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Numerical Solution of Systems of Partial Differential Equation
Using New Variational Homotopy Perturbation Method

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Abstract: This research paper deals with the systems of partial differential equations by using the New Variational Homotopy Perturbation Method. The New Method does not require discretization, linearization or any restrictive assumption of any form in providing analytical or approximate solutions to linear and nonlinear equation. Theses virtues make it to be reliable and its efficiency is demonstrated with numerical examples.

Mathematics subject classification: 35k15, 35C05, 65D99, 65M99

Keywords: systems of partial differential equations; Variational Iteration Method, Homotopy Perturbation Method, New Variational Homotopy Perturbation Method, Boundary Value Problems; Initial Value Problems.

I. Introduction

System of partial differential equations have attracted much attention in a variety of applied Sciences. The general ideas and the essential features of these systems are of wide applicability. These system were formally derived to describe wave propagation, to control the shallow water waves and to examine the chemical reaction-diffusion model of Brusselator. The method of characteristics, the Riemann invariants, Adomian decomposition method [1-3], Homotopy perturbation method [4-9], Homotopy analysis method and Laplace decomposition method [10-17], were the commonly used methods. In order to improve on existing method of solution we introduce the New Variational Homotopy Perturbation Method for System of partial differential equations which is time cost effective and users friendly.

II. New Variational Homotopy Perturbation Method

We extent the new scheme to systems of partial differential equations of form:

\[ L_1 U(x,t) + N_1(U(x,t), V(x,t)) = f(x,t) \]
\[ L_2 U(x,t) + N_2(U(x,t), V(x,t)) = g(x,t) \]  

(1)

Where \( L_1 \) and \( L_2 \) are linear differential operators, with respect to time; \( N_1 \) and \( N_2 \) are non-linear operators and \( f(x,t) \) and \( g(x,t) \) are given functions. According to the variational iteration method, we construct a correction functionals as follows [18]:

\[ U_{n+1}(x,t) = U_n(x,t) + \int_0^\tau \lambda_1(\tau)L_n U_n(x,\tau) + N_1(\overline{U}_n(x,\tau),\overline{V}_n(x,\tau)) - f(x,\tau) d\tau \]  

(2)

\[ V_{n+1}(x,t) = V_n(x,t) + \int_0^\tau \lambda_2(\tau)L_n U_n(x,\tau) + N_2(\overline{U}_n(x,\tau),\overline{V}_n(x,\tau)) - g(x,\tau) d\tau \]  

(3)

where \( \lambda_1 \) and \( \lambda_2 \) are general Lagrange multipliers, which can be identified, optimally, via a variational theory [2]. The second term on the right-hand side of equations (2) and (3) is called the correction and the subscript ‘n’ denotes the \( n^{th} \) order approximation. \( \overline{U}_n \) and \( \overline{V}_n \) are considered as a restricted variation, one can assume that the above correction functionals are stationary i.e. \( \delta \overline{U}_n = 0 \) and \( \delta \overline{V}_n = 0 \), then, the Lagrange multipliers can be identified.

Now we apply the Homotopy Perturbation method to the correction functional in equations (2) and (3) above. With the introduction of the power series into the correction functionals, we have the following:
Numerical Solution of Systems of Partial Differential Equation Using New Variational Homotopy

\[ U_0 + PU_1 + P^2U_2 + \ldots = U_0 - P \int \lambda_1(\tau) [L_1(U_0 + PU_1 + P^2U_2 + \ldots)] d\tau \]
\[ + N_1((\overline{U}_0 + P\overline{U}_1 + P^2\overline{U}_2 + \ldots) + (\overline{V}_0 + P\overline{V}_1 + P^2\overline{V}_2 + \ldots)) d\tau \]
\[ - \int \lambda_1(\tau)f(x, \tau) d\tau \]
\[ V_0 + PV_1 + P^2V_2 + \ldots = V_0 - P \int \lambda_2(\tau) [L_2(U_0 + PU_1 + \ldots)] d\tau \]
\[ + N_2((\overline{U}_0 + P\overline{U}_1 + \ldots)(\overline{V}_0 + P\overline{V}_1 + \ldots)) d\tau \]
\[ - \int \lambda_2(\tau)g(x, \tau) d\tau \]

which can be expressed as:
\[ \sum_{n=0}^{\infty} P^n U_n = U_0(x, t) + P \int \lambda_1(\tau) \left[ L_1 \sum_{n=0}^{\infty} P^n U_n + N_1 \left( \sum_{n=0}^{\infty} P^n \overline{U}_n, \sum_{n=0}^{\infty} P^n \overline{V}_n \right) \right] d\tau \]
\[ - \int \lambda_1(\tau)f(x, \tau) d\tau \]
\[ \sum_{n=0}^{\infty} P^n V_n = V_0(x, t) + P \int \lambda_2(\tau) \left[ L_2 \sum_{n=0}^{\infty} P^n U_n + N_2 \left( \sum_{n=0}^{\infty} P^n \overline{U}_n, \sum_{n=0}^{\infty} P^n \overline{V}_n \right) \right] d\tau \]
\[ - \int \lambda_2(\tau)g(x, \tau) d\tau \]

Hence, equations (6) and (7) represent the New Variational Homotopy Perturbation method for systems of partial differential equations.

The comparison of the coefficients of like powers of \( p \) gives solution of various orders. This implies:

\[ P^0 : U_0 = U_0(x, t) - \int \lambda_1(\tau)f(x, \tau) d\tau \]
\[ V_0 = V_0(x, t) - \int \lambda_2(\tau)g(x, \tau) d\tau \]
\[ P^1 : U_1 = \int \lambda_1(\tau)(L_1 U_0 + N_1(\overline{U}_0, \overline{V}_0)) d\tau \]
\[ V_1 = \int \lambda_2(\tau)(L_2 U_0 + N_2(\overline{U}_0, \overline{V}_0)) d\tau \]
\[ P^2 : U_2 = \int \lambda_1(\tau)(L_1 U_1 + N_1(\overline{U}_1, \overline{V}_1)) d\tau \]
\[ V_2 = \int \lambda_2(\tau)(L_2 U_1 + N_2(\overline{U}_1, \overline{V}_1)) d\tau \]
\[ P^n : U_n = \int \lambda_1(\tau)(L_1 U_{n-1} + N_1(\overline{U}_{n-1}, \overline{V}_{n-1})) d\tau \]
\[ V_n = \int \lambda_2(\tau)(L_2 U_{n-1} + N_2(\overline{U}_{n-1}, \overline{V}_{n-1})) d\tau \]

Therefore, the series solutions are given as:
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\[ U(x,t) = U_0 + U_1 + U_2 + \ldots U_n \]
\[ V(x,t) = V_0 + V_1 + V_2 + \ldots V_n \]  \hfill (9)

Hence,
\[ U(x,t) = U_0(x,t) + \int_0^t \lambda_1(\tau) \{L_1 U_0 + N_1(U_0, \nabla U_0)\} d\tau + \]
\[ \int_0^t \lambda_2(\tau) \{L_2 U_0 + N_2(U_0, \nabla U_0)\} d\tau + \]
\[ \ldots \]
\[ \int_0^t \lambda_n(\tau) \{L_n U_0 + N_n(U_0, \nabla U_0)\} d\tau \]
\[ - \int_0^t \lambda_1(\tau) f(x,\tau) d\tau \]  \hfill (10)

and
\[ V(x,t) = V_0(x,t) + \int_0^t \lambda_2(\tau) \{L_2 U_0 + N_2(U_0, \nabla U_0)\} d\tau + \]
\[ \int_0^t \lambda_2(\tau) \{L_2 U_0 + N_2(U_0, \nabla U_0)\} d\tau + \]
\[ \ldots \]
\[ \int_0^t \lambda_n(\tau) \{L_n U_0 + N_n(U_0, \nabla U_0)\} d\tau \]
\[ - \int_0^t \lambda_2(\tau) g(x,\tau) d\tau \]  \hfill (11)

III. Numerical Examples

Example 1:
We consider the model equations for the coupled Schrodinger-kdv equation given by Doosthoseini and Shahmohamadi (2010) [8]:
\[ U_i = U_{xx} + UW \]
\[ U_i = 6WW_x + W_{xx} = \left( U^2 \right)_x \]  \hfill (12)

Where \( i = \sqrt{-1} \)
By using \( U = u + iv \), one can separate equation (12) into real and imaginary parts demonstrated by Doosthoseini and Shahmohamadi (2010). Therefore, one can get a \((1+1)\)-dimensional tripled system in the following form:
\[ U_i = V_{xx} + VW = 0 \]
\[ V_i = U_{xx} + UW = 0 \]
\[ W_i = 6WW_x + W_{xx} - 2UU_x - 2VV_x = 0 \]  \hfill (13)

We construct a correction functional as follows:
\[ U_{n+1}(x,t) = U_n(x,t) + \int_0^t \lambda_1 \left[ \frac{\partial U_n(x,\tau)}{\partial \tau} - \frac{\partial^2 V_n(x,\tau)}{\partial x^2} - \nabla_n(x,\tau) W_n(x,t) \right] d\tau \]  \hfill (14)
\[ V_{n+1}(x,t) = V_n(x,t) + \int_0^t \lambda_2 \left[ \frac{\partial V_n(x,\tau)}{\partial \tau} - \frac{\partial^2 U_n(x,\tau)}{\partial x^2} - \nabla_n(x,\tau) W_n(x,t) \right] d\tau \]  \hfill (15)
\[ W_{n+1}(x,t) = W_n(x,t) + \int_0^t \lambda_3 \left[ \frac{\partial W_n(x,\tau)}{\partial \tau} + 6W_n(x,\tau) \frac{\partial W_n(x,\tau)}{\partial x} - \frac{\partial^3 W_n(x,\tau)}{\partial x^3} \right. \]
\[ \left. - 2U_n(x,\tau) \frac{\partial U_n(x,\tau)}{\partial x} - 2V_n(x,\tau) \frac{\partial V_n(x,\tau)}{\partial x} \right] d\tau \]  \hfill (16)
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where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are general Lagrange multipliers.

Its stationary conditions can be obtained as follows:

\[
\begin{align*}
1 + \lambda_1(\tau)|_{\tau=1} + \lambda_2(\tau) = 0 \\
1 + \lambda_2(\tau)|_{\tau=1} + \lambda_2(\tau) = 0 \\
1 + \lambda_3(\tau)|_{\tau=1} + \lambda_3(\tau) = 0
\end{align*}
\]

(Hence \( \lambda_1 = \lambda_2 = \lambda_3 = -1 \) and the correction functional becomes

\[
\begin{align*}
U_{n+1} &= U_n - \int_{0}^{1} \left( \frac{\partial U_n}{\partial \tau} + \frac{\partial^2 V_n}{\partial x^2} - V_n W_n \right) d\tau \\
V_{n+1} &= V_n - \int_{0}^{1} \left( \frac{\partial V_n}{\partial \tau} - \frac{\partial^2 U_n}{\partial x^2} - U_n W_n \right) d\tau \\
W_{n+1} &= W_n - \int_{0}^{1} \left( \frac{\partial W_n}{\partial \tau} - 6W_n \frac{\partial^2 W_n}{\partial x^2} - \frac{\partial^3 W_n}{\partial x^3} - 2U_n \frac{\partial U_n}{\partial x} - 2V_n \frac{\partial V_n}{\partial x} \right) d\tau
\end{align*}
\]

Applying Homotopy Perturbation to equations (18), (19) and (20), we have:

For equation (18)

\[
\begin{align*}
P^0 : U_0 &= \cos x \\
P^1 : U_0 &= t \left( \frac{\partial \cos x}{\partial \tau} - \frac{\partial^2 \sin x}{\partial \tau} - \sin x \left( \frac{3}{4} \right) \right) \\
&= -\left( \frac{t^3}{4} \right) \sin x
\end{align*}
\]

For equation (19)

\[
\begin{align*}
P^0 : V_0 &= \sin x \\
P^1 : V_0 &= t \left( 0 - \cos x + \frac{3}{4} \cos x \right) \\
&= \frac{t^4}{4} \cos x
\end{align*}
\]

For equation (20)

\[
\begin{align*}
P^0 : W_0 &= \cos x \\
P^1 : W_0 &= t \left( \frac{\partial \cos x}{\partial \tau} + \frac{\partial^2 \sin x}{\partial \tau} + \sin x \left( \frac{3}{4} \right) \right) \\
&= -\left( \frac{t^4}{4} \right) \sin x
\end{align*}
\]
Comparing the coefficients of like power of \( p \):

\[
P^0 : W_0 = \frac{3}{4}
\]

\[
P^1 : W_1 = -t \left( 0 - \left( \frac{3}{4} \right) 0 \right) + (0) - 2 \cos x (- \sin x) - 2 \sin x \cos x
\]

\[= 0\]

Following the same procedure, we have

\[
P^2 : U_2 = -t \left( \frac{\partial U_1}{\partial \tau} - \frac{\partial^3 U_2}{\partial x^2} - V W_1 \right)
\]

\[= -\frac{t}{4} \sin x - \frac{t^2}{4} \cos x\]

\[
P^2 : V_2 = -t \left( \frac{\partial V_1}{\partial \tau} - \frac{\partial^3 V_2}{\partial x^2} - U W_1 \right)
\]

\[= -\frac{t}{4} \cos x - \frac{t^2}{4} \sin x\]

\[
P^2 : W_2 = -t \left( \frac{\partial W_1}{\partial \tau} - 6 W_1 \frac{\partial^2 W_1}{\partial x^2} - \frac{\partial^3 W_1}{\partial x^3} - 2 U \frac{\partial^2 U_1}{\partial x^2} - 2 V \frac{\partial V_1}{\partial x} \right)
\]

\[= 0\]

Therefore the solutions are:

\[
U(x, t) = \cos x - \frac{t}{4} \sin x + \frac{t}{4} \sin x - \frac{t^2}{4} \cos x = \cos x \left( 1 - \frac{t^2}{4} \right)
\]

\[= \frac{3}{4}\]

\[\text{(24)}\]

\[
V(x, t) = \sin x - \frac{t}{4} \cos x + \frac{t}{4} \cos x - \frac{t^2}{4} \sin x = \sin x \left( 1 - \frac{t^2}{4} \right)
\]

\[\text{(25)}\]

\[
W(x, t) = \frac{3}{4} + 0 + 0 = \frac{3}{4}
\]

\[\text{(26)}\]
Table 2 for example 1 for V:

<table>
<thead>
<tr>
<th>X</th>
<th>EXACT</th>
<th>VIM/HPM</th>
<th>ERROR</th>
<th>NV/HPM</th>
<th>ERROR</th>
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<tr>
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<td>0.010908</td>
<td>0.133790</td>
<td>1.22882x10</td>
<td>0.009272</td>
<td>1.636x10</td>
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<td>0.107014</td>
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<td>0.007261</td>
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<td>0.006545</td>
<td>0.080250</td>
<td>9.8287x10</td>
<td>-2</td>
<td>0.005354</td>
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<td>0.053495</td>
<td>-2</td>
<td>0.003526</td>
</tr>
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<td>0.001750</td>
</tr>
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<td>0.0000000</td>
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<td></td>
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</table>

Table 3 for Example 1:

<table>
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<tr>
<th>X</th>
<th>EXACT</th>
<th>VIM/HPM</th>
<th>ERROR</th>
<th>NV/HPM</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
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<td>-0.5</td>
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<td>0.7500000</td>
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<td>0.7500000</td>
<td>0.0000000</td>
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<tr>
<td>-0.3</td>
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<td>0.0000000</td>
<td>0.7500000</td>
<td>0.0000000</td>
</tr>
<tr>
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</tr>
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<td>0.0000000</td>
</tr>
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<td>0.7500000</td>
<td>0.7500000</td>
<td>0.0000000</td>
<td>0.7500000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.7500000</td>
<td>0.7500000</td>
<td>0.0000000</td>
<td>0.7500000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>0.2</td>
<td>0.7500000</td>
<td>0.7500000</td>
<td>0.0000000</td>
<td>0.7500000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7500000</td>
<td>0.7500000</td>
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<td>0.0000000</td>
</tr>
<tr>
<td>0.4</td>
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<td>0.0000000</td>
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<td>0.0000000</td>
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<tr>
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<td>0.7500000</td>
<td>0.0000000</td>
<td>0.7500000</td>
<td>0.0000000</td>
</tr>
</tbody>
</table>

Example 2:
We consider a coupled system of nonlinear physical equations given by Abdoul et al (2009) [2]:

\[
\frac{\partial U(x,t)}{\partial t} = U(1-U^2-V) + U_{xx}, \quad t > 0
\]

\[
\frac{\partial V(x,t)}{\partial t} = V(1-U-V) + V_{xx},
\]

With initial conditions

\[
U(x,0) = \frac{e^{xkr}}{1 + e^{xkr}}
\]

\[
V(x,0) = \frac{1 + \frac{3}{4}e^{xkr}}{1 + e^{xkr}}^2,
\]

and the exact solutions are

\[
U(x,t) = \frac{e^{x(kr+t)}}{1 + e^{x(kr+t)}}
\]

\[
V(x,t) = 1 + \frac{3}{4}e^{x(kr+t)}
\]

For \( k = 1, 2, 3, 4 \) and \( t = 0.01 \)
The correction functionals are given as

$$U_{n+1} = U_n(x,t) - \int_0^t \left[ \frac{\partial U_n(x,\tau)}{\partial \tau} - \frac{\partial^2 U_n(x,\tau)}{\partial x^2} - U_n(x,t) + U_n^3(x,t) \right] d\tau$$

$$V_{n+1}(x,t) = V_n(x,t) - \int_0^t \left[ \frac{\partial V_n(x,\tau)}{\partial \tau} - \frac{\partial^2 V_n(x,\tau)}{\partial x^2} - V_n(x,t) + V_n^2(x,t) \right] d\tau$$

Applying the Homotopy Perturbation method to equations (30) & (31), we have

$$U_0 + PU_1 + P^2U_2 + ... = U_0 - P \int_0^t \left[ \frac{\partial}{\partial \tau} (U_0 + PU_1 + ...) - \frac{\partial^2}{\partial x^2} (U_0 + PU_1 + ...) \right] d\tau$$

$$V_0 + PV_1 + P^2V_2 + ... = V_0 - P \int_0^t \left[ \frac{\partial}{\partial \tau} (V_0 + PV_1 + ...) - \frac{\partial^2}{\partial x^2} (V_0 + PV_1 + ...) \right] d\tau$$

Comparing the coefficients of like powers of $p$, we have

$$P^0 : U_0 = \frac{e^{kx}}{1 + e^{3kx}}$$

$$: V_0 = \frac{1 + \left(\frac{3}{4}\right)e^{kx}}{1 + e^{kx}}^2$$

$$P^0 : U_1 = \frac{4k^2 e^{kx} t - 4k^2 e^{3kx} t + 5e^{2kx} t + 5e^{3kx} t}{4\left[1 + e^{kx}\right]^4}$$

$$: V_1 = \frac{16ke^{kx} t - 32e^{kx} t + 2le^{2kx} t + 16e^{3kx} t}{16\left[1 + e^{kx}\right]^4}$$

Therefore, the series solutions are

$$U = U_0 + U_1 + ....$$

$$= \left[4e^{kx}(1 + e^{kx})^3 + (4k^2 e^{kx} t - 4k^2 e^{3kx} t + 5e^{2kx} t + 5e^{3kx} t) \right] / 4(1 + e^{kx})^4$$

and

$$V = V_0 + V_1 + ....$$

$$= \left[16 + 44e^{kx} t + 40e^{kx} t + 12e^{3kx} t + 16ke^{kx} t + 32e^{kx} t + 2le^{2kx} t + 16e^{3kx} t \right] / 16(1 + e^{kx})^4$$
In this paper, New Variational Homotopy Perturbation Method has been successfully applied to find the solutions of system of partial differential equations and the results obtained were compared with the two conventional variational iteration and Homotopy Perturbation Method. It can be concluded that the NVHPM is very powerful and efficient technique for finding approximation solutions for wide classes of problems. It is worth mentioning that the Method is computational cost friendly.

IV. Conclusion

TABLE 1 FOR EXAMPLE 2 FOR U:

<table>
<thead>
<tr>
<th>X</th>
<th>EXACT</th>
<th>VIM/HPM</th>
<th>ERROR</th>
<th>NVHPM</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
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<td>0.54983</td>
<td>0.27629</td>
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<td>0.540098</td>
<td>9.732x10^-3</td>
</tr>
<tr>
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<td>0.59869</td>
<td>0.30535</td>
<td>2.933x10^-1</td>
<td>0.68032</td>
<td>8.163x10^-2</td>
</tr>
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</tr>
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<tr>
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<td>0.73106</td>
<td>0.41218</td>
<td>3.189x10^-1</td>
<td>0.73637</td>
<td>5.311x10^-2</td>
</tr>
</tbody>
</table>

TABLE 2 FOR EXAMPLE 2 FOR V:

<table>
<thead>
<tr>
<th>X</th>
<th>EXACT</th>
<th>VIM/HPM</th>
<th>ERROR</th>
<th>NVHPM</th>
<th>ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.17479</td>
<td>0.003182</td>
<td>1.716x10^-1</td>
<td>0.161797</td>
<td>1.299x10^-2</td>
</tr>
<tr>
<td>0.2</td>
<td>0.13695</td>
<td>0.002567</td>
<td>1.344x10^-1</td>
<td>0.135013</td>
<td>1.937x10^-3</td>
</tr>
<tr>
<td>0.3</td>
<td>0.10529</td>
<td>0.002050</td>
<td>1.032x10^-1</td>
<td>0.111854</td>
<td>6.564x10^-3</td>
</tr>
<tr>
<td>0.4</td>
<td>0.07954</td>
<td>0.001621</td>
<td>7.929x10^-2</td>
<td>0.092062</td>
<td>1.252x10^-4</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05911</td>
<td>0.001270</td>
<td>5.784x10^-2</td>
<td>0.075330</td>
<td>1.622x10^-4</td>
</tr>
</tbody>
</table>

REFERENCES: