolute error function for Example 3 by VIM (case 1) with \( n = 12 \)

Figure 8. Plot of absolute error function for Example 3 by VIM (case 2) with \( n = 12 \)

B: Implementation of VIM

Graphs of absolute error functions obtained by VIM for this equation are shown in Figs. 7, 8 and 9.
5. Conclusion

We have successfully applied homotopy perturbation and variational iteration methods to obtain the exact or approximate solutions of the second-order hyperbolic telegraph equation. The main advantage of the methods shows that they provide an analytical approximation, in many cases exact solution, in a rapidly convergent sequence with elegantly computed terms. Their small sizes of computation in comparison with the computational size required in other numerical methods and their rapid convergence show that the methods are reliable and introduce a significant improvement in solving partial differential equations over existing methods. Illustrative examples are included to demonstrate the validity and applicability of these techniques.

References

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