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THE SHAPE OF SELF-MOTION PERCEPTION—I. EQUIVALENCE CLASSIFICATION FOR SUSTAINED MOTIONS

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Abstract—Two completely different motions of a subject relative to the earth can induce exactly the same stimuli to the vestibular, somatosensory and visual systems. When this happens, the subject may experience disorientation and misperception of self-motion. We have identified large classes of motions that are perceptually equivalent, i.e. indistinguishable by the subject, under three sets of conditions: no vision, with vision and earth-fixed visual surround, and with vision during possible movement of the visual surround. For each of these sets of conditions, we have developed a classification of all sustained motions according to their perceptual equivalences. The result is a complete list of the possible misperceptions of sustained motion due to equivalence of the forces and other direct stimuli to the sensors under the given conditions.

This research expands the range of possible experiments by including all components of linear and angular velocity and acceleration. Many of the predictions in this paper can be tested experimentally. In addition, the equivalence classes developed here predict perceptual phenomena in unusual motion environments that are difficult or impossible to investigate in the laboratory.

Key words: spatial orientation, mathematics, vestibular system, space flight, models, gravitation.

"The pilot took a waveoff as he attempted to land the helicopter on a spot lighted probably by four flashlights. Because of the extreme denseness of the fog he was unable to find the spot. As he circled to attempt a second landing apparently he became completely disoriented. While he was in vertiginous state, he circled to the right but thought he was turning to the left. Although he was on instruments he does not remember altitude or airspeed. As he crashed he stated he became a passenger and rode it in."[16]

The pilot's confusion between "circling to the right" and "turning to the left" illustrates the perceptual ambiguity that can occur during motion, especially when information from the vestibular system is not supplemented by accurate information from the visual system. Even accurate information from the visual system does not always lead to accurate perception of self-motion, and disorientation can occur during good as well as bad visibility conditions.

The vestibular and visual systems play major roles in perception of self-motion, along with the somatosensory, auditory and other systems. The vestibular hairs of the otolith organs deflect in response to linear accelerations, and the hairs of the semicircular canals deflect in conjunction with pressure on the cupulae in response to angular accelerations; thus, the vestibular system acts as an acceleration transducer during low frequency and sustained motion.[6] The cells of the vestibular nuclei, which receive primary vestibular input, are also known to be modulated by optokinetic stimuli.[5] In fact, both linear and angular motions of the visual surround have been shown to induce the sensation of self-motion during absence of vestibular indication of motion.[5]

Cases of self-motion misperception, whether for pilots, astronauts, vestibular patients or passengers of trains, planes, automobiles or experimental apparatus, are often explained through careful analysis of the vestibular and/or visual stimulus in each case.[5,7] For example, the helicopter pilot's misperception of self-motion could be explained by calculations showing that a constant velocity coordinated turn to the right in an aircraft imposes the same accelerations on the vestibular endorgans as a constant velocity coordinated turn to the left. Therefore, the receptor response is the same in each case; the nervous system cannot distinguish between the two motions, and may perceive one when the actual motion is the other. Perception is more specific than the evidence that it is based upon, and strangely durable, given its fragile base.

While a separate analysis of each self-motion perception phenomenon is useful in explaining the phenomena, a more unified approach would provide not just explanations but also predictions and a means to relate many different types of motions. A

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unified approach has been made accessible by recent work[11] on the coordinate systems inherently used by sensing, perceiving organisms (e.g., humans). These "subject-coincident coordinate systems" have unusual properties when compared with the coordinate systems standardly used in physics and engineering, and an investigation of these properties leads to a complete classification of the set of possible sustained motions of a subject.[11] A sustained motion is one in which the linear, angular and gravitational accelerations remain fixed relative to the subject's head, while the linear and angular velocities can change in magnitude but not direction. A key property of sustained motion is that the stimulus to the vestibular endorgans is constant and can cause saturation or adaptation within the vestibular system. One of the fundamental approaches in this paper is to develop and use mathematics that fit closely with the empirical situation.

In the present paper, we develop a complete classification of sustained motions that are perceptually indistinguishable due to having the same effect on the sensory receptors. This classification is developed with the purpose not only of explaining many individual phenomena, but also of predicting new phenomena and identifying classes of motions that the nervous system cannot distinguish. Three cases are investigated: no vision, with vision and earth-fixed visual surround, and with vision during possible movement of the visual surround. Vestibular and somatosensory information is available in all cases. We begin by explaining the subject-coincident coordinate system, defining sustained motion precisely and presenting the categorization of sustained motions in the first section. This is followed by a section on the "dark-equivalence classes" of perceptually indistinguishable sustained motions in the dark (or with occluded vision). The third and fourth sections cover the "light-equivalence classes" of perceptually indistinguishable motions with vision available, first with earth-fixed visual surround, then with mobile visual surround.

SUSTAINED MOTIONS

The purpose of this section is to introduce sustained motions, which form the main set of motions investigated in the present paper. The basic idea of a sustained motion is clear from the name—a sustained motion can be sustained over time. However, in order to pursue a rigorous investigation of the perceptual properties of sustained motions, we must start with precise definitions. We begin by introducing the subject-coincident coordinate system, within which motions of a subject are measured, then follow this by discussions of motions and sustained motions, ending the section with a complete categorization of the set of sustained motions. This categorization is used throughout the paper.

The subject-coincident coordinate system

When discussing motion of a subject in terms of velocities and accelerations, it is generally accepted that the coordinate system used should be oriented with the subject, regardless of the subject's orientation relative to the earth or other objects. This convention reflects the fact that the sensory organs of the subject are fixed within the subject, and are necessarily oriented with the subject. We take a standard coordinate system used for accelerations and expand it to include velocities, thereby creating a "subject-coincident coordinate system". By using a subject-coincident coordinate system, we not only have the means to specify all the necessary components of motion, but automatically have the means to specify all these necessary components from the point of view of the subject, capturing the subject's vantage point during self-motion perception.

Acceleration is specified according to the standard[10] coordinate system oriented with the head, as shown in Fig. 1. Forward and backward linear accelerations are along the x-axis with forward being in the positive x direction, leftward and rightward linear accelerations are along the y-axis with leftward being positive, while upward and downward are along the z-axis with upward being positive. The

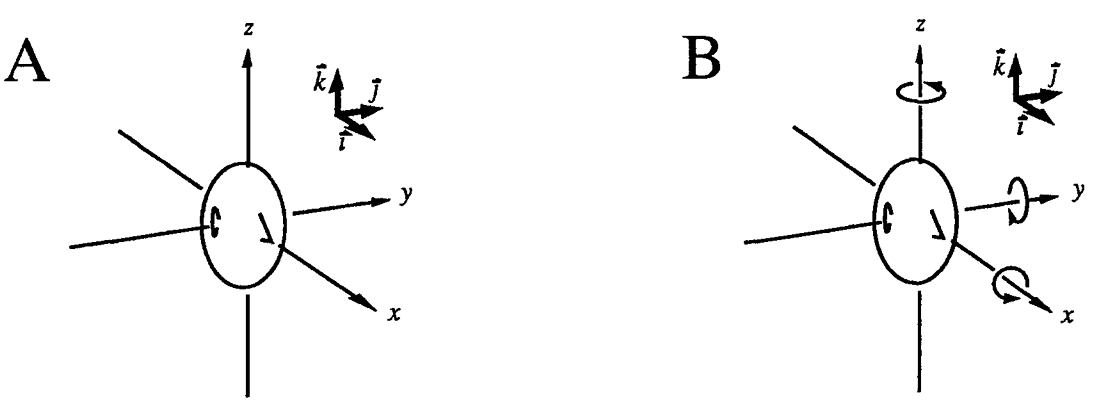

Fig. 1. Standard coordinate axes for specification of head motion. (A) Coordinate axes for linear motion, and standard unit vectors. (B) Coordinate axes for angular motion, and standard unit vectors.

convention of having unit vectors **i**, **j** and **k** in the x, y and z directions, respectively, is followed. Similarly, angular accelerations are specified in the xyz coordinate system, using the right-hand rule. In particular, rightward roll is given by a vector in the positive x direction, forward pitch is given by a vector in the positive y direction and leftward yaw is given by a vector in the positive z direction. Even though these coordinates do not correspond exactly with the axes of the vestibular endorgans, they serve as a means to discuss and relate various motions from the perspective of the subject, because, like the endorgans, they span the three-dimensional space.

This standard coordinate system was introduced in vestibular research for the purpose of discussing accelerations, but we need to discuss velocities in addition to accelerations. Therefore, we expand the system by specifying linear and angular velocities of the subject according to the same x, y and z coordinate axes. All motions are described as head motions, but it is understood that the body as a whole may be moving; definitions and results in this section apply equally to head motion and whole-body motion. Results in later sections apply best to whole-body passive motion, since certain stated criteria are required for the analyses.

Surprisingly, the introduction of velocities into the standard coordinate system throws us beyond textbook physics and textbook engineering. Acceleration is no longer the derivative of velocity, and our coordinate system is an unusual hybrid of the traditional fixed and moving coordinate systems. Nevertheless, the basic properties of these coordinate systems have been worked out in detail,[11] capturing the subject's vantage point during movement. The key point is that the "subject-coincident coordinate system" we use is not moving, in the traditional sense, with the subject. If it were truly moving with the subject, then the subject would always be at the origin of the coordinate system; therefore, we would always measure zero velocity and zero acceleration. Since a moving subject clearly has non-zero velocity and possibly acceleration, measurements of velocity and acceleration are made by instantaneously fixing coordinate axes relative to the earth in a position that coincides with that of the subject. In general, we make the following definition.

Definition. A subject-coincident coordinate system is given by a set of coordinate axes fixed in position relative to the object in question (such as the head), and for which movement of the object is specified in the following way: at each point in time, aspects of motion such as the linear velocity, linear acceleration, angular velocity and angular acceleration of the object are given by their measurements in the (traditional) coordinate system whose axes coincide with the coordinate axes of the object, but are fixed in space relative to a predetermined "reference object" (such as the earth).

Subject-coincident coordinate systems describe motion from the subject's point of view. In this way, subject-coincident coordinate systems differ from those traditional in physics and engineering, particularly when velocities and accelerations are specified simultaneously.

The system of Fig. 1, when use for aspects of motion such as velocities and accelerations, is a subject-coincident coordinate system. Examples of motions specified using this subject-coincident coordinate system are given below, after the formal definition of "motion".

Motions

In addition to linear and angular velocities and accelerations, orientation relative to the earth (or other host planet or moon) is important because of the gravitational field. The gravitational pull induces an upward reactionary force on the soles of the feet when standing, for example, and a downward deflection of the hairs of the saccular macula of the inner ear. This z-directed force on the body, with opposite otolith hair deflection, is the same as that occurring during acceleration in the positive z direction. In this way, gravity behaves exactly like linear acceleration (this fact being Einstein's Principle of Equivalence), and we indicate orientation, or "attitude", relative to the gravitational field by a vector $\mathbf{a}_g$, the attitude vector, pointing away from the ground with magnitude equal to the acceleration due to gravity of a free-falling body (which is approximately 9.8 m/s^2 at the surface of the earth). When the head is earth-upright, as illustrated in Fig. 2, this vector points in the z direction; it is possible for this vector to point in the x, y or any other direction, depending on the position of the head.

In order to specify the subject's perspective of self-motion at a given point in time, the following formal definition is made.

Definition. A motion, referring to head motion at a given point in time, is specified by five vectors in the subject-coincident coordinate system of Fig. 1, those for:

1. Linear velocity, denoted by $\mathbf{v}$.
2. Linear acceleration, denoted by $\mathbf{a}$.
3. Angular velocity, denoted by $\boldsymbol{\omega}$.
4. Angular acceleration, denoted by $\boldsymbol{\alpha}$.
5. Attitude, denoted by $\mathbf{a}_g$.

Since each of these five vectors has three components, a motion is technically a 15-dimensional vector. The set of all possible motions forms a 15-dimensional vector space which we denote by M.

Examples. Forward travel at 225 m/s (=810 km/h) while seated in a passenger jet can be specified by the appropriate vectors $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ and $\mathbf{a}_g$, as shown in Fig. 3A. A 60°/s counterclockwise rotation while seated in a rotating chair is specified as shown in Fig. 3B. Rotation in a 1 m radius experimental centrifuge at 90°/s is specified as shown in Fig. 3C; in

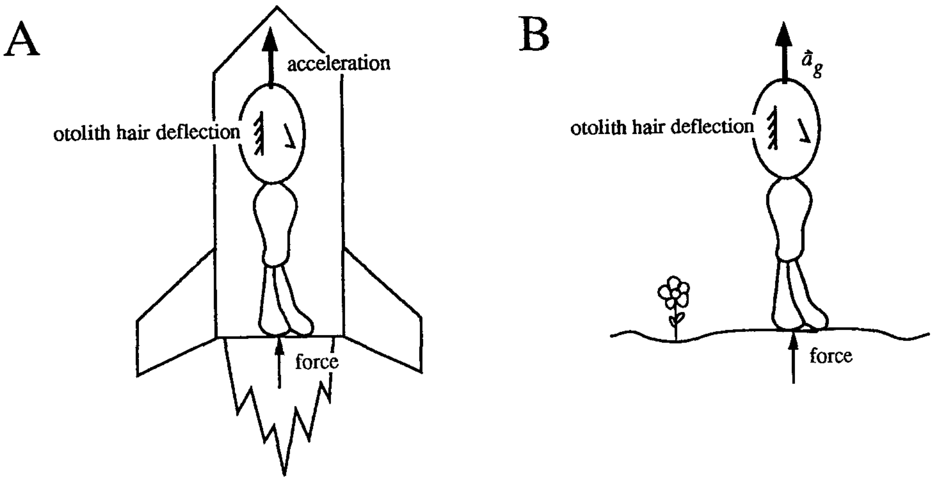

Fig. 2. Equivalence of linear acceleration and gravity. (A) Linear acceleration. (B) Gravity. The vector $\mathbf{a}_g$ is the attitude vector, indicating "apparent" acceleration due to the gravitational field.

this centrifuge rotation, linear acceleration is not the derivative of linear velocity, illustrating an unusual property of the subject's perspective of self-motion.

Each of the motions in Fig. 3 can be sustained over time. In contrast, there are certain motions whose specified vectors cannot be sustained over time; for example, an upright position ($\mathbf{a}_g$ in the z direction) paired with forward pitch (ω in the y direction) causes the head to face downward at some point, hence the resulting attitude is not in agreement with the original vector $\mathbf{a}_g$. In formally defining "sustained motion", we want as many of the vectors describing such a motion to be sustainable over time. However, because non-zero accelerations can cause velocities to change over time, it is unreasonable to expect all velocities and accelerations to remain constant; instead, we require accelerations to remain constant and allow velocities to change only in simple ways as follows.

Definition. A sustained motion is a motion (given by $\mathbf{v}$, $\mathbf{a}$, ω, α, $\mathbf{a}_g$) with $|\mathbf{a}_g| = 1$ g-unit (≈ 9.8 m/s^2), and for which the linear acceleration, the angular acceleration and the attitude would remain equal to $\mathbf{a}$, α and

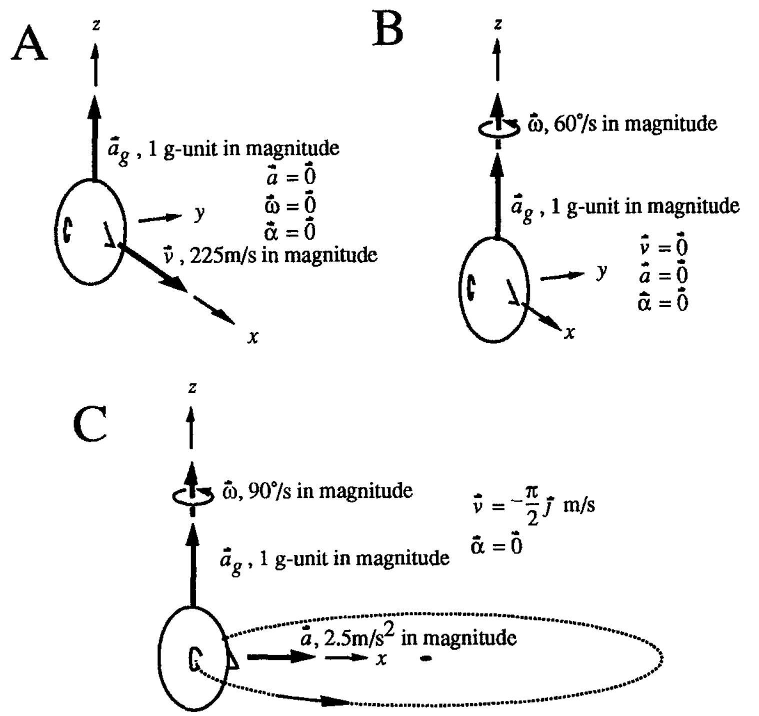

Fig. 3. Three motions belonging to the set M. (A) Forward travel at 225 m/s (=810 km/h). (B) Counterclockwise rotation at 60°/s. (C) Rotation in a 1 m radius experimental centrifuge at 90°/s. Note that because $\mathbf{v}$, ω and $\mathbf{a}_g$ use different units of measure, no direct comparison of their lengths is intended.

$\mathbf{a}_g$, respectively, throughout performance of the prescribed movement, while the linear velocity and the angular velocity would remain parallel to $\mathbf{v}$ and ω, respectively; in addition, if $\mathbf{a}$ is non-zero, then the specified $\mathbf{v}$ is required to be non-zero, and if α is non-zero, then the specified ω is required to be non-zero. In analogy with the notation M for the set of motions, we denote the set of sustained motions by M_S.

We focus on sustained motions, as defined above, in this paper. It should be noted, however, that other definitions of "sustained" are also possible. For situations without vision available, another reasonable definition would be to require linear acceleration plus attitude to remain equal to $\mathbf{a} + \mathbf{a}_g$ and angular acceleration to remain equal to α throughout performance of the prescribed movement, keeping the detected accelerations constant. The formal definition above adds further requirements, precluding tumbling motions, for example.

The definition of "sustained" used here applies to many situations, including those with vision available, and describes a relatively large and non-trivial class of motions while keeping the number of motions manageable. Taking an outside observer's viewpoint, the sustained motions fall into nine categories[11] according to the different properties of the motions: "Is the subject moving in a straight line? Is the subject turning? Is the subject accelerating? etc." The nine categories listed below give each possible combination of linear velocity, linear acceleration, angular velocity and angular acceleration. Figure 4 gives a schematic description of the nine categories, and Table 1 summarizes the permitted values and the interdependencies between $\mathbf{v}$, $\mathbf{a}$, ω, α and $\mathbf{a}_g$ for each of the nine categories.

In each of the following descriptions, any velocity or acceleration not mentioned and not arising from those mentioned is assumed to be zero.

Categories of sustained motions

(1) Fixed position. Examples: (i) pitch-back 45° (reclined in a chair), (ii) right ear down (lying on side).

(2) Linear velocity. Linear velocity is non-zero and constant. Examples: (i) pitch-back 45°, moving 25 m/s in direction of chin (reclined in a moving car), (ii) upright, moving 10 m/s in the $-z$ direction (descending in an elevator).

(3) Linear acceleration. Non-zero linear acceleration is constant and parallel to the linear velocity. Examples: (i) Face downward, moving 2 m/s in the x direction with 9.8 m/s^2 acceleration (falling, face downward), (ii) upright, moving 4 m/s in the y direction with deceleration of 2 m/s^2 (sitting sideways in a stopping bus).

(4) Angular velocity. Non-zero constant angular velocity occurs about an earth-vertical axis. The vertical axis is required in order for the attitude vector to remain fixed relative to the head. (The head itself need not be vertical.) Linear velocity may arise, but only in the form of tangential velocity, as may

Table 1. Permitted values of linear and angular velocities and accelerations in the nine categories of sustained motion

	Attitude vector $\mathbf{a}_g$	Linear velocity $\mathbf{v}$	Linear acceleration $\mathbf{a}$	Angular velocity ω	Angular acceleration α
(1) Fixed position	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
(2) Linear velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Unrestricted, $\neq\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
(3) Linear acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Unrestricted, $\neq\mathbf{0}$	Parallel to $\mathbf{v}$	$\mathbf{0}$	$\mathbf{0}$
(4) Angular velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Depends on ω and radius of rotation	Depends on ω and radius of rotation	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	$\mathbf{0}$
(5) Angular and linear velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Partially depends on angular motion, $\neq\mathbf{0}$	Depends on ω and radius of rotation	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	$\mathbf{0}$
(6) Angular velocity and linear acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	$\mathbf{0}$
(7) Angular acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	$\mathbf{0}$	$\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$
(8) Angular acceleration and linear velocity	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	$\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$
(9) Angular and linear acceleration	$\lvert\mathbf{a}_g\rvert = 1$ g-unit	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$	Parallel to $\mathbf{a}_g$, $\neq\mathbf{0}$

