State-dependent anisotropy: Comparisons of quasi-analytical solutions with stochastic results for steady gravity drainage

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Abstract. Anisotropy in large-scale unsaturated hydraulic conductivity of layered soils changes with the moisture state. Here, state-dependent anisotropy is computed under conditions of large-scale gravity drainage. Soils represented by Gardner's exponential function are perfectly stratified, periodic, and inclined. Analytical integration of Darcy's law across each layer results in a system of nonlinear equations that is solved iteratively for capillary suction at layer interfaces and for the Darcy flux normal to layering. Computed fluxes and suction profiles are used to determine both upscaled hydraulic conductivity in the principal directions and the corresponding "state-dependent" anisotropy ratio as functions of the mean suction. Three groups of layered soils are analyzed and compared with independent predictions from the stochastic results of Yeh et al. (1985b). The small-perturbation approach predicts appropriate behaviors for anisotropy under nonarid conditions. However, the stochastic results are limited to moderate values of mean suction; this limitation is linked to a Taylor series approximation in terms of a group of statistical and geometric parameters. Two alternative forms of the Taylor series provide upper and lower bounds for the state-dependent anisotropy of relatively dry soils.

1. Introduction

Soil hydraulic properties vary in magnitude at different locations. Previous studies [Byers and Stephens, 1983; Russo and Bouron, 1992] revealed significant variability at scales of centimeters to meters, and Childs et al. [1957] reported periodic layering of clay and fine sand, each about 5 mm thick. Deposition and weathering on slopes also can cause the stratification to be inclined relative to gravity [e.g., Lotts and Smart, 1953]. Many practical problems require description and prediction of hydrologic behavior over scales larger than the observed scale of variability with depth, i.e., hydraulic properties averaged over multiple layers are needed. Similarly, most numerical simulators use discrete blocks at scales larger than individual layers. Thus hydraulic properties of many locally homogeneous layers must be spatially averaged ("upscaled") to the scale of interest.

The present analysis pertains to steady unsaturated flow during large-scale (mean) gravity drainage, i.e., zero gradient in the mean soil water or capillary potential. During large-scale gravity drainage in layered systems, the local capillary gradients within a layer can be significant and even exceed the unit gravitational gradient. For example, Figure 1 shows two profiles of simulated capillary suction within soil layers of finite thickness (hydraulic properties of each layer are given below). In this example, four soil layers repeat periodically in the vertical direction; Figure 1 shows two periods. For each suction profile the vertical flux is steady and thus constant over depth. The mean suction (dashed line) over a period of four layers is the same for both periods; this defines "mean gravity drainage" for periodic soils. At a given mean capillary suction the local suction $\psi$, water content $\theta$, and unsaturated hydraulic conductivity $K$ may vary considerably throughout the layered soil profile. Spatial variability in suction, and thus $\theta(\psi)$ and $K(\psi)$, changes with the mean suction $\bar{\psi}$. Generally, a direct average (e.g., arithmetic or harmonic mean) of the local unsaturated hydraulic conductivities at the mean suction does not yield the correct upscaled conductivity.

An upscaling procedure requires solution of the small-scale (local) flow equation before computing spatial averages of the hydraulic properties. Darcy's law must be solved with the local soil properties, and the upscaled $K$ can be determined from the resulting Darcy flux and mean head gradient. Yeh et al. [1985b] studied three-dimensional, unsaturated flow through statistically stratified soils using a spectral/stochastic approach with a linear perturbation method. The analysis provides functions for upscaled $K$, which is defined by the large-scale Darcy's law:

$$E[|q_i|] = -\bar{K}_{ij}J_i,$$

where $E[|q_i|]$ and $J_i$ are the expected values of the Darcy flux and the mean head gradient, respectively, in the $i$th direction and $\bar{K}_{ij}$ is the tensorial notation for upscaled $K$.

Yeh et al. [1985b] found that large-scale anisotropy in $\bar{K}_{ij}$ can change significantly with the mean suction. The state-dependent (or suction-dependent) behavior of anisotropy is qualitatively consistent with the findings of other investigators [Zaslavsky and Sinai, 1981; Mudd, 1984; Bear et al., 1987; Stephens and Heermann, 1988; McCord and Stephens, 1987].
McCord et al., 1991; Wallach and Zaslavsky, 1991]. Physical experiments showing reasonable agreement with the stochastic results for $K_{ij}$ and suction-dependent anisotropy have been performed in the laboratory using alternating layers of only two soil types in columns (one-dimensional analysis) [Yeh and Harvey, 1990] and in sand tanks (two dimensions) [Stephens and Heermann, 1988]. Field studies [Zaslavsky and Struii, 1981; McCord and Stephens, 1987; Russo and Bouton, 1992] and numerical experiments in one dimension [Yeh, 1989], two dimensions [Wallach and Zaslavsky, 1991], and three dimensions [Polman et al., 1991] provide further evidence that the stochastic results are qualitatively observed. Despite such promising agreement between experiments and the stochastic theory, the accuracy of the stochastic results for multidimensional flow in perfectly stratified soils has not been quantified over a broad range of moisture states.

Presently, an analytical method is pursued as an alternative approach to physical and numerical experiments for testing the predictive capability of the stochastic results of Yeh et al. [1985b]. Because a pure analytical solution is not available, our quasi-analytical method uses analytical integration and an iterative numerical solver. The conceptual model for this quasi-analytical method is similar to that of Wallach and Zaslavsky [1991], who numerically simulated unsaturated flow through dipping, layered soils of infinite lateral extent. No two-dimensional analytical solutions that quantitatively compare with the stochastic results for state-dependent anisotropy have been published prior to the present work.

Unsaturated hydraulic conductivity at the Darcy scale is often modeled as a closed-form function of suction. The present study requires a simple function to integrate Darcy's law for each soil layer and determine the suction-head distribution in space. Gardner's exponential function [Gardner, 1958; Pullan, 1990], characterizes the local soils' unsaturated $K$ functions at the local Darcy scale:

$$K(\psi) = K^* \exp (-\alpha \psi),$$

where $K^*$ is the saturated hydraulic conductivity and $\alpha$ is a fitting parameter.

Figure 1. Profiles of local capillary suction and corresponding mean suction for large-scale gravity drainage through two periods of four layers of Bet Dagan soils (group 2). Layer numbers correspond to the soils in Table 1.

The stochastic analysis of Yeh et al. [1985b] invoked Gardner's function at the local scale, but the resulting upscaled unsaturated hydraulic conductivity curve $K_0(\psi)$ generally does not coincide with Gardner's function. An upscaled conductivity model must include characteristics of the spatial variability in soil hydraulic properties in addition to the mean parameters. For perfectly stratified soils, Yeh et al.'s results give $K_0$ as a closed-form function of the mean suction and of the mean values, variances, and spatial autocorrelations of Gardner's $\alpha$ and $K^*$. The tensorial or directional properties of $K_0$ (i.e., $\bar{K}$) also depend on the angle between the mean head gradient and the direction normal to stratification. Under mean gravity drainage this angle is simply the dip angle of the layering, and the mean flow through tilted strata has a lateral component given by the off-diagonal components of $\bar{K}$. Tilted strata also alter the local suction distribution compared with the suction resulting from vertical flow through horizontal layers; thereby, in turn, affects the conductivity and anisotropy ratio.

Assumptions of Yeh et al.'s stochastic method (discussed below) can be violated by the actual behavior of local $K$ properties, the variance in Gardner parameters, and extrapolation to high capillary suction. The purposes of this paper are to (1) evaluate the accuracy of the stochastic results under ideal conditions for the theory (Gardner-type local soils, small parameter variances, and relatively low suction, i.e., not extremely dry soils) and (2) extrapolate the results beyond the stated limits (larger variances in the Gardner parameters and drier conditions).

The present work tests extrapolation of the anisotropy function to arid field conditions with the constraint of Gardner's function to represent the unsaturated hydraulic conductivity of local soils. In the field and laboratory, Gardner's function cannot represent many soils' $K(\psi)$ functions over the entire range of relevant suction. Deviations are most pronounced near full saturation and below saturations of about 0.3, but very few data exist in the drier range [Mualem, 1976a]. Previous comparisons of the stochastic results with physical experiments have not quantified the consequences of misrepresenting the local $K(\psi)$ curves. By using Gardner's function, this study isolates errors in the stochastic approach from errors associated with the characterization of $K$ at the local scale.

2. Stochastic Analysis

Yeh et al. [1985a, b] solved for the expected values in (1) using a stochastic analysis, but they made several approximations to yield closed-form models for $\bar{K}$. Most of the assumptions and simplifications needed to predict $\bar{K}$ are explicitly stated in the literature (called "explicit limitations" here), while others are not explicitly stated ("implicit limitations").

The present work invokes some of the limitations used by Yeh et al. [1985b]. Our perfectly stratified soils correspond to Yeh et al.'s infinite ratio of correlation lengths parallel and perpendicular to layering. Although both analyses use perfectly stratified soils, our analysis assumes periodic layering of a finite number of soil layers having discrete thicknesses. Such abrupt changes at layer interfaces do not conform at the local scale with the concept of a correlation length in a second-order stationary random field. Layers dip at a given angle, and both analyses invoke a unit mean head gradient in the vertical direction, which defines mean gravity drainage.

The stochastic results give expected values of the hydraulic conductivities over an infinite ensemble of realizations, and the assumption of stationarity implies an infinite domain in space. Application of the stochastic results involves equating a statis-
tical ensemble average with a spatial average, usually over a finite domain. Russo [1992] included nonstationary conditions to solve the finite domain problem, but his analyses did not yield closed-form solutions. Russo defined both a block size and a length scale of nonuniformity of the average flow that should be much larger (10 to 20 times) than the scale of soil heterogeneity in order for stationarity to apply.

The stochastic approach of Yeh et al. [1985b] assumes unbounded domains and uniform average flow. The present domains are periodic in the direction perpendicular to layering, and the average flow is uniform over the scale of one period of soil layers. Periodicity is consistent with the spectral representation of stationarity, except that variability exists over a finite range of scales, rather than at all scales. To apply spectral analyses to stochastic processes, Priestley [1981, pp. 206–215] derived stationary solutions from periodicity by taking the limit as the period goes to infinity. Kitadis [1990] used a spatial average for periodic media to derive an effective saturated hydraulic conductivity that is identical to the stochastic results for an ensemble average [Gutjahr et al., 1978]. Thus ensemble averages can be used to predict spatial averages in periodic media.

2.1. Small-Perturbation Approach and Terminology

Small-perturbation methods in stochastic analyses limit the form and amount of variability in both the Gardner parameters and capillary suction. Yeh et al. [1985a, equations (4a)–(4c)] represented each parameter or variable in (2) by its arithmetic mean and zero-mean perturbations about the mean:

$$
\psi = \bar{\psi} + \hat{\psi}
$$

(3a)

$$
\alpha = \bar{\alpha} + \hat{\alpha}
$$

(3b)

$$
\ln K^{\text{stat}} = F + f
$$

(3c)

Local soil properties $K^{\text{stat}}$ and $\alpha$ are defined by Gardner’s model (2), and $\psi$ is a local suction value. Perturbations ($\hat{\psi}, \hat{\alpha}, F, f$) are considered to be small compared with the means ($\bar{\psi}, \bar{\alpha}, F, f$), such that products of the perturbations were neglected in solving for the expected values in the steady state form of Richards’ equation [Richards, 1951].

2.2. Taylor Series Expansions

In addition to the small-perturbation limitation, Yeh et al. dropped terms beyond second order in the expected values [Yeh et al., 1985a, equation (12)]. They attempted to account for these lost higher-order terms in the expressions for $K$ by invoking an “exponential generalization” [Gelhar and Axness, 1983, p. 168]. The exponential form becomes important for extrapolation to dry soils, even when the variance in Gardner’s conductivity parameters is not large, because the variance in $K(\psi)$ increases with the mean suction.

Analytical expressions for the upscaled hydraulic conductivities in the principal directions take the forms [Yeh et al., 1985b, p. 462, equation (33)]

$$
\bar{K}_{11} = K_m(1 - \chi) = K_m e^{-\chi}
$$

(4)

$$
\bar{K}_{22} = K_m(1 + \chi) = K_m e^{2\chi}
$$

(5)

where, now, the subscripts 11 and 22 refer to the directions perpendicular and parallel to the layering, respectively. $K_m$ is a mean unsaturated conductivity that changes with the mean suction [Yeh et al., 1985a, p. 451]. The dimensionless term $\chi$ is a function of the variability in hydraulic properties, their spatial correlation length, the dip angle relative to the mean head gradient, and the mean suction:

$$
\chi = \chi_a = \frac{\sigma^2 + \sigma^2 \bar{\psi}^2}{2(1 + \bar{\alpha} \lambda \cos \omega)}
$$

(6)

or

$$
\chi = \chi_{sc} = \frac{\sigma^2(1 - \bar{\theta} \bar{\psi})^2}{2(1 + \bar{\alpha} \lambda \cos \omega)}
$$

(7)

for the cases of statistical independence (si) and perfect correlation (pc) between $\alpha$ and $\ln K^{\text{stat}}$. Overbars indicate the arithmetic mean values of $\alpha$ and $\psi$. The correlation length perpendicular to layering for both $\alpha$ and $\ln K^{\text{stat}}$ is $\lambda$, and $\omega$ is the dip angle for stratification. For the case of perfect correlation, the autocorrelation function for $\alpha$ is a constant $\xi^2$ multiplied by the autocorrelation function for $\ln K^{\text{stat}}$ (i.e., $\xi^2 = \sigma^2/\sigma^2_f$).

The stochastic analysis using the small-perturbation method results in this dimensionless group ($\chi$) of soil and geometric parameters. For upscaled conductivity, the middle terms in (4) and (5) are the direct result of the stochastic analysis, including truncation of terms higher than first order in $\chi$. The right-hand side of each equation reflects a second approximation; this exponential generalization assumes that second- and higher-order $\chi$ terms are missing from a truncated Taylor series of an exponential expression. Gelhar and Axness [1983] used the exponential function to extend the analyses of saturated flow to cases of moderately large variance in $\ln K^{\text{stat}}$. For unsaturated flow, $\chi$ is a function of both the media and the moisture state (i.e., $\psi$). The literature to date places no explicit limitations on the mean capillary suction $\bar{\psi}$.

Figure 2 shows predictions (including extrapolation to large $\chi$ values) of state-dependent anisotropy, $U$, as functions of $\chi$ using three possible functional forms of the stochastic results. The first curve, $U = (1 + \chi)/(1 - \chi)$, is a ratio of the direct (first-order) stochastic results for $K$ in the principal directions. It cannot be used for extrapolation because $\bar{K}_{11}$ in (4) would become negative for $\chi > 1$. The second curve, $U = e^{2\chi}$, is a ratio of the exponential generalizations (equations (4) and (5)) of the stochastic results. The third curve, $U = 1 + 2\chi + 2\chi^2$, would be a candidate for extrapolation.
is an alternative form used for extrapolation below. All three functional forms are equivalent for small $\chi$, but moderately dry field soils require extrapolation up to $\chi$ values significantly greater than 1. It is very important, therefore, to choose an appropriate form of $U$ for extrapolation of the stochastic results.

### 2.3. Anisotropy Ratios

The anisotropy ratio and its Taylor series approximation for the small-perturbation result are

$$\frac{K_{22}}{K_{11}} \approx \frac{1 + \chi}{1 - \chi} = 1 + 2\chi + 2\chi^2 + 2\chi^3 + \cdots$$  \hspace{1cm} (8)

Yeh et al. [1985b, p. 462] computed state-dependent anisotropy using a ratio of exponential approximations to the directional conductivities:

$$\frac{K_{22}}{K_{11}} = e^x = e^{2\chi} = 1 + 2\chi + 2\chi^2 + \frac{4}{3}\chi^3 + \frac{2}{3}\chi^4 + \cdots$$  \hspace{1cm} (9)

We propose an alternative form that agrees with both the above Taylor series through the second-order term. This truncated Taylor series for state-dependent anisotropy $U(\chi)$ is

$$U(\chi) \approx 1 + 2\chi + 2\chi^2$$  \hspace{1cm} (10)

The following comparisons of quasi-analytical solutions with stochastic results consider both forms of $U(\chi)$ given in (9) and (10).

### 3. Methodology for Quasi-Analytical Solutions

We solve for steady, two-dimensional unsaturated flow through dipping, layered soils of infinite lateral extent. Soil layers are assumed to be periodic in the direction normal to the dip. Figure 3 shows one period of finite-thickness soil layers with the geometry and boundary conditions used to analyze mean gravity drainage through inclined soil layers. The z coordinate is oriented vertically downward and aligned with gravity and x is the horizontal direction, while n is normal to layering and y points downslope. The angle of inclination is $\omega$, and each layer has a thickness of $b$. Uniform values of suction $\psi_n$ along the upper and lower boundaries are equivalent to simulating mean gravity drainage through periodic soil layers. In this case the mean suction and upscaled hydraulic properties are the same for every period, so computation over one period is sufficient.

#### 3.1. Governing Equations for Mean Gravity Drainage Through Periodic, Layered Soils

After Wallach and Zaslavsky [1991] the infinite lateral extent and constant boundary suction make the problem quasi-one-dimensional normal to the slope [ct. Philip, 1991], i.e.,

$$\frac{\partial \psi}{\partial s} = 0$$ \hspace{1cm} (11)
$$\frac{\partial q_n}{\partial s} = 0.$$ \hspace{1cm} (12)

At steady state,

$$\nabla \cdot q = 0 \Rightarrow \frac{\partial q_n}{\partial n} = 0.$$ \hspace{1cm} (13)

In this manner we can analytically integrate Darcy's law across each homogeneous layer using Gardner's function for $K(\psi)$. The local expressions for Darcy's law in each direction are

$$q_i = K_i(\psi) \sin \omega$$ \hspace{1cm} (14)
$$q_n = K_i(\psi) \left( \frac{\partial \psi}{\partial n} + \cos \omega \right)$$ \hspace{1cm} (15)

where $i$ designates a soil type in Figure 3 at a distance $n$ normal to stratification and $\psi = \psi(n)$. Darcy's law and continuity above constitute the steady state form of Richards' equation also used by Yeh et al. [1985a, equation (2)].

#### 3.2. Analytical Integration and Simultaneous Equations

Analytical integration of Darcy's law across each homogeneous layer allows us to compute suction at layer interfaces and flux normal to stratification using continuity of flux and head across layers. Note that $\partial q_n/\partial n \neq 0$, and (14) gives a solution for $q_n$ only after solving for $\psi(n)$. Conversely, $q_n$ is constant throughout the domain, so we rearrange and integrate (15):

$$\int_{n=n_{i-1}}^{n_{i+1}} dn = \int_{\psi_{i-1}}^{\psi_{i+1}} \frac{d\psi}{[q_i/K_i(\psi)] \cos \omega}$$ \hspace{1cm} (16)

where $i$ is the layer index for $i = 1, \cdots, M$ soil layers. The unknowns are $q_n$ and $\psi_1, \psi_2, \cdots, \psi_{M-1}$ at the layer interfaces. Also, $K_i(\psi) = K^\infty \exp (-\alpha_i \psi)$.

Analytical integration and exponentiation to remove a natural logarithm yields a system of $M$ equations,

$$\Gamma_i = \left( \frac{q_n}{K^\infty_i e^{\alpha_i \psi_{i-1}} - \cos \omega} \right) e^{\alpha_i \psi_{i-1} - \frac{1}{2} \sin \omega} - 1 = 0$$ \hspace{1cm} (17)

Iteration is required to solve for the zero roots of the functions $\Gamma_i$ for $i = 1, \cdots, M$. The derivatives, $\partial \Gamma_i/\partial \psi_{i-1}, \partial \Gamma_i/\partial \psi_{i}, \partial \Gamma_i/q_n$, constitute the elements of a Jacobian matrix. This matrix allows us to solve a system of nonlinear, simultaneous equations for the flux and interface suctions using the Newton-Raphson method. Convergence is quite sensitive to the initial
estimates of the unknowns, so the method requires trial-and-error solution, especially for many layers.

Our procedure for determining the initial estimate uses physical information about suction gradients. If \( q_n \) is known, the direction of monotonic suction gradient within each layer can be determined for a given suction at the top of the layer. For \( i = 1 \), \( \psi_{i-1} = \psi_0 \) and \( K_x(\psi_0) \) are known. If \( q_n \) \( \geq K_x(\psi_0) \), a positive suction gradient is needed in addition to the unit gravitational gradient. We use this knowledge to constrain the solution, such that \( \psi_i > \psi_0 \). Conversely, if \( 0 < q_n < K_x(\psi_0) \), we know that \( \psi_i < \psi_0 \) and the magnitude of the vertical suction gradient cannot exceed unity, so \( \psi_0 - \psi_i < b/\cos \omega \). This alternative constraint is \( \psi_0 \) \( < (b/\cos \omega) \times \psi_1 < \psi_0 \) Suction values at each interface \( \psi_i \) are computed stepping from \( i = 1, \ldots, M \). For mean gravity drainage we must iterate using different values of \( q_n \), until \( \psi_M \approx \psi_0 \). If \( \psi_M < \psi_0 \), the present value of \( q_n \) is too low, and conversely, \( q_n \) is too large if \( \psi_M > \psi_0 \). The process is repeated for different values of \( q_n \) using this logic and the bisection method.

### 3.3. Computing Upscaled Hydraulic Properties
From Known \( \psi \) Distributions

Analytical solution of individual integrals similar to (16) gives the function \( n(\psi) \) at regular \( \psi \) intervals between known values of \( \psi \) at the layer interfaces. To get the inverse relation, \( \phi(n) \), at regular increments in \( n \), we interpolate within each layer. Computing a sufficient number of discrete \( n(\psi) \) values per layer ensures accuracy of the interpolated \( \phi(n) \) distribution. Finally, we solve the large-scale versions of the directional Darcy’s law (equations (14) and (15)) for \( \bar{K}_{22} \) in the \( s \) direction and for \( \bar{K}_{11} \) in the \( n \) direction. Noting that the mean suction gradient is zero, components of the unit vertical, mean head gradient in the \( n \) and \( s \) directions are \( \cos \omega \) and \( \sin \omega \), respectively. Because \( q_n \) is constant (\( q_n = q_n \)), rearrangement of the large-scale Darcy’s law yields

\[
\bar{K}_{11}(\psi) = \frac{q_n}{\cos \omega}.
\]

However, \( q_n \) varies with \( n \), so we must compute its arithmetic average, \( \bar{q}_n \), to determine the upscaled \( \bar{K} \) parallel to layering,

\[
\bar{K}_{22} = \frac{\bar{q}_s}{\sin \omega} = \frac{1}{B} \int_{\omega=0}^{\infty} K_x(\psi) \, dn,
\]

where \( B = Mb \) is the thickness of one period of soil layers.

### 3.4. Application of the Quasi-Analytical Solution

The computed upscaled conductivities and their anisotropy ratios are very accurate, but not exact solutions. Analytical integration of (16) is exact, but numerical round-off error associated with the iterative solution to (17) and integration of the conductivity distribution (19) make the solutions presented below quasi-analytical. For brevity, results of the present quasi-analytical method are referred to as “analytical” results below.

Analytical results are given for three test cases: (1) coarse and medium sand layers packed in the laboratory by Yeh and Harvey [1990]; (2) sandy soils from a trench in Bet Dagan, Israel [Russo and Bouton, 1992]; and (3) hypothetical soils with perfect correlation between Gardner’s \( \alpha \) and \( \ln K_{75} \) based on three California soils (J. Constantz, personal communication, 1993). Table 1 shows Gardner’s parameters for each homogeneous soil along with sample statistics for each group of soils.

<table>
<thead>
<tr>
<th>Table 1. Local Gardner Soil Parameters and Group (Sample) Statistics</th>
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<tr>
<td><strong>Soils</strong></td>
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<tr>
<td>Lab sands</td>
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<td>medium</td>
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<td>Bet Dagan soils</td>
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<td>layer 2</td>
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<td>layer 4</td>
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<td>Group 3</td>
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Sample variance of \( \ln K_{75}(\bar{\psi}) \), sample standard deviation of \( \alpha \) (\( s_f \)), and sample mean of \( \alpha \) (\( m_{s1} \)) are used in the stochastic results to predict the upscaled behavior.

We explain other relevant details about the soils data in the results section.

The quasi-analytical approach allows us to compute upscaled \( K \) in the principal directions and the associated anisotropy over a range of mean suction. We compare the analytical results with independent predictions from the stochastic results using sample statistics computed from the known soil properties in Table 1. Given the exact hydraulic properties and known layer geometries, the quasi-analytical solutions for each soil group yield the “true” (considered to be exact) upscaled \( K \) and suction-dependent anisotropy. The stochastic results, on the other hand, are not exact and represent ensemble behaviors. They are used to predict the “true” space-averaged behavior using known soil parameter statistics and apparent correlation lengths.

### 3.5. Error Analysis of the Stochastic Predictions

Section 4 shows prediction accuracies for both \( \bar{K} \) and \( U \) graphically, rather than using an integral measure of goodness of fit [e.g., Mualem, 1976]. Semilog plots show differences in \( \log \bar{K} \) and \( \log U \) over the computed range of \( \bar{\psi} \). In this manner, prediction errors can be compared visually among \( \log \bar{K}_{11} \), \( \log \bar{K}_{22} \), and \( \log U \). Using superscripts \( p \) and \( d \) to denote stochastic predictions and analytical results, respectively, the prediction error in \( \log U \) is related to errors in upscaled conductivity as follows:

\[
\log U_p - \log U_d = \log \left( \frac{U_p}{U_d} \right) = \log \left( \frac{K_{22}}{K_{11}} \right) = (\log K_{22} - \log \bar{K}_{22}) - (\log K_{11} - \log \bar{K}_{11})
\]
or using \( \varepsilon \) to designate the prediction error in quantity \( X \),

\[
\varepsilon_{\log \psi} = \varepsilon_{\log K_{11}} - \varepsilon_{\log K_{22}}
\]

(20)

Errors in anisotropy can be less than the individual errors in upscaled \( K \) if \( K_{11} \) and \( K_{22} \) are both overpredicted or both underpredicted.

4. Results

4.1. Group 1: Coarse and Medium Sands

Yeh and Harvey [1990] conducted drainage experiments in the laboratory using alternating layers of coarse and medium sands in a vertical column. Their reported bulk densities (1.45–1.56 g/cm³) and high porosities (0.411–0.454) indicate that the sands are relatively loosely packed. Both soils have the same reported \( K_{\text{sat}} \), but the fitted values of \( K_{\text{sat}} \) and slopes of the \( \log K \) curves differ significantly as shown in Table 1. The contrast in fitted values of \( K_{\text{sat}} \) for Gardner’s model results in an extremely high sample variance in ln \( K_{\text{sat}} \) (\( \sigma^2 = 11.85 \)) used for the stochastic result.

The stochastic predictions use a correlation length of half the layer thickness (\( \lambda = 7.5 \) cm) for both cases of statistical independence (si) and perfect correlation (pc) between ln \( K_{\text{sat}} \) and \( \alpha \). Perfect correlation implies a common point of intersection for every local \( K(\psi) \) curve. Any two soils with intersecting \( K(\psi) \) curves have a common \( K \) at some value of \( \psi \) where the layered system becomes isotropic. Thus the stochastic prediction for perfect correlation between ln \( K_{\text{sat}} \) and \( \alpha \) should apply to group 1.

Figure 4 shows the analytical results for \( \bar{K} \) in both principal directions computed with \( \omega = 0 \), the laboratory results of Yeh and Harvey, and the stochastic results for perfect correlation (pc). The stochastic results yield independent predictions based on the statistics of Gardner parameters in Table 1. In the applicable range of suctions above the air entry values for both soils (\( \psi > 10 \) cm), the analytical results for the upscaled vertical conductivity \( K_{11} \), agree with the laboratory data within likely measurement error. The stochastic result underestimates \( K_{11} \), as also shown by Yeh and Harvey. Furthermore, the stochastic result consistently underestimates the analytical values for \( K_{22} \) in this suction range.

This case of simple, periodic layering of two sands illustrates several characteristics of the analytical results for upscaled conductivity and anisotropy in comparison with the stochastic results. Although Yeh and Harvey conducted one-dimensional tests, we compute \( \bar{K} \) in two dimensions for horizontal layering. When the layers are tilted, the mean flow has a lateral component, and \( \bar{K} \) changes with the angle of inclination. We test the effect of the dip angle on \( \bar{K} \) and the resulting anisotropy with a moderately steep slope (\( \omega = 30^\circ \)). Figure 5 shows components of \( \bar{K} \) in the principal directions and their ratio \( U \) for both horizontal and inclined layering. The predicted effect from (4)–(7) is that \( \chi(\psi) \) increases as \( \omega \) increases, so the dip causes a decrease in \( K_{11} \) and an increase in \( K_{22} \). The analytical results show no significant difference in \( K_{22} \) between the two dip angles, while \( K_{11} \) is lower for the inclined case than for the horizontal case. Therefore \( U \) for \( \omega = 30^\circ \) is greater than \( U \) for \( \omega = 0 \).

Of particular interest in Figure 5 are the differences in accuracy between the stochastic predictions for \( \bar{K} \) and \( U \). First, we look at the relatively small values of \( \psi \) above air entry in the range of 10–18 cm of suction. Here, the stochastic result predicts \( U(\psi) \) well, but it underpredicts \( \bar{K} \) in both directions. In both the lower range of \( \psi \) and at higher suctions, the stochastic
prediction is better for $U$ than for $\bar{K}$. As expected, the stochastic result is not reliable for $\chi > 1$ ($\phi > 18$ cm). Other possible reasons for prediction errors in $U$ at both very low and high suctions are the large sample variance ($\chi^2 = 11.85$), which violates the small-perturbation assumption, and the use of only two soil types (layers) per period. In this scenario, however, the analytically computed anisotropy remains bounded by the two stochastic cases of cross correlation (pc and si) as suggested by Yeh et al. [1985b].

Figure 5 illustrates how the direct average overestimates the suction-dependent anisotropy. The “direct average” in Figure 5 is a ratio of the arithmetic divided by harmonic mean values of $K$ for each layer at the mean suction. The curve displays significant overestimation of the anisotropy at high values of $\bar{\psi}$, despite yielding isotropic behavior at the correct suction. The contrast in $K(\psi)$ values between the two sands causes significant variation of the suction within each layer. The spatial distribution of $K(\psi)$ values associated with this variability in $\psi$ cannot be represented by a single value of $K(\bar{\psi})$ for each soil type, as is assumed by using a direct average of $K$ at a uniform suction.

4.2. Group 2: Bet Dagan Soils

For the second scenario we use Gardner parameters for four sandy soils sampled at 10-cm depth intervals at Bet Dagan, Israel [Russo and Bouton, 1992]. Russo and Bouton estimate the correlation length in the vertical direction for a larger number of samples to be 19 cm for $K^{\alpha}$. Table 1 lists values of the Gardner parameters and group sample statistics for the four soils. Russo and Bouton collected these sandy soils in relatively undisturbed cores to preserve the in situ properties. The mean Gardner parameter $\alpha$ is lower for the Bet Dagan soils than for Yeh and Harvey’s lab soils, and consequently the range of analyzed suctions is greater. Figure 1 shows two suction profiles over two periods of these four soil layers at relatively low values of $\bar{\psi}$.

Figure 6 shows the analytical results for $\bar{K}$ and $U$ using dip angles of 0° and 45°. For $\omega = 0^\circ$, the stochastic result accurately predicts $U(\bar{\psi})$ in the range $0^\circ < \bar{\psi} < 30$ cm. Note that the Gardner parameters for these four soils are very closely (not perfectly) correlated, and the computed sample statistics provide a good prediction without any curve fitting. Although the prediction for anisotropy in this range is accurate for both $\omega = 0^\circ$ and 45°, the predictions of both $\bar{K}$ and $\bar{U}$ are much worse, underpredicting both components of $\bar{K}$ again. For graphical clarity, stochastic predictions are shown for the case of perfect correlation (pc) only. The dip angle significantly affects $\bar{K}$ but has very little effect on the analytical results for $\bar{U}$. The stochastic results do not predict these relative changes well for each direction, but they predict relative changes in $U(\bar{\psi})$ very well. The anisotropy curve assuming statistical independence (not shown in Figure 6) overpredicts anisotropy over the entire curve, similar to group 1 in Figure 5.

Extrapolation of the exponential form of the stochastic results diverges significantly from the analytical results at higher values of $\bar{\psi}$ as shown for $\omega = 45^\circ$ in Figure 7. In addition to increasing with $\bar{\psi}$, the value of $\chi$ in the stochastic result increases as $\omega$ is changed from 0° to 45°. Physically, this increase in $\chi$ is associated with an increase in the spatial variability in $\psi$ and, thus, increased spatial variability in $K(\psi)$. Figure 7 compares both forms of the Taylor series approximation for suction-dependent anisotropy, $U = e^{\chi^2}$ and $U = 1 + 2\chi + 2\chi^2$, and shows the exponential predictions for $\bar{K}$, assuming perfect correlation (pc) between in $K^{\alpha}$ and $\alpha$. The two forms of predicted $U$ start to noticeably diverge from each other at about 20 cm of suction, which corresponds to $\chi_{pc} = 0.79$ as shown. The exponential form provides good predictions of $U$ up to $\bar{\psi} = 40$ cm ($\chi_{pc} = 4.1$). At $\bar{\psi} = 55$ cm ($\chi_{pc} = 8.3$), the stochastic prediction of $\bar{K}$ begins to diverge from the analytical solution, which results in physically unreasonable increases in $\bar{K}$ and associated $U$ values at higher suctions.

For the Bet Dagan soils the two alternative closed-form functions for suction-dependent anisotropy bound the analytical results. In the range of extrapolation to relatively dry conditions ($30 < \psi < 60$ cm), the exponential form is more accurate than the truncated form; both have the same order of accuracy at lower values of $\chi$, i.e., under wetter conditions. Although the prediction for $\bar{K}$ becomes physically infeasible above $\bar{\psi} = 60$ cm, Gardner’s model would not apply under such extremely dry conditions for these soils because vapor conductivity would dominate below $\bar{K}$ with $\psi = 10^{-9}$ cm/h [Rose, 1971]. However, the present results are independent of such physical considerations, and divergence of the extrapolation is due to the approximations of the stochastic method. The stochastic results are implicitly limited to moderate values of $\bar{\psi}$.

Direct averages for $\bar{K}$ and $\bar{K}$ are the arithmetic and harmonic means, respectively, and their ratio equals $U$. The upper graph in Figure 7 for conductivity shows both the arithmetic and harmonic means. The arithmetic mean of $K(\psi)$ is a much better predictor of the analytical result for $\bar{K}$ than the
stochastic result is. And, the harmonic mean is approximately equivalent to the stochastic prediction of $K_{11}$ in this case. Both the harmonic mean and the stochastic result underestimate $K_{11}$ by about an order of magnitude. As a result, the ratio of direct averages overestimates the anisotropy over most of the curve. At suction less than about 35 cm, the stochastic prediction of $U$ is better than the direct average.

For both groups 1 and 2, prediction errors for log $U$ are smaller than individual errors in log $K$ at relatively low suction. The ratio of $K_{22}$ to $K_{11}$ is most accurate in these cases because both $K_{22}$ and $K_{11}$ are underpredicted using mean values of $\alpha$, i.e., the slope of $K_m(\tilde{\psi})$ in (4) and (5) is too steep for these two examples.

4.3. Group 3: Perfectly Correlated Soils

Both of the previous layered soil groups follow the pc curves more closely than the si curves. Generally, $K_{\text{sat}}$ tends to be correlated with the slope of the unsaturated $K$ curve because both tortuosity, which affects $K_{\text{sat}}$, and capillary water retention increase with smaller grain and pore sizes. We have generated a 10-layer group of pc soils based on the log $K_{\text{sat}}$-$\alpha$ relationship of three California soils: Cayucos clay loam, Hanford sandy loam, and Delhi sand (J. Constanz, personal communication, 1993). Figure 8 shows log $K_{\text{sat}}$ versus $\alpha$ for each soil. Ten Gardner soils that fall along the straight line are generated by sampling 10 random $\alpha$ values from a normal distribution with the sample mean and variance of the California soils. Then, $K_{\text{sat}}$ is computed from the log-linear relation with each generated $\alpha$ value. Layers are ordered randomly with no designated spatial structure, so the correlation length used for prediction is half the layer thickness (compare group 1 [Yeh and Harvey, 1990]).

The stochastic results indicate that the correlation length $\lambda$ and dip angle $\omega$ can have offsetting effects if

$$\lambda_1 \cos \omega_1 = \lambda_2 \cos \omega_2,$$

where the subscripts 1 and 2 correspond to two different geometries for the same soil types. Although the correlation length $\lambda$ is not precisely defined by a given layer thickness $b$, changes in $b$ should correspond to the same relative changes in $\lambda$. Therefore we test the effects of $\lambda$ and $\omega$ on the upscaled properties by taking

$$\omega_2 = \cos^{-1} \left( \frac{b_1}{b_2} \cos \omega_1 \right).$$

Following this approach, a steep slope ($\omega = 60^\circ$) and increased layer thickness ($b = 20$ cm) are used for comparison with $\omega = 0^\circ$ and $b = 10$ cm. Figure 9 shows the upscaled $K$ and anisotropy in the range $0 < \tilde{\psi} < 150$ cm for the 10 pc soils with layer thicknesses of 10 and 20 cm. Our results confirm the parametric grouping of $\lambda$ and $\cos \omega$ in (7) for both $K$ and $U$ in this case. A steep dip angle balances the doubling in layer thickness. In Figure 9 the agreement of the perfectly correlated (pc) stochastic predictions with the analytical results is good. The effects of layer thickness and/or dip angle become significant at moderate values of $\tilde{\psi}$ ($X_{pc} = 1$ at $\tilde{\psi} = 120$ cm). The stochastic results predict relative changes in anisotropy, despite the absolute errors. However, relative changes in conductivity with layer thickness are overpredicted for $K_{22}$ and underpredicted for $K_{11}$.

Figure 10 shows that extrapolation to the dry range (high $\tilde{\psi}$) leads to large errors, even though the stochastic results are accurate in the wetter range. For both $\tilde{K}$ and $U$ we compute solutions for the dry range using the tilted equivalents ($b = 12.5$ cm, $\omega = 36.8^\circ$, and $b = 20$ cm, $\omega = 60^\circ$) to horizontal, 10-cm-thick layers. Once again, errors in the stochastic extrapolation become very large, and the stochastic prediction for $K_{22}$ grows with increasing $\tilde{\psi}$, which is physically infeasible.
For these pc soils the exponential generalization overpredicts the analytical results for suction-dependent anisotropy, and the second-order truncated form of the stochastic result underpredicts the anisotropy. On this semilog plot, log \( U \) increases approximately linearly with \( \bar{\psi} \) for \( \bar{\psi} > 150 \text{ cm} \), where \( \chi_{pc} > 2.0 \) in the stochastic analysis. In that portion of the curve, we could fit another anisotropy function using the relationship \( U = c_1 \exp (c_2 \bar{\psi}) \). However, we do not pursue the subject of curve fitting here, using the nonphysical constants \( c_1 \) and \( c_2 \).

5. Discussion

The quasi-analytical results above provide a quantitative basis for comparison with the stochastic results for mean gravity drainage from near saturation to very dry conditions. The selected laboratory, field, and theoretical soil scenarios cover a limited variety of possible soil heterogeneities and geometries, but the selected soils display some of the important differences and similarities in upscaled hydraulic properties between the various soils, dip angles, and layer thicknesses. In our experience the assumption of perfect correlation between \( \ln K^{x,s} \) and \( \alpha \) appears to be a reasonable model. The alternative assumption of statistical independence can cause large errors even at low \( \bar{\psi} \), and the anisotropy will always be overestimated because of the likelihood of correlation for real soils. The si and pc cases do appear to provide upper and lower bounds, respec-

![Figure 9](image_url)  
**Figure 9.** Upscaled \( K \) and anisotropy versus mean suction for 10 pc soils with 10- and 20-cm layer thicknesses. Stochastic results for anisotropy are the exponential form: \( U = e^{x_{pc}} \).

![Figure 10](image_url)  
**Figure 10.** Upscaled \( K \) and anisotropy versus mean suction for 10 pc soils with 10- and 20-cm layer thicknesses. For horizontal (\( \omega = 0^\circ \)), 10-cm layers, the correlation length is \( \lambda = 5 \text{ cm} \). Extrapolations of the stochastic results and associated values of \( \chi_{pc} \) are shown.

tively, for the suction-dependent anisotropy up to \( \chi \) values near unity.

5.1. Small-Perturbation Limitation

Agreement between the stochastic results and analytical results at relatively low values of \( \bar{\psi} \) for soil groups with relatively small variances in \( \ln K^{x,s} \) (groups 2 and 3) indicates that the parametric groupings for \( \chi \) based on the small-perturbation assumption are appropriate. The result for \( \chi_{pc} \) in (7) has three terms that can be discussed separately in light of the previous comparisons.

The shape of the anisotropy curve is related to \( (1 - \bar{\bar{\psi}})^2 \) in the numerator. In every case the analytical results for \( U \) are lowest at \( \bar{\psi} \) values where \( (1 - \bar{\bar{\psi}})^2 = 0 \). That is, the pc result accurately predicts the mean suction corresponding to the minimum value of anisotropy in every case. For soils generated to meet the pc assumption, the anisotropy equals one at its minimum, as predicted by the stochastic result at \( \bar{\psi} = 1/\xi = \sigma_{\bar{\psi}}/\bar{\psi}_\omega \).

The sample variance in fitted values of saturated hydraulic conductivity (\( x_f \)) scales \( \chi_{pc} \) for the entire \( U(\bar{\psi}) \) curve, but should not affect the relative suction dependence of log anisotropy. Soil groups 1 and 2 are the high (\( x_f = 11.85 \)) and low (\( x_f = 0.556 \)) cases, respectively. Both groups have high values of \( \sigma_{\bar{\psi}} \), but the suction-dependent anisotropy of the Bet Dagan soils (group 2) is more accurately predicted than that of the
laboratory sands (group 1). When $\kappa^2$ was large, the stochastic prediction using a reasonable correlation length did not match the analytical results for anisotropy at low suctions. The primary difference between soil statistics for the two groups is the large variance in $\ln K^{\ast}$ for group 1, which indicates that the errors are related to variation of the small-perturbation limitation and the associated fact that $\chi$ is not small near saturation for this case. For group 2 the variance in $\ln K^{\ast}$ is not large, and the stochastic predictions for suction-dependent anisotropy are good. Group 3 has a moderate value of $\kappa^2$ near unity, and the upscaled properties are predicted very well in the lower $\psi$ (wetter) range.

The utility of the stochastic results for layered systems is further supported by the fact that the balance between dip angle and layer thicknesses observed for analytical systems is consistent with the stochastic result. The term $\alpha \cos \omega$ occurs in the denominator of both (6) and (7). Using group 3, a steep dip angle compensates for an increase in the layer thickness as quantitatively predicted by the stochastic analysis. This compensating effect continues into the dry range, even though the absolute extrapolation error becomes significant. The stochastic result also predicts relative changes in $U$ due to changes in $\omega$ or $b$. For small values of $\alpha$, the denominator of $\chi$ approaches a minimum of 2.0 where further increases in the dip angle and/or reductions in the layer thickness do not affect the upscaled properties.

Finally, the general shape of $U(\psi)$ and its rate of increase agree favorably with the stochastic results when $\chi < 1$. If the predictions of $\bar{K}$ are good in both principal directions, then the prediction of $U$ is good. However, it does not necessarily follow that prediction of $U$ is poor when the predictions of $\bar{K}_{11}$ and $\bar{K}_{22}$ are poor. For example, both $\bar{K}_{11}$ and $\bar{K}_{22}$ are significantly underpredicted for both the Yeh and Harvey sands (group 1) and the Bet Dagan soils (group 2), so the predictions for $U$ are more accurate in the same range of suctions. No general conclusions can be drawn from the present study as to whether $\bar{K}_{11}, \bar{K}_{22}$, or their anisotropy ratio $U$ is predicted most accurately by the exponential form of the stochastic results.

### 5.2. Extrapolation to Dry Soils

Differences between the two alternative forms of the Taylor series approximation for suction-dependent anisotropy become significant at fairly low anisotropy values (e.g., $U \approx 5$), which correspond to $\chi$ values near unity. Soil groups 2 and 3 illustrate how the two alternative forms bound the analytical results ($1 + 2\chi + 2\chi^2 < U < e^{2\chi}$) at high suction values (dry soils). Neither form appears to be consistently better than the other, and alternative functions may be proposed for soils represented by Gardner's model. The analytical results show a nearly linear relation for log $U$ versus $\psi$ in the drier range, where the stochastic result grows artificially large using the exponential generalization ($U = e^{2\chi}$) for extrapolation. An exponential function in terms of $\psi$ (not $\chi$ or $\bar{\psi}$) can be fit to the analytical results at high suctions, but it is not useful for independent predictions based on local soil hydraulic properties. The log-linear growth in $U(\psi)$ extends to high capillary suctions, where the hydraulic conductivity in both directions becomes very small, and Gardner's model for the local $K(\psi)$ is not valid. Such extreme extrapolation becomes purely theoretical, but part of the log-linear portion is important for arid sites. Local relative conductivity models other than Gardner's model may not display the same log-linear relation.

### 6. Summary

In this paper we derived a quasi-analytical solution for steady, mean gravity drainage through inclined, perfectly stratified soils. Gardner's exponential function represents the unsaturated hydraulic conductivities of local soils in both the stochastic and present analytical analyses. Analytical integration of Darcy's law resulted in a system of nonlinear equations that was solved iteratively for capillary suctions at the layer interfaces and for the Darcy flux normal to layering. The computed suction profiles were used to determine solutions for the upscaled $K$ in the principal directions, along with the corresponding anisotropy at various values of $\psi$.

Three groups of layered soils were analyzed and compared with predictions from the stochastic results of Yeh et al. [1985b]. The grouping of statistical and geometric parameters ($\chi$) resulting from the small-perturbation approximation predicted the appropriate behaviors for state-dependent anisotropy under relatively wet conditions. State-dependent anisotropy of soils represented by Gardner's function is bounded by the ideal cases of perfect correlation and statistical independence between the Gardner parameters, when $\chi$ is less than one. Ultimately, the variance in $K(\psi)$ or $\ln K(\psi)$, which is greatest for very dry soils, causes significant prediction errors at high mean suctions in every case. High values of variance in $\alpha$ simply cause the variance in $K(\psi)$ to increase rapidly as the mean suction increases. The Taylor series approximation in terms of $\chi$ creates an implicit limitation of the stochastic results to moderate values of the mean suction. The exponential form ($e^{2\chi}$) of the closed-form function for anisotropy postulated by Yeh et al. [1985b] and the truncated form ($1 + 2\chi + 2\chi^2$) postulated here provide upper and lower bounds, respectively, for anisotropy computed at high suctions (dry soils).

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**References**


McCord, J. T., D. B. Stephens, and J. L. Wilson, Toward validating state-dependent macroscopic anisotropy in unsaturated media:


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