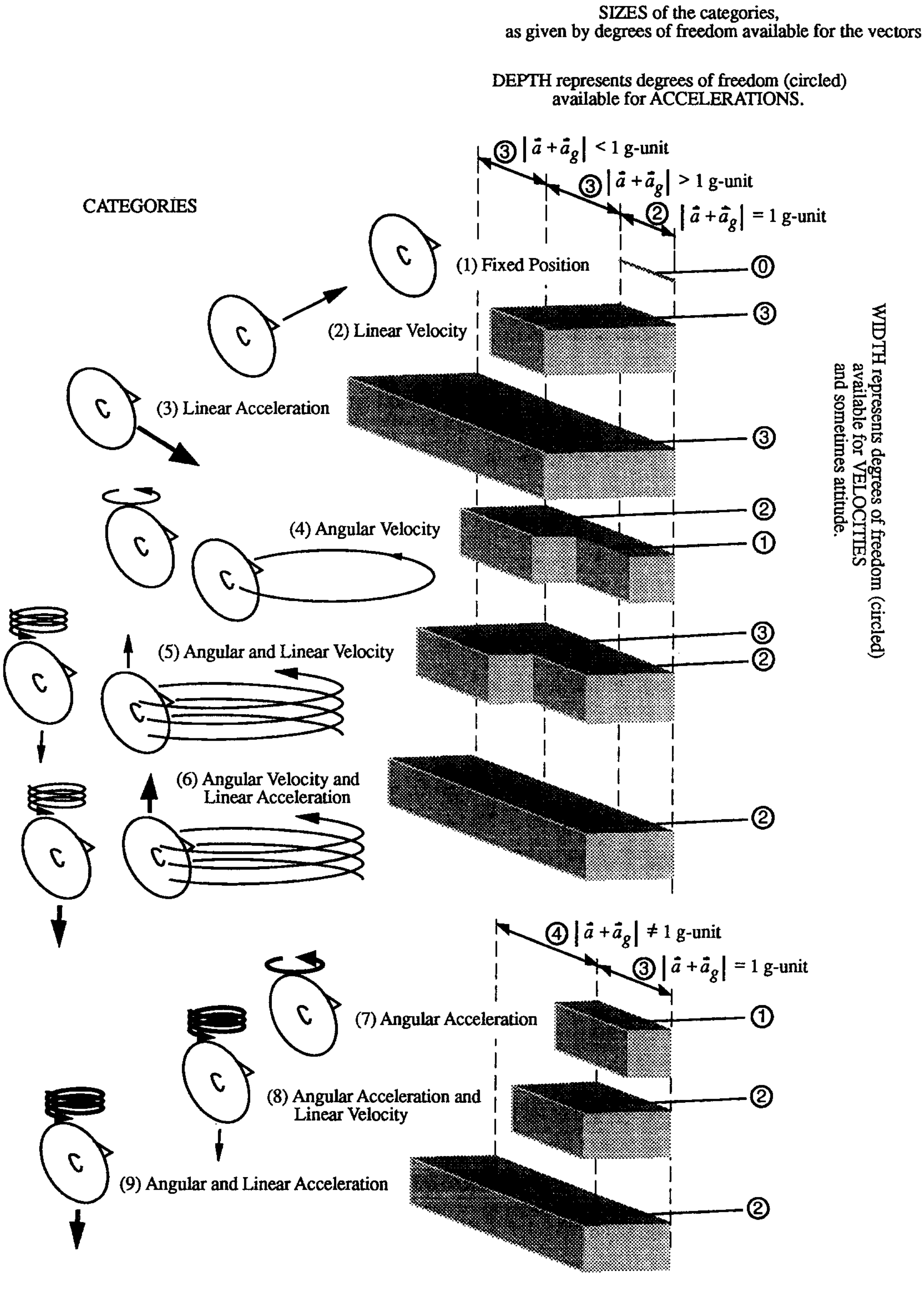

Fig. 4. The nine categories of sustained motions. The heads illustrate examples of motions from each of the categories. Each rectangular box represents a category, the larger boxes "contain" many different motions, as compared to the smaller boxes which "contain" fewer motions. The number of motions in a box (i.e. in a category) depends on the number of degrees of freedom available for the velocities and accelerations. Depth represents degrees of freedom available for accelerations, and width represents degrees of freedom available for velocities and attitude. Detailed discussions of these velocities and accelerations can be found in Holly.[11]

linear acceleration only in the form of centripetal acceleration. Examples: (i) upright, spinning on-axis at 90°/s (sitting in a rotating chair), (ii) upright, moving clockwise about an axis that is 10 m away in a direction halfway between x and $-y$, with tangential velocity of 4 m/s (looking toward the right while making a right turn in a car).

(5) Angular and linear velocity. Non-zero constant angular velocity occurs about an earth-vertical axis, and the linear velocity has a non-zero earth-vertical component in addition to a possible tangential component due to angular velocity at a non-zero radius. (Linear acceleration may arise in the form of centripetal acceleration.) Example: slight leftward tilt position, moving in the x direction at 225 m/s while turning counterclockwise about an earth-vertical axis 1600 m to the left, with additional 30 m/s linear velocity in the earth-downward direction (in an airplane during a constant-velocity version of the deceiving and dangerous "graveyard spiral").

(6) Angular velocity and linear acceleration. Non-zero constant angular velocity occurs about an earth-vertical axis through the head, and non-zero linear acceleration is earth-vertical and constant. Example: upright, spinning clockwise on-axis at 15°/s, while moving downward at 21 m/s with acceleration of 7 m/s^2 (sitting in a rotating chair in a falling elevator).

(7) Angular acceleration. Non-zero constant angular acceleration occurs about an earth-vertical axis through the head. Example: upside-down, with a 5°/s^2 decelerating on-axis spin of 30°/s (hanging upside down from a rope, spinning but slowing down).

(8) Angular acceleration and linear velocity. Non-zero constant angular acceleration occurs about an earth-vertical axis through the head, and linear velocity is non-zero, constant and earth-vertical. Example: upside-down, with a 5°/s^2 decelerating on-axis spin of 30°/s, while moving upward at 0.5 m/s (hanging upside down from a rope, spinning but slowing down, while being pulled upward).

(9) Angular and linear acceleration. Non-zero constant angular acceleration occurs about an earth-vertical axis through the head, and linear acceleration is non-zero, constant and earth-vertical. Example: upright, with a 5°/s^2 accelerating on-axis spin of 15°/s, while moving downward at 21 m/s with acceleration of 7 m/s^2 (sitting in a chair with accelerating rotation in a falling elevator).

This complete categorization of the set of sustained motions is referred to throughout the paper, in discussing the various sustained motions that are perceptually indistinguishable by the subject.

DARK-EQUIVALENCE

The goal of this section is to classify those sustained motions that are perceptually indistinguishable in the dark. Disorientation and misperception of self-motion are not uncommon when the eyes are closed, in darkness or when the visual surround is obscured. In general, the nature of self-motion perception and the laws of physics necessitate perceptual ambiguity, or "equivalence", between motions. One common example occurs during travel in a passenger jet: while seated with eyes closed during calm flight at 810 km/h, we have a sense of stationarity just as though the aircraft were going nowhere. Experience tells us that the flight usually reaches its destination (even if our luggage does not) so that a speed of 810 km/h is credible, but the speed cannot be detected in that particular sensory context.

Many other examples of perceptually indistinguishable motions exist, some of which are analysed in motion studies, and others of which are exploited in vestibular and related research. A classic example of the latter is the use of a centrifuge for studies that require particular linear accelerations. A subject seated inside a large centrifuge at a distance of r meters from the vertically-oriented rotation axis will experience a centripetal acceleration of $r(\omega\pi/180)^2$ m/s^2 when the centrifuge rotates at a sustained angular velocity of ω°/s. Therefore, as long as the visual surround is obscured, sustained forward linear acceleration of 2.5 m/s^2 in a motor vehicle, for example, can be simulated by sitting at a radius of 6 m facing the center of a centrifuge rotating at 37°/s; the sustained forward linear acceleration and the sustained rotation in a centrifuge are perceptually equivalent motions.

While each of the above examples consists of two particular motions that are perceptually indistinguishable (assuming the motions are sustained so that start-up sensory and perceptual transients have died away), there are in fact large classes of motions such that those motions belonging to the same large class cannot be distinguished from one another in the given sensory context. These classes are called equivalence classes, and perceptually indistinguishable motions are termed equivalent in this paper. In order to identify equivalence classes of motions, we first identify the stimuli and sensory systems involved, and set up the necessary mathematical formulations in terms of equivalence relations. From there, the desired dark-equivalence classification is developed.

The vestibular system

For situations in which optokinetic input is not available, the vestibular system is crucial for self-motion perception. Physically, the vestibular system is directly affected by accelerations,[6] and in certain frequency ranges of motion it acts as a velocity transducer because of the physical dynamics of the system.[6] However, during sustained constant acceleration (including zero acceleration), the deflections of vestibular sensory hairs correlate only with acceleration because dynamics are not significant. Because of the lack of velocity information in the dark during a sustained motion, many motions become indistinguishable.

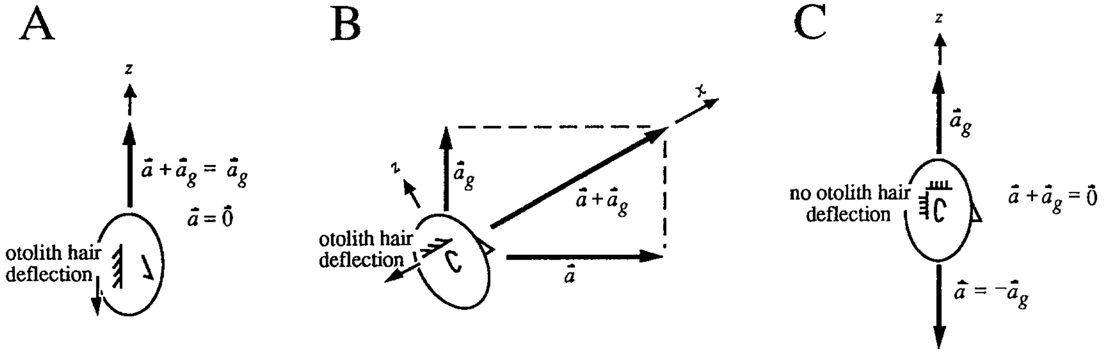

Fig. 5. Examples of the detections of $\mathbf{a} + \mathbf{a}_g$ by the otolith organs. (A) Head upright, no linear acceleration. (B) Head tilted back, horizontal forward linear acceleration. (C) Head upright, linear acceleration downward of magnitude equal to that of $\mathbf{a}_g$. In this situation, the subject is in free-fall, so the otolith organs simply fall with the head and exhibit no deflection.

For the investigation of the "dark" situation, we assume that the vestibular system and somatosensory systems are available, along with proprioception; however, we assume that no complicating factors exist, such as sound or air currents that aid in localization or velocity judgement. (Large movements can be presumed to take place in enclosed vehicles, for example.) In addition, we assume that the subject is not moving the limbs during the motion, which could affect the sensation by introducing Coriolis forces, for example. This restriction is also necessary because motor commands may play a role in self-motion perception.[9] In fact, relative positions of the head, trunk and limbs can affect a subject's perception of orientation,[13] so the present study assumes that two motions are only compared when the subject's body position is the same.

The vestibular system is constructed in such a way as to respond to angular acceleration by means of the semicircular canals, and linear acceleration by means of the otolith organs, although the response-evoking "linear acceleration" is actually the vector sum of $\mathbf{a}$ and the "apparent" acceleration $\mathbf{a}_g$ caused by the gravitational field, as illustrated in Fig. 5. According to Einstein's Principle of Equivalence, an accelerometer such as the inner ear cannot distinguish between $\mathbf{a}$ and $\mathbf{a}_g$, nor can it reconstruct $\mathbf{a}$ and $\mathbf{a}_g$ from their sum. This physical fact contributes to the perceptual misinterpretation of motion and orientation.

Definition. The perceived acceleration, $\mathbf{a} + \mathbf{a}_g$, will be called the resultant acceleration and will be denoted by $\mathbf{A}$. That is,

$$\mathbf{A} = \mathbf{a} + \mathbf{a}_g$$

resultant acceleration =

linear acceleration + attitude vector.

The somatosensory system duplicates the type of information given by the vestibular system. Force on the surface of the body is detected, and since force is directly related to acceleration ($F = ma$), the somatosensory system simply gives another way to detect the resultant acceleration. In addition, proprioception such as that of the neck also simply duplicates this acceleration information. Neck muscles may contract to hold the head in line with the body during an acceleration, but just like the vestibular endorgans, they can only respond to the resultant acceleration.

Since accelerations as described above are detected during sustained self-motion in the dark, but velocities are not, the following terminology is used.

Definition. If sustained motion B is specified by the vectors $\mathbf{v}^B$, $\mathbf{a}^B$, $\boldsymbol{\omega}^B$, $\boldsymbol{\alpha}^B$, $\mathbf{a}_g^B$, and sustained motion C by $\mathbf{v}^C$, $\mathbf{a}^C$, $\boldsymbol{\omega}^C$, $\boldsymbol{\alpha}^C$, $\mathbf{a}_g^C$, then the motions B and C are termed dark-equivalent if

$$\mathbf{a}^B + \mathbf{a}_g^B = \mathbf{a}^C + \mathbf{a}_g^C$$

(i.e. $\mathbf{A}^B = \mathbf{A}^C$, resultant accelerations equal), and

$$\boldsymbol{\alpha}^B = \boldsymbol{\alpha}^C$$

(angular accelerations equal).

Because accelerations are equal, dark-equivalent motions are indistinguishable when vision is not available (such as in the dark); a possible exception is that of fast spins, as explained here. The statement that dark-equivalent motions are indistinguishable in the dark assumes that linear acceleration is the same at all points on the subject's body. This assumption holds perfectly during linear motion, and is a good approximation if angular velocity is not too large. However, during rotation at high speeds, two points at different radii from the axis of rotation may have significantly different centrifugal forces imposed on them. For example, during on-axis rotation of an upright subject, the right and left otolith organs may detect oppositely-directed linear accelerations. Whether the vestibular system performs a comparison of such accelerations in detecting angular velocity is still open to experimental investigation. It is known that after start-up transients have died away, subjects in the dark do not detect constant angular velocity at speeds of at least 240°/s.[3]

The present investigation assumes that the sustained motion takes place long enough for the vestibular and other sensory and perceptual start-up transients to die away. The vestibular system and its related perceptions have the remarkable ability to "forget" start-up cues after a sufficient time has elapsed; for example, subjects carefully estimate angular velocity as zero when seated and spinning on-axis approximately 50 s after a 10-s acceleration at 24°/s².[3] Shorter duration sustained motion is sufficient when the start-up acceleration is shorter or of smaller magnitude. Perceptions during the changing portions of motion are discussed in the accompanying paper.[12]

Equivalence classes and equivalence relations

We will demonstrate that dark-equivalence partitions the set M_S of sustained motions into classes, termed dark-equivalence classes, that do not overlap with each other. The point is that the classes do not overlap: a motion is dark-equivalent to every motion in its class, and two motions in different classes are never dark-equivalent. The formal demonstration of this claim uses the mathematical language of equivalence relations and equivalence classes. These concepts belong to basic mathematics, but we review them here for the sake of completeness and rigor.

Equivalence relations are illustrated in Fig. 6, with formal definition given as follows.

Definition. With a given set in mind (e.g. of persons, motions or pens), a binary relation "$\sim$" (used on pairs in the set, such as "lives in the same house as", "is dark-equivalent to" or "is the same brand as") is an equivalence relation if the following three conditions hold:

1. For each element X in the set, $X \sim X$ is true (e.g. "X is the same brand as itself").
2. Whenever two elements X and Y in the set have $X \sim Y$, then $Y \sim X$ is also true (e.g. "If X is the same brand as Y, then Y is the same brand as X").
3. Whenever three elements X, Y and Z in the set have $X \sim Y$ and $Y \sim Z$, then $X \sim Z$ is also true (e.g. "If X is the same brand as Y, and Y is the same brand as Z, then X is the same brand as Z").

As shown in Fig. 6, to "play tennis with" is not necessarily an equivalence relation among a set of persons. It may be the case that Greta plays tennis with Larry, and Larry plays tennis with Conrad, but Greta does not play tennis with Conrad. On the other hand, to "live in the same house as" is an equivalence relation in the figure.

For the set M_S of sustained motions, the binary relation "is dark-equivalent to" is an equivalence relation. Conditions 1 and 2 in the definition of "equivalence relation" clearly hold. To show condition 3, we use superscripts X, Y and Z on the vectors specifying the motions X, Y and Z, respectively.

If $X \sim Y$ and $Y \sim Z$, this means that

$$\mathbf{a}^X + \mathbf{a}_g^X = \mathbf{a}^Y + \mathbf{a}_g^Y \text{ and } \boldsymbol{\alpha}^X = \boldsymbol{\alpha}^Y, \text{ and}$$

$$\mathbf{a}^Y + \mathbf{a}_g^Y = \mathbf{a}^Z + \mathbf{a}_g^Z \text{ and } \boldsymbol{\alpha}^Y = \boldsymbol{\alpha}^Z.$$

From this it follows that

$$\mathbf{a}^X + \mathbf{a}_g^X = \mathbf{a}^Z + \mathbf{a}_g^Z \text{ and } \boldsymbol{\alpha}^X = \boldsymbol{\alpha}^Z,$$

therefore $X \sim Z$. Since conditions 1, 2 and 3 hold, dark-equivalence is an equivalence relation.

A fundamental mathematical result about any equivalence relation on a set S is the Main Theorem on Equivalence Relations: the set S splits into non-overlapping subsets such that for each member X of S, X is equivalent to every member of the subset containing X, and is not equivalent to any members of S outside that subset. These subsets of S are called equivalence classes.

The Main Theorem on Equivalence Relations, stating that equivalence classes cover the set and do not overlap, is a basic mathematical fact, and is used implicitly in the analyses that follow.

Dark-equivalence classification

The set M_S of all sustained motions can therefore be partitioned (without overlap) into dark-

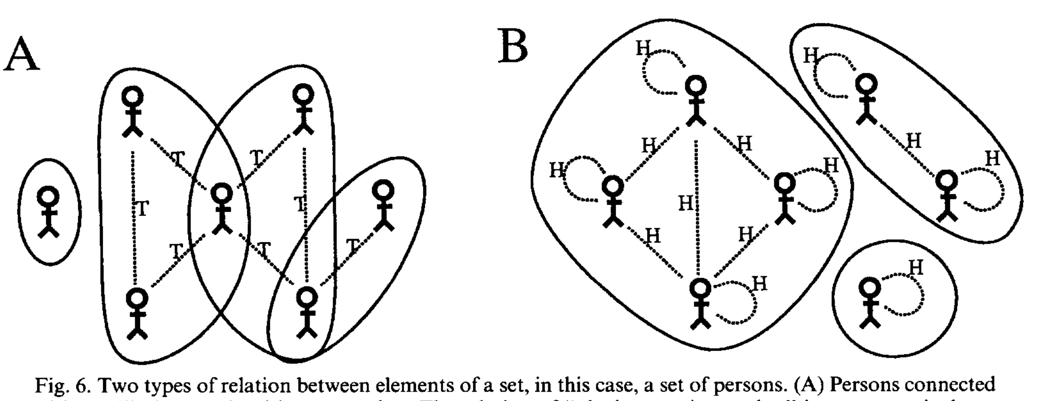

Fig. 6. Two types of relation between elements of a set, in this case, a set of persons. (A) Persons connected with a "T" play tennis with one another. The relation of "playing tennis together" is not an equivalence relation. (B) Persons connected with an "H" live in the same house. The relation of "living in the same house" is an equivalence relation here, and allows one to neatly partition the set of persons into equivalence classes.

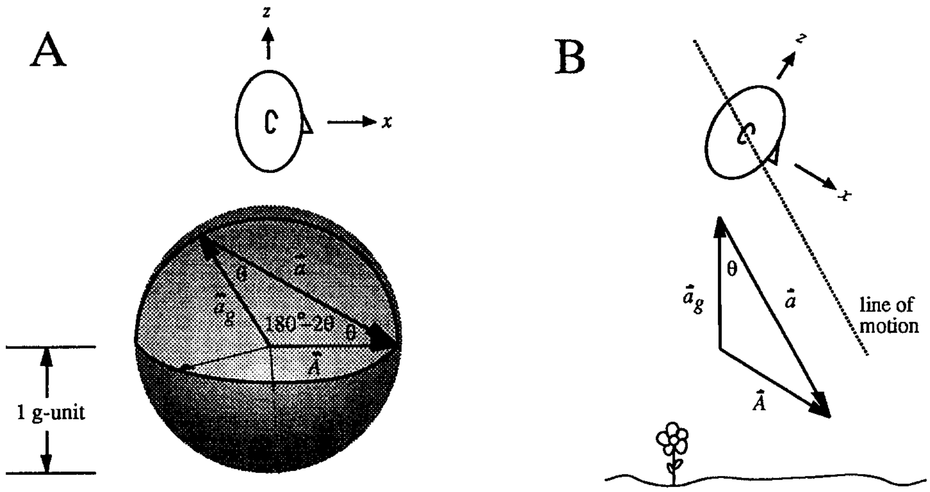

Fig. 7. (A) Relationship between the accelerations **A**, $\mathbf{a}_g$, and **a** for a Category 3 motion (linear acceleration) in a One-G dark-equivalence class, with θ denoting the angle between **a** and straight earth-downward. For a given **A**, e.g. in the x direction as indicated by the head shown, the vector $\mathbf{a}_g$ of length 1 g-unit can take any direction except exactly that of **A**. An example is labeled along with the corresponding $\mathbf{a}(=\mathbf{A}-\mathbf{a}_g)$, and the thin lines give other examples of possible $\mathbf{a}_g$. (B) The example of part (A), as viewed by an earth-upright observer.

equivalence classes; all motions in the same dark-equivalence class are perceptually indistinguishable in the dark, under the given assumptions. While the categories of sustained motion presented in the previous section give a way to discuss the various kinds of sustained motion from an observer's viewpoint (such as the viewpoint telling whether a pilot is spiraling downward toward a crash), the dark-equivalence classes slice through the categories and tell which motions are indistinguishable to the subject.

In order to describe the exact composition of the dark-equivalence classes, we first observe that each dark-equivalence class is identified by two vectors, resultant acceleration **A** and angular acceleration $\boldsymbol{\alpha}$. More precisely, if a sustained motion has resultant acceleration $\mathbf{A}=\mathbf{a}+\mathbf{a}_g$ and angular acceleration $\boldsymbol{\alpha}$, then the dark-equivalence class containing that motion consists exactly of all sustained motions with that same resultant acceleration and angular acceleration (by definition of "dark-equivalence"). In general, an individual dark-equivalence class can be specified by simply stating values for **A** and $\boldsymbol{\alpha}$.

For example, if the values $\mathbf{A}=\mathbf{k}$ (1 g-unit in the z direction) and $\boldsymbol{\alpha}=\mathbf{0}$ are given, we have the dark-equivalence class consisting of all sustained motions in which a person feels resultant acceleration of 1 g-unit upward and no angular acceleration. One such motion is simply to be upright and stationary on the surface of the earth. Another such motion is to be upright and traveling forward at a constant velocity of 810 km/h. We do a full investigation of this dark-equivalence class under "Examples in the dark."

For technical reasons, it is important to reiterate that sustained motions are the present topic of discussion. Because "dark-equivalence" is technically defined only for sustained motions, a dark-equivalence class is a subset of the set M_S of sustained motions. Therefore, the present classification tells when sustained motions are dark-equivalent, but does not say whether other, non-sustained motions are perceptually indistinguishable from the present sustained motions. This topic is explored further in the Discussion.

We present a complete classification in Table 2 of the set of sustained motions according to dark-equivalence, with Figs 7 and 8 supplementing the description for two of the cases. Mathematical derivation of the more complicated portions is contained in the Appendix. Because there is a continuous range of values for the vectors involved, there are infinitely many dark-equivalence classes, but we group them according to manageable and meaningful types by the different ranges of values for **A** and $\boldsymbol{\alpha}$. Since some sustained motion categories differ from one another only in velocity, and dark-equivalence is oblivious to velocity, a single dark-equivalence class may contain motions from several of the categories of sustained motion. A schematic representation of the equivalence classification is given in Fig. 9, showing the relative sizes of the dark-equivalence classes.

Examples in the dark

Earth gravity. The perception of being upright and stationary in the dark or with eyes closed is caused by a resultant acceleration vector of magnitude 1 g-unit in the z direction, with no angular acceleration. While this fact seems innocent enough, it turns out that many other motions also have this resultant acceleration vector, standardly denoted **k** (the g-unit being the current unit of measure), along with no angular acceleration. These motions are perceptually indistinguishable from being upright and stationary.

Table 2. The dark-equivalence classes of sustained motions

	One-G type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{\alpha} = \mathbf{0}$	Hypo-G type $\lvert\mathbf{A}\rvert < 1$ g-unit $\boldsymbol{\alpha} = \mathbf{0}$	Hyper-G type $\lvert\mathbf{A}\rvert > 1$ g-unit $\boldsymbol{\alpha} = \mathbf{0}$	Torquing type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{\alpha} \neq \mathbf{0}$ $\boldsymbol{\alpha} \parallel \mathbf{A}$	Corkscrew type $\lvert\mathbf{A}\rvert \neq 1$ g-unit $\boldsymbol{\alpha} \neq \mathbf{0}$ $\boldsymbol{\alpha} \parallel \mathbf{A}$
(1) Fixed position	$\mathbf{a}_g = \mathbf{A}$ (1)	None	None	None	None
(2) Linear velocity	$\mathbf{a}_g = \mathbf{A}$ (∞)	None	None	None	None
(3) Linear acceleration	$\mathbf{a}$ differs θ from downward $\lvert\mathbf{a}\rvert = \dfrac{\sin(180° - 2\theta)}{\sin\theta}$ as shown in Fig. 7* (∞)	$\mathbf{a}$ with downward component such that $\mathbf{a} + \mathbf{a}_g = \mathbf{A}$ (∞)	$\mathbf{a} + \mathbf{a}_g = \mathbf{A}$ (∞)	None	None
(4) Angular velocity	On-axis $\mathbf{a}_g = \mathbf{A}$ (∞)	None	$\left(\lvert\boldsymbol{\omega}\rvert \dfrac{\pi}{180}\right)^2 r = 9.8\sqrt{(\lvert\mathbf{A}\rvert^2 - 1)}$ (r = radius) $\mathbf{a}$ in direction of $\mathbf{A} - \mathbf{a}_g$ Vectors shown in Fig. 8* (∞)	None	None
(5) Angular and linear velocity	On-axis $\mathbf{a}_g = \mathbf{A}$ (∞)	None	$\left(\lvert\boldsymbol{\omega}\rvert \dfrac{\pi}{180}\right)^2 r = 9.8\sqrt{(\lvert\mathbf{A}\rvert^2 - 1)}$ (r = radius) $\mathbf{a}$ in direction of $\mathbf{A} - \mathbf{a}_g$ (∞)	None	None
(6) Angular velocity and linear acceleration	$\mathbf{a}_g = -\mathbf{A}$ $\mathbf{a} = 2\mathbf{A}$ downward (∞)	$\mathbf{a}$ with downward component such that $\mathbf{a} + \mathbf{a}_g = \mathbf{A}$ (∞)	$\mathbf{a} + \mathbf{a}_g = \mathbf{A}$ (∞)	None	None
(7) Angular acceleration	None	None	None	$\mathbf{a}_g = \mathbf{A}$ (∞)	None
(8) Angular acceleration and linear velocity	None	None	None	$\mathbf{a}_g = \mathbf{A}$ (∞)	None
(9) Angular and linear acceleration	None	None	None	$\mathbf{a}_g = -\mathbf{A}$ $\mathbf{a} = 2\mathbf{A}$ downward (∞)	$\mathbf{a}_g \parallel \mathbf{A}$ vertical $\mathbf{a}$ such that $\mathbf{a} + \mathbf{a}_g = \mathbf{A}$ (∞)

*Mathematical derivation given in the Appendix.

There are infinitely many dark-equivalence classes, but they fall into the five types listed at the top of the table. A dark-equivalence class is associated with particular values for **A** and $\boldsymbol{\alpha}$ (as explained in the text), which determine the type. In certain cases, an additional restriction on $\boldsymbol{\alpha}$ and **A** is necessary; the symbol "‖" stands for "parallel" in the figure. The motions that belong to a dark-equivalence class (with its particular associated values of **A** and $\boldsymbol{\alpha}$) are those described in the column below the appropriate type header. A dark-equivalence class typically contains motions from several different categories; for each category the table contains a description of the contained motions. Each such description is given in terms of the criterion or criteria that a motion in the category would have to satisfy in order to belong to the dark-equivalence class of interest, followed by a number in parentheses indicating the number of motions in the category that satisfy the criteria. In addition to satisfying the requirements on $\mathbf{a}_g$, **a** and **A** listed in the table, a motion in a dark-equivalence class must have angular acceleration as specified by $\boldsymbol{\alpha}$ for the dark-equivalence class in question. Often, infinitely many motions in a category satisfy the listed criteria, in which case the number indicator "∞" is used in parentheses, and Fig. 9 gives more detail about the "size" of the infinities. In this case, the dark-equivalence class contains infinitely many motions from the category in question, along with any motions from other categories fitting the associated descriptions. "None" indicates that no motions of an indicated category belong to the dark-equivalence class. In order to fully interpret the descriptions of motions within the categories here, it is necessary to use this table in conjunction with an understanding of the categories themselves, through use of Table 1 or reference to category descriptions in the text. The vectors are given with units of measure as follows: **A**, **a** and $\mathbf{a}_g$ in g-units, $\boldsymbol{\alpha}$ in °/s², $\mathbf{v}$ in m/s and $\boldsymbol{\omega}$ in °/s. "Downward" and "vertical" mean "earth-downward" and "earth-vertical", respectively.

Examples include (i) being upright with forward velocity of 810 km/h, (ii) being upright and rotating at 60°/s, and even (iii) being completely upside-down and heading 90 km/h straight toward the earth with twice the acceleration of gravity. Misperception in such a situation can be dangerous!

Using the description in Table 2 of dark-equivalence classes, and in particular, the description for One-G Type, we determine the entire class of sustained motions that are dark-equivalent to being upright and stationary; the class is given by $\mathbf{A} = \mathbf{k}$ and $\boldsymbol{\alpha} = \mathbf{0}$.

Besides sitting upright and stationary (Category 1), there are infinitely many motions in which one is upright while moving at constant velocity (Category 2), such as "forward velocity of 810 km/h", "upward velocity of 0.001 m/s", "velocity of 90 km/h in a direction 30° to the left of forward", etc.

There are also many attitudes relative to the gravitational field that are allowed, as long as the linear acceleration leads to the resultant acceleration vector $\mathbf{k}$ (Category 3). In fact, any attitude is possible: if one chooses a direction for $\mathbf{a}_g$ giving the attitude, then the motion specified by $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{a}_g$ as chosen, $\mathbf{a} = \mathbf{k} - \mathbf{a}_g$, and "some" $\mathbf{v}$, is dark-equivalent to being upright. By "some" $\mathbf{v}$, we mean that there are choices; any vector $\mathbf{v}$ parallel to $\mathbf{a}$ is allowed, so there are actually infinitely many motions possible, even after choosing an attitude. Note that $|\mathbf{a}_g| = |\mathbf{k}| = 1$, so that the vector subtraction here results in $\mathbf{a}$ being an acceleration toward the earth, possibly at an angle.

Another collection of motions dark-equivalent to being upright and stationary are the on-axis rotations with a vertical axis (Category 4). Either direction and any magnitude of angular velocity is allowed (however, small magnitudes are desired—see the note following the definition of "dark-equivalent").

In addition, such a rotation can take place while traveling upward or downward with constant velocity (Category 5) or while accelerating downward at twice the acceleration of gravity in an upside-down position (Category 6). Once again, there are infinitely many motions here dark-equivalent to being upright and stationary.

In summary, a person who sits with eyes closed and apparently stationary may actually be:

(i) moving at a non-zero velocity;
(ii) accelerating toward the earth while oriented upside down, possibly at an angle;
(iii) rotating;
(iv) rotating while also moving up or down; or
(v) rotating while accelerating downward, upside down.

These are illustrated in Fig. 10.

Micro-gravity. When an astronaut experiences zero gravity, what everyday motions might be associated with the resulting perception?

Both the resultant acceleration vector $\mathbf{A}$ and the angular acceleration vector $\boldsymbol{\alpha}$ are zero, so the dark-equivalence class is of the Hypo-G type. Since only Categories 3 and 6 have members in Hypo-G

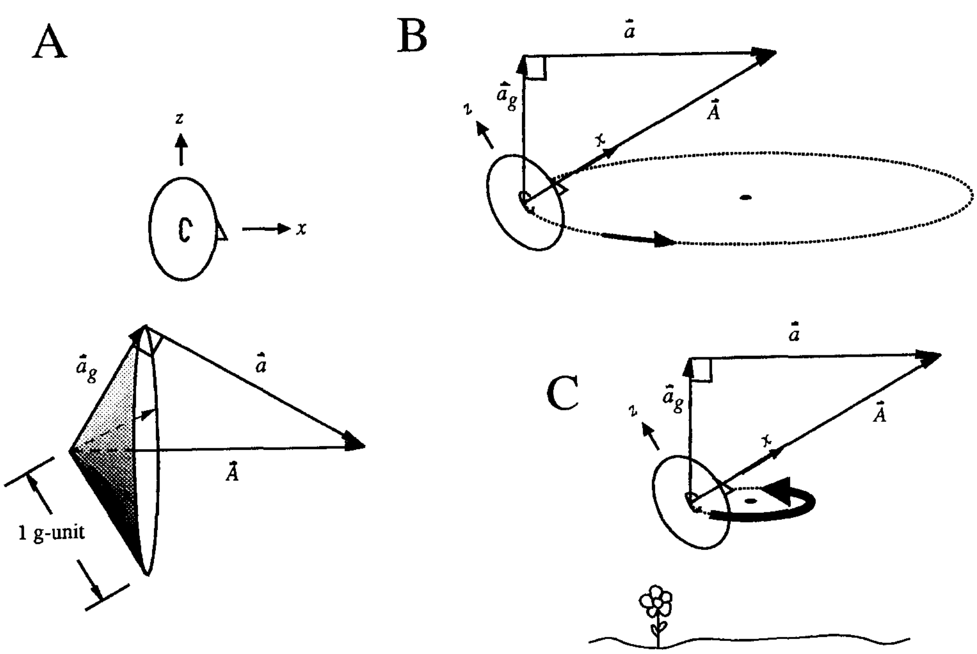

Fig. 8. (A) Relationship between the accelerations $\mathbf{A}$, $\mathbf{a}_g$ and $\mathbf{a}$ for a Category 4 motion (angular velocity) in a Hyper-G dark-equivalence class. For a given $\mathbf{A}$, the direction of $\mathbf{a}_g$ differs from that of $\mathbf{A}$ by an angle prescribed by the fact that $\mathbf{a}$ and $\mathbf{a}_g$ must be orthogonal. An example is labeled, $\mathbf{A}$ being in the x direction, and other examples of possible $\mathbf{a}_g$ shown by thin lines. (B) The example of A, achieved by a motion with large radius of rotation and small angular velocity. (C) The example of A, achieved by a motion with small radius of rotation and large angular velocity.

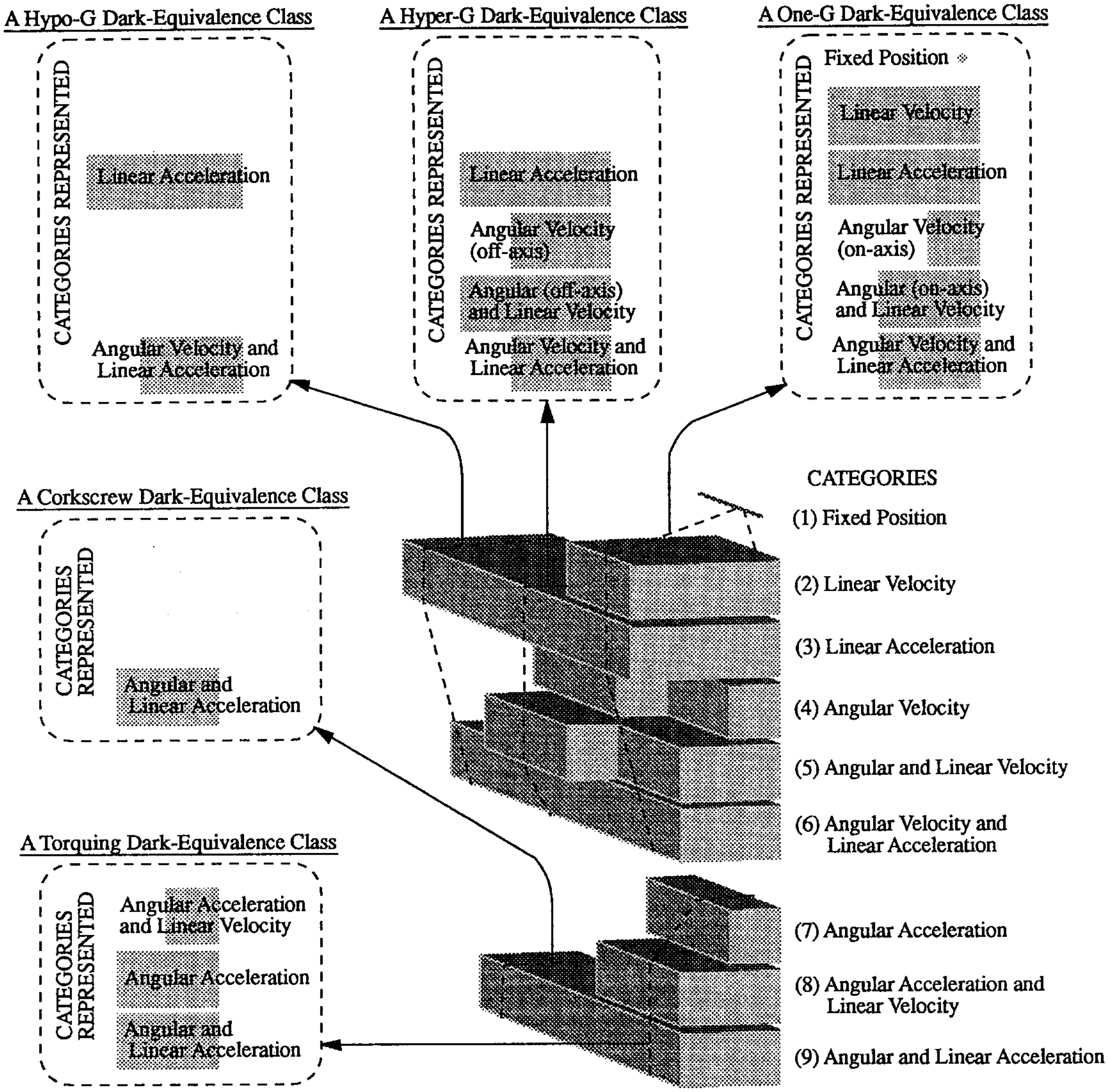

Fig. 9. The five types of dark-equivalence class. Each dark-equivalence class consists of all the motions from a slice through the categories when the category boxes (from Fig. 4) are lined up. Sample dark-equivalence classes of each type are shown schematically here. Each class consists of many motions, possibly from several different categories. Within each dark-equivalence class, the relative numbers of motions from the different categories are indicated by the relative sizes of the shaded boxes. Widths of the shaded boxes represent degrees of freedom available for velocities and attitude. See also Fig. 4.

dark-equivalence classes, the class with **A** and α both zero is relatively small.

Earth-downward acceleration of 1 g-unit, usually associated with falling, is required for sustained motions in the present class. The motion can have zero spin (Category 3), or have a constant spin about a vertical axis through the head (Category 6). There are actually many different motions here; besides there being various magnitudes of spin possible, there are also many ways to fall: "feet-first", "head-first", "face-first", etc. In addition, 1 g-unit earth-downward acceleration also occurs during the upward movement after jumping or being thrown.

Therefore, the dark-equivalence class with **A** and α both zero consists of all the different attitudes of falling along with all the possible downward velocities and on-axis vertical spins, in addition to all attitudes, velocities and on-axis vertical spins during "free-fall" upward movement.

LIGHT-EQUIVALENCE WITH EARTH-FIXED SURROUND

When an earth-fixed visual surround is available, much less disorientation and misperception of self-motion occurs than does in the dark. The main addition that the visual system provides to the perceptual picture drawn by the vestibular system is information about velocity. The goal of this section is to investigate the consequences of velocity input, and classify those sustained motions that are perceptually indistinguishable in the light, when vision is available and the visual surround is earth-fixed. As in many experiments on perception, no provisions are taken here for controlling eye movements.

With vision, including the sense of eye movement within the head, a subject can detect differences in velocity between the head and a full-field visual surround; such a difference usually means, and is perceived to mean, that the head is moving. In each part of the visual field, the flow may be converging, diverging, receding, looming or curling, and all of these components are combined to produce the information of linear and angular velocity. These facts are demonstrated by many studies of self-motion perception with vision available.[5] With a visual stimulus of a constant velocity rotating visual surround, such as with a surrounding rotating drum, subjects experience a sensation of self-rotation called circular vection. The axis of the rotation sensation coincides with the axis of visual-surround rotation, even leading to such experiences as roll vection about the naso-occipital axis and pitch vection about the interaural axis. Similarly, linear vection is induced by constant velocity linear motion of the visual surround, with direction of velocity sensation opposite that of the visual stimulus. The visual stimulus need not consist of motion of the entire visual surround; objects move within the visual field in everyday life without causing a sensation of self-motion. Two portions of the visual surround that have been determined important to the perception of self-motion are the periphery of the visual field and the background in visual space.[5] The details explaining self-motion perception by vision are a topic of debate, with many factors playing a role.[5,18]

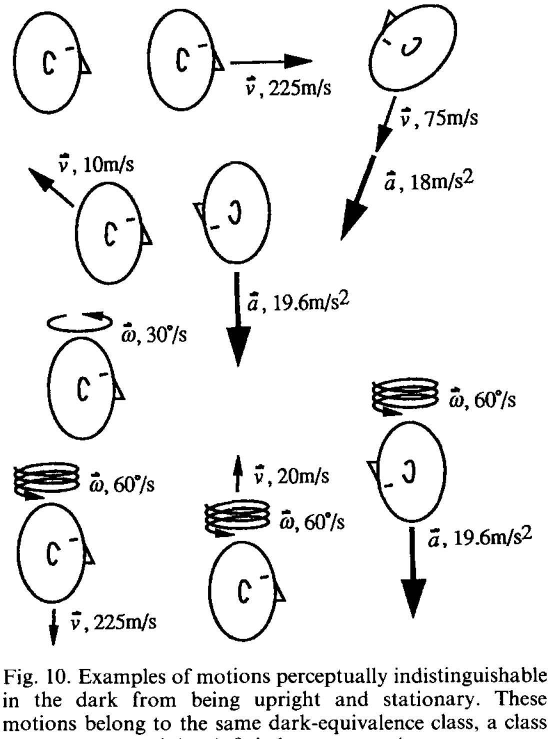

Fig. 10. Examples of motions perceptually indistinguishable in the dark from being upright and stationary. These motions belong to the same dark-equivalence class, a class containing infinitely many motions.

We assume that the subject has a wide view of the visual surround as in the studies mentioned above, so the visual system provides the input of linear velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$. By assuming this, we do not mean that the subject can necessarily estimate linear and angular velocities accurately at the conscious level, but that different linear velocities of the same visual surround can be distinguished by the visual system, as can different angular velocities. This assumption is most valid at speeds that are not excessive; it is known that visual surround angular velocities of at least 360°/s are distinguishable by a subject,[2] although the exact limits would depend on the particular nature of the visual surround. A discussion of this issue and of other possible restrictions on the visual input can be found in the Discussion.

In addition, we must decide whether to allow a view of a horizontal ground or floor in the visual input, and if so, whether to suppose that the value of $\mathbf{a}_g$ is then provided as input. If $\mathbf{a}_g$ is considered to be given in this way, it turns out that no ambiguity in sustained motion can ever occur because the vectors $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, $\mathbf{a}_g$ are uniquely determined by the values of $\mathbf{A}$ and $\boldsymbol{\alpha}$ (from vestibular) with $\mathbf{v}$, $\boldsymbol{\omega}$ and $\mathbf{a}_g$ (from visual), since the only vector not clearly given, $\mathbf{a}$, is simply computed by taking the difference between two given vectors: $\mathbf{a} = \mathbf{A} - \mathbf{a}_g$. However, misperceptions do occur even when visual input is available (especially when a horizontal ground is not available, as in hilly terrain or during flight above or among clouds), so we make the assumption that $\mathbf{a}_g$ is not provided; optokinetic input to self-motion perception in this case provides only $\mathbf{v}$ and $\boldsymbol{\omega}$.

In summary, the vestibular system provides the values of $\mathbf{A}$ $(= \mathbf{a} + \mathbf{a}_g)$ and $\boldsymbol{\alpha}$ during a sustained motion, and the visual system is considered to provide the values of $\mathbf{v}$ and $\boldsymbol{\omega}$. With this in mind, we make the following definition.

Definition. If sustained motion B is specified by the vectors $\mathbf{v}^B$, $\mathbf{a}^B$, $\boldsymbol{\omega}^B$, $\boldsymbol{\alpha}^B$, $\mathbf{a}_g^B$, and sustained motion C by $\mathbf{v}^C$, $\mathbf{a}^C$, $\boldsymbol{\omega}^C$, $\boldsymbol{\alpha}^C$, $\mathbf{a}_g^C$, then the motions B and C are termed light-equivalent if

$$\mathbf{a}^B + \mathbf{a}_g^B = \mathbf{a}^C + \mathbf{a}_g^C$$

(i.e. $\mathbf{A}^B = \mathbf{A}^C$, resultant accelerations equal),

$$\boldsymbol{\alpha}^B = \boldsymbol{\alpha}^C$$

(angular accelerations equal),

$$\mathbf{v}^B = \mathbf{v}^C$$

(linear velocities equal), and

$$\boldsymbol{\omega}^B = \boldsymbol{\omega}^C$$

(angular velocities equal).

Just as dark-equivalence reflects perceptual indistinguishability in the dark, light-equivalence reflects perceptual indistinguishability in the light. Light-

Table 3. The light-equivalence classes of sustained motions when the visual surround is earth-fixed

(A)	One-G stationary type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{v} = \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} = \mathbf{0}$	One-G moving type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} = \mathbf{0}$ Case $\boldsymbol{v} \perp \mathbf{A}$	One-G moving type Case $\boldsymbol{v} \not\perp \mathbf{A}$	Hypo-G moving type $\lvert\mathbf{A}\rvert < 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} = \mathbf{0}$	Hyper-G moving type $\lvert\mathbf{A}\rvert > 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} = \mathbf{0}$ Smallest angle θ between $\mathbf{A}$ and $\boldsymbol{v}$ is $\leq$ Arcsin $1/\lvert\mathbf{A}\rvert$ Case $\theta =$ Arcsin $1/\lvert\mathbf{A}\rvert$	Hyper-G moving type Case $\theta <$ Arcsin $1/\lvert\mathbf{A}\rvert$
(1) Fixed position	$\mathbf{a}_g = \mathbf{A}$ (1)	None	None	None	None	None
(2) Linear velocity	None	$\mathbf{a}_g = \mathbf{A}$ (Fig. 11A)* (1)	$\mathbf{a}_g = \mathbf{A}$ (1)	None	None	None
(3) Linear acceleration	None	None	$\mathbf{a}_g = \mathbf{A} - c\boldsymbol{v}$ c non-zero real, $\mathbf{a} = c\boldsymbol{v}$ (Fig. 11B)* (1)	$\mathbf{a}_g = \mathbf{A} - c\boldsymbol{v}$ c real, $\mathbf{a} = c\boldsymbol{v}$ (Fig. 11C) (2)	$\mathbf{a}_g = \mathbf{A} - c\boldsymbol{v}$ c real, $\mathbf{a} = c\boldsymbol{v}$ (Fig. 11D) (1)	$\mathbf{a}_g = \mathbf{A} - c\boldsymbol{v}$ c real, $\mathbf{a} = c\boldsymbol{v}$ (Fig. 11E) (2)

(B)	One-G rotating type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{v} = \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ $\mathbf{A} \parallel \boldsymbol{\omega}$	One-G curl type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ $\mathbf{A} \parallel \boldsymbol{v} \parallel \boldsymbol{\omega}$	Hypo-G curl type $\lvert\mathbf{A}\rvert < 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ $\mathbf{A} \parallel \boldsymbol{v} \parallel \boldsymbol{\omega}$	Hyper-G curl type $\lvert\mathbf{A}\rvert > 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} = \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ with additional requirements for each case* Case $\boldsymbol{v} \parallel \boldsymbol{\omega}$	Hyper-G curl type Case $\boldsymbol{v} \perp \boldsymbol{\omega}$	Hyper-G curl type Case $\boldsymbol{v} \nparallel \boldsymbol{\omega}$, $\boldsymbol{v} \not\perp \boldsymbol{\omega}$
(4) Angular velocity	$\mathbf{a}_g = \mathbf{A}$ (1)	None	None	None	Off-axis rotation with vectors as given (exact equations in Appendix)* (1)	None
(5) Angular and linear velocity	None	$\mathbf{a}_g = \mathbf{A}$ (1)	None	None	None	Off-axis rotation with vectors as given (exact equations in Appendix)* (1)
(6) Angular velocity and linear acceleration	None	$\mathbf{a}_g = -\mathbf{A}$ $\mathbf{a} = 2\mathbf{A}$ All vectors vertical (1)	$\mathbf{a}_g \parallel \mathbf{A}$ $\mathbf{a} = \mathbf{A} - \mathbf{a}_g$ All vectors vertical (2)	$\mathbf{a}_g \parallel \mathbf{A} \parallel \boldsymbol{v} \parallel \boldsymbol{\omega}$ $\mathbf{a} = \mathbf{A} - \mathbf{a}_g$ All vectors vertical* (2)	None	None

(C)	Torquing rotating type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{v} = \mathbf{0}$ $\boldsymbol{\alpha} \neq \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ $\mathbf{A} \parallel \boldsymbol{\alpha} \parallel \boldsymbol{\omega}$	Torquing curl type $\lvert\mathbf{A}\rvert = 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} \neq \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ $\mathbf{A} \parallel \boldsymbol{\alpha} \parallel \boldsymbol{v} \parallel \boldsymbol{\omega}$	Corkscrew curl type $\lvert\mathbf{A}\rvert \neq 1$ g-unit $\boldsymbol{v} \neq \mathbf{0}$ $\boldsymbol{\alpha} \neq \mathbf{0}$ $\boldsymbol{\omega} \neq \mathbf{0}$ $\mathbf{A} \parallel \boldsymbol{\alpha} \parallel \boldsymbol{v} \parallel \boldsymbol{\omega}$
(7) Angular acceleration	$\mathbf{a}_g = \mathbf{A}$ (1)	None	None
(8) Angular acceleration and linear velocity	None	$\mathbf{a}_g = \mathbf{A}$ All vectors vertical (1)	None
(9) Angular and linear acceleration	None	$\mathbf{a}_g = -\mathbf{A}$ $\mathbf{a} = 2\mathbf{A}$ All vectors vertical (1)	$\mathbf{a}_g \parallel \mathbf{A}$ $\mathbf{a} = \mathbf{A} - \mathbf{a}_g$ All vectors vertical (2)

*Mathematical derivation is given in the Appendix.

There are infinitely many light-equivalence classes, but they fall into the 11 types listed at the tops of the three parts of the table. A light-equivalence class is associated with particular values for $\mathbf{A}$, $\boldsymbol{\alpha}$, $\boldsymbol{v}$ and $\boldsymbol{\omega}$ (as explained in the text), which determine the type. In certain cases, additional restrictions on the vectors for a type are necessary; the symbol "$\parallel$" stands for "parallel" and the symbol "$\perp$" stands for "perpendicular". The motions that belong to a light-equivalence class are those described in the column below the appropriate type header. A light-equivalence class may contain motions from more than one category; for each category the chart contains a description of the contained motion or motions. Each such description is given in terms of the criterion or criteria that a motion in the category would have to satisfy in order to belong to the light-equivalence class of interest, followed by a number in parentheses indicating the number of motions in the category that satisfy the criteria. In addition to satisfying the requirements on $\mathbf{a}_g$, $\mathbf{a}$ and $\mathbf{A}$ listed in the chart, a motion in a dark-equivalence class must have linear velocity $\boldsymbol{v}$, angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ as specified by the light-equivalence class. "None" indicates that no motions of an indicated category belong to the light-equivalence class. More empty categories are avoided by breaking the table into blocks, rather than keeping them together as in Table 2. In contrast to the dark-equivalence classes of sustained motions, the present equivalence classes each contain only a finite number of sustained motions. In order to fully interpret the descriptions of motions within the categories here, it is necessary to use this table in conjunction with an understanding of the categories themselves, through use of Table 1 or reference to category descriptions in the text. The vectors are given with units of measure as follows: $\mathbf{A}$, $\mathbf{a}$ and $\mathbf{a}_g$ in g-units, $\boldsymbol{\alpha}$ in °/s^2, $\boldsymbol{v}$ in m/s and $\boldsymbol{\omega}$ in °/s. "Vertical" means "earth-vertical". (A) Classes without angular motion. (B) Classes with angular velocity but not acceleration. (C) Classes with angular acceleration.

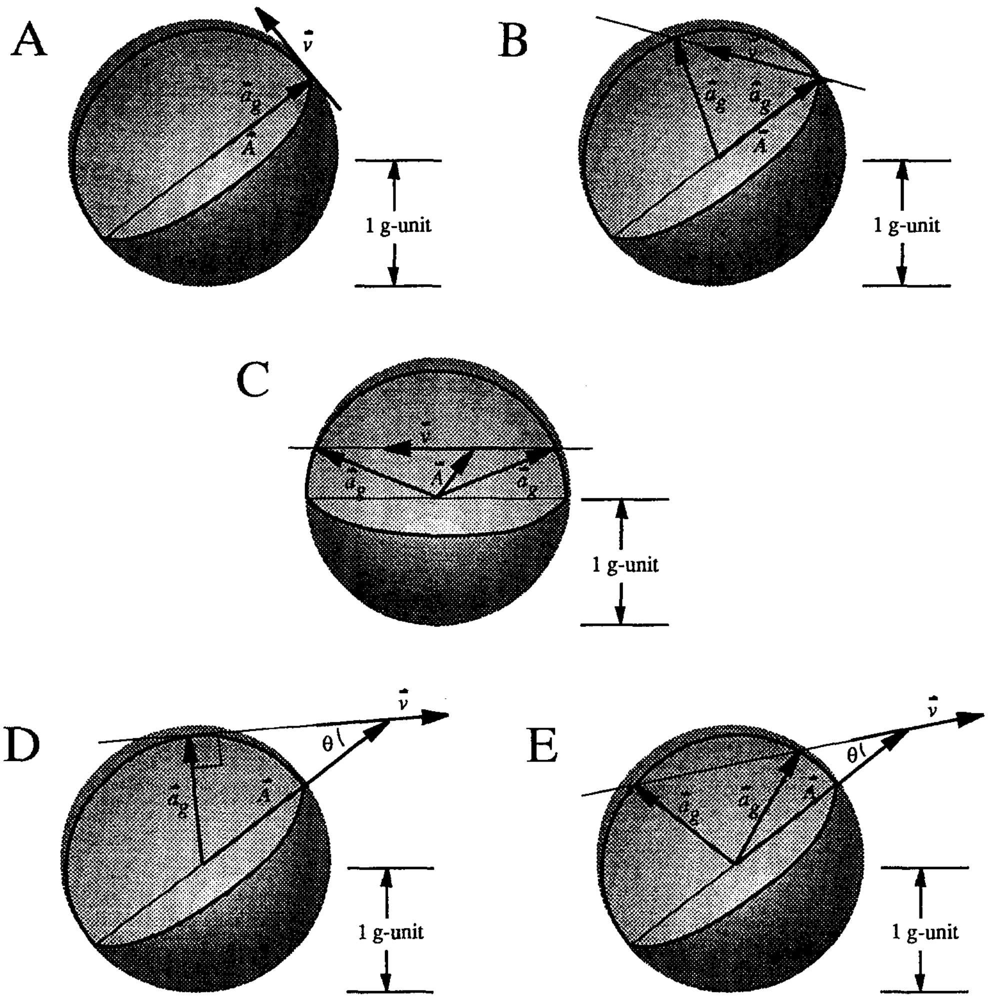

Fig. 11. Examples of vectors **A** and **v** for certain light-equivalence classes, showing the possible values for $\mathbf{a}_g$ $(= \mathbf{A} - c\mathbf{v}$ for real numbers $c)$. (A) For a One-G moving light-equivalence class, Case $\mathbf{v} \perp \mathbf{A}$. (B) For a One-G moving light-equivalence class, Case $\mathbf{v} \not\perp \mathbf{A}$. (C) For a Hypo-G moving light-equivalence class. (D) For a Hyper-G moving light-equivalence class, Case $\theta = \text{Arcsin } 1/|\mathbf{A}|$. (E) For a Hyper-G moving light-equivalence class, Case $\theta < \text{Arcsin } 1/|\mathbf{A}|$.

equivalence is an equivalence relation on the set M_S of sustained motions, implying that the set M_S of sustained motions can be partitioned, this time into light-equivalence classes. Just as each dark-equivalence class is specified by stating the values **A** and $\boldsymbol{\alpha}$, each light-equivalence class is specified by stating the values of the four vectors **A**, $\boldsymbol{\alpha}$, **v** and $\boldsymbol{\omega}$.

Light-equivalence classification

We present a complete classification in Table 3 of the set of sustained motions according to light-equivalence when the visual surround is fixed relative to the earth, with Fig. 11 supplementing the description for three of the cases. Mathematical derivation of the more complicated portions is contained in the Appendix. Just as in the dark-equivalence classification, we list the equivalence classes by manageable and meaningful types. Since there is less perceptual ambiguity with optokinetic input than in the dark, light-equivalence classes are much smaller than dark-equivalence classes. In face, since **A**, $\boldsymbol{\alpha}$, **v** and $\boldsymbol{\omega}$ are specified here, a light-equivalence class contains more than one member only when there is more than one way to decompose **A** into **a** and $\mathbf{a}_g$, so that the vectors obey the restrictions on relative directions (parallel, perpendicular, etc.). Figure 12 gives a schematic representation of the light-equivalence classes.

Examples with earth-fixed visual surround

Earth gravity. When upright and stationary with an earth-fixed surround visible, healthy subjects are usually confident that they are not moving. This confidence is justified, as can be seen by inspection of the relevant light-equivalence class.

The vectors of interest here are $\mathbf{A} = \mathbf{k}$ (upward-directed of magnitude 1 g-unit, due to upright orientation), $\alpha = 0$, $\mathbf{v} = 0$ and $\omega = 0$, so the light-equivalence class is of the One-G Stationary type. From Table 3, it is immediate that the light-equivalence class containing "being upright and stationary" contains no other motions. There is no perceptual ambiguity.

Moving upward. When supine, if constant upward velocity of 100 m/s is occurring, then the vectors have values $\mathbf{A} = \mathbf{i}$, $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{v} = 100\mathbf{i}$ (units for velocity being m/s) and $\boldsymbol{\omega} = \mathbf{0}$. The resulting light-equivalence class is determined from Table 3.

This light-equivalence class is of the One-G Moving type, and $\mathbf{v}$ is not orthogonal to $\mathbf{A}$. One member is from Category 2 (linear velocity) and is upward travel of 100 m/s. The other member is from Category 3 (linear acceleration), and the only way for $\mathbf{a}_g$ to equal $\mathbf{A} - c\mathbf{v}$ for non-zero real c is for $\mathbf{a}_g = -\mathbf{i}$. In other words,

$$\mathbf{a}_g = \mathbf{A} - c\mathbf{v}$$

$$-\mathbf{i} = \mathbf{i} - \frac{2}{100}(100\mathbf{i}).$$

Then $\mathbf{a} = c\mathbf{v} = 2\mathbf{i}$, and the motion described is travel in a face-down position heading toward the earth at 100 m/s, with twice the acceleration of gravity. This light-equivalence class is displayed in Fig. 13.

LIGHT-EQUIVALENCE WITH MOBILE VISUAL SURROUND

When the entire visual surround moves independently of the earth, misperception of self-motion is likely to occur. The goal of this section is to classify motions that are perceptually indistinguishable in the light, with a visual surround that may be moving. When we changed our focus of investigation from dark-equivalence to light-equivalence with earth-fixed visual surround, vision added velocities to the list of vectors available to the subject. In changing the focus from light-equivalence with earth-fixed visual surround to light-equivalence with mobile visual surround, there are no new vectors available to the subject (thus the name "light-equivalence" still applies), but we expand the domain of study, as shown schematically in Fig. 14. For this new, larger domain, the motion of the subject and the velocities of the visual surround are included in the description of each motion; the sustained motions with earth-fixed visual surround form a subset of this larger domain because they are precisely those elements for which the visual surround velocities are equal to zero.

Two new parameters, the linear velocity $\mathbf{v}_{vs}$ and angular velocity $\boldsymbol{\omega}_{vs}$ of the visual surround relative to the earth, at the position of the subject, must be included in specifying the conditions for each motion. A sustained motion with possibly moving visual

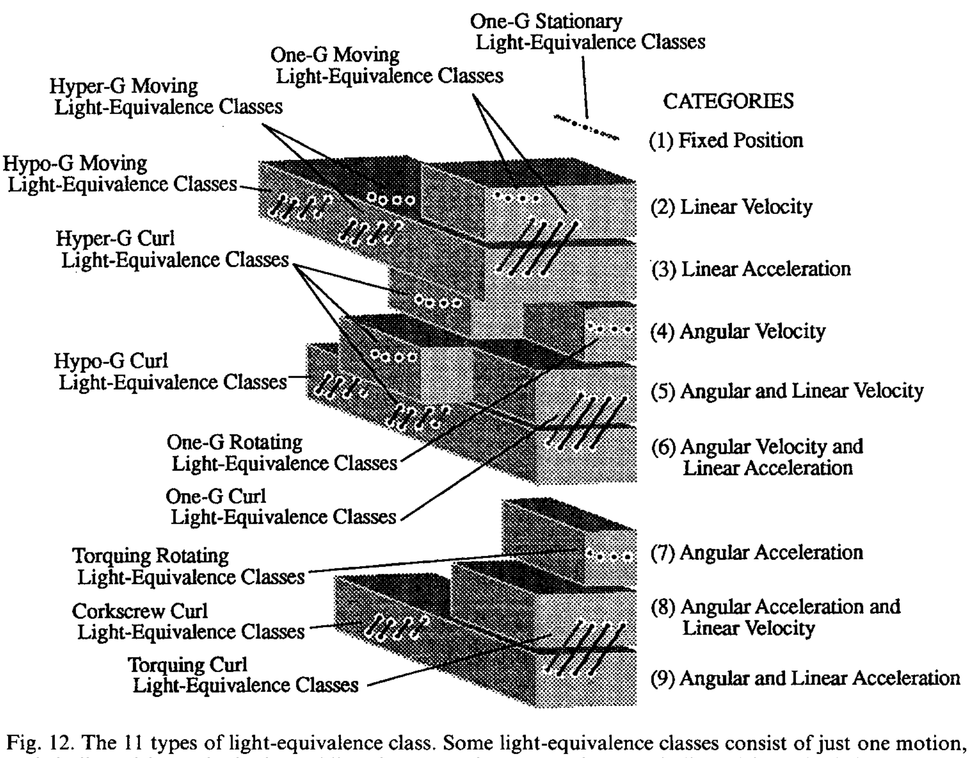

Fig. 12. The 11 types of light-equivalence class. Some light-equivalence classes consist of just one motion, each indicated by a single dot, while others contain two motions, as indicated by paired dots. Several sample equivalence classes of each type are shown, with each dot representing a motion in the category indicated. The light-equivalence classes are shown in groups of four for illustrative purposes, but there are actually infinitely many of each kind. The category boxes and their parts are discussed in greater detail in Fig. 4.

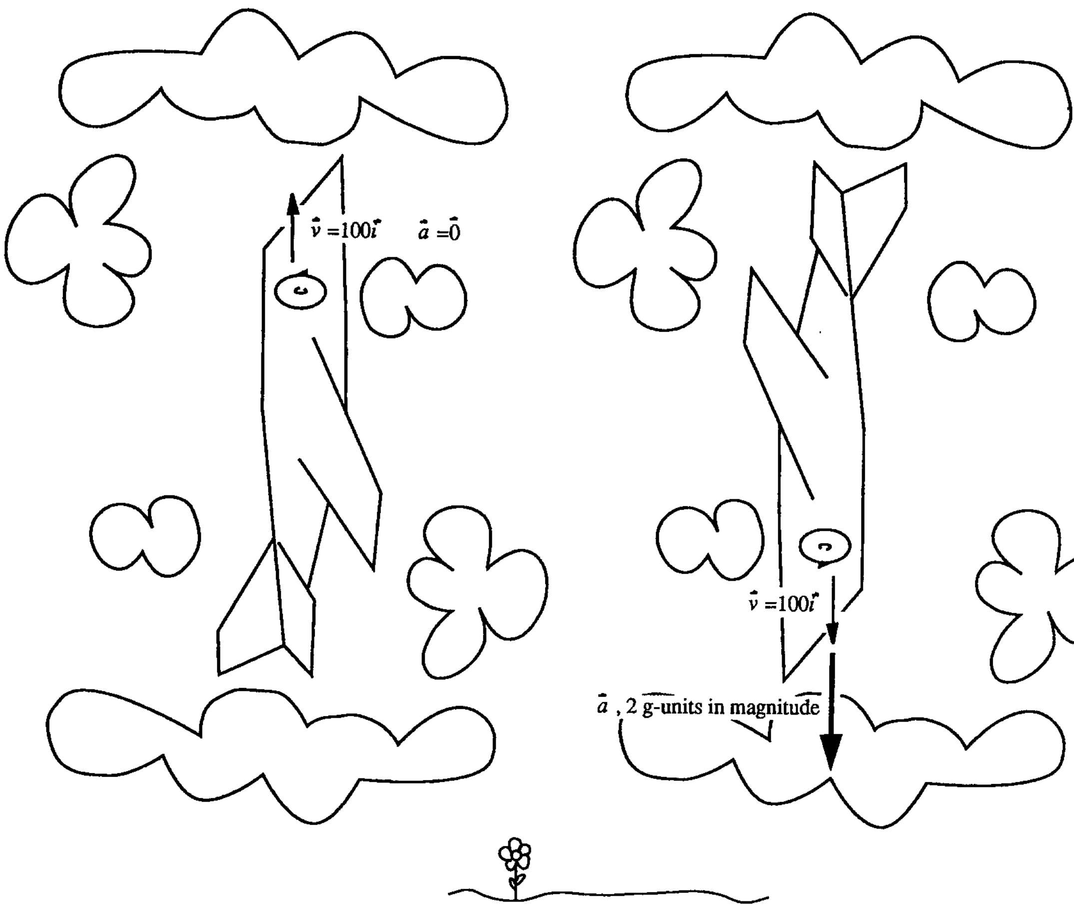

Fig. 13. Two perceptually indistinguishable motions with vision available but no view of the ground. These two motions constitute a one-G moving light-equivalence class.

surround is given by the collection of vectors $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, $\mathbf{a}_g$, $\mathbf{v}_{vs}$ and $\boldsymbol{\omega}_{vs}$ (all given in subject-coincident coordinates as shown in Fig. 1).

Sustained movement of the entire visual surround generally causes an illusion of self-motion in the opposite direction.[5] In technical terms, optokinetic input indicates head velocities of $\mathbf{v}$ and $\boldsymbol{\omega}$ if the visual surround is earth-fixed, but of $\mathbf{v}-\mathbf{v}_{vs}$ and $\boldsymbol{\omega}-\boldsymbol{\omega}_{vs}$ if the visual surround is moving relative to the earth. We call $\mathbf{v}-\mathbf{v}_{vs}$ and $\boldsymbol{\omega}-\boldsymbol{\omega}_{vs}$ the apparent linear velocity and the apparent angular velocity, respectively. The cases of interest are those in which a sustained motion is paired with a visual surround movement in such a way that the resulting "apparent" motion also falls into the set of sustained motions. Such a situation, given by the appropriate vectors $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, $\mathbf{a}_g$, $\mathbf{v}_{vs}$ and $\boldsymbol{\omega}_{vs}$, will be called a surround-mobile motion. Note that we do not require $\mathbf{v}_{vs}$ or $\boldsymbol{\omega}_{vs}$ to be non-zero; each sustained motion is also a surround-mobile motion with $\mathbf{v}_{vs}=\mathbf{0}$ and $\boldsymbol{\omega}_{vs}=\mathbf{0}$.

The definition of "light-equivalence" can now be extended to cover all surround-mobile motions.

Definition. If surround-mobile motion B is specified by the vectors $\mathbf{v}^B$, $\mathbf{a}^B$, $\boldsymbol{\omega}^B$, $\boldsymbol{\alpha}^B$, $\mathbf{a}_g^B$, $\mathbf{v}_{vs}^B$, $\boldsymbol{\omega}_{vs}^B$ and surround-mobile motion C by $\mathbf{v}^C$, $\mathbf{a}^C$, $\boldsymbol{\omega}^C$, $\boldsymbol{\alpha}^C$, $\mathbf{a}_g^C$, $\mathbf{v}_{vs}^C$, $\boldsymbol{\omega}_{vs}^C$, then the motions B and C are termed light-equivalent if

$$\mathbf{a}^B+\mathbf{a}_g^B=\mathbf{a}^C+\mathbf{a}_g^C$$

(i.e. $\mathbf{A}^B=\mathbf{A}^C$, resultant accelerations equal),

$$\boldsymbol{\alpha}^B=\boldsymbol{\alpha}^C$$

(angular accelerations equal),

$$\mathbf{v}^B-\mathbf{v}_{vs}^B=\mathbf{v}^C-\mathbf{v}_{vs}^C$$

(apparent linear velocities equal), and

$$\boldsymbol{\omega}^B-\boldsymbol{\omega}_{vs}^B=\boldsymbol{\omega}^C-\boldsymbol{\omega}_{vs}^C$$

(apparent angular velocities equal).

Once again, light-equivalent motions are perceptually indistinguishable since they cause the same sensation of self-motion. If the subject does not know that the visual surround is moving, or even if the subject knows that the visual surround may move but does not know its direction and/or magnitude, this light-equivalence between motions can easily lead to misperception of self-motion. On the other hand, if the subject knows the exact movement conditions, a cognitive factor can affect the subject's belief of self-motion (as opposed to the sensation of self-motion). We investigate the sensation of self-motion, thereby focusing on those cases in which the subject does not know the direction and/or magnitude of the visual surround movement. At the same time, this investigation has implications in cases where the subject has knowledge of the movement; sometimes sensation will override knowledge of the movement, such as when a pilot has trouble believing the aircraft instruments.[16]

Algorithm to compute the light-equivalence class of a surround-mobile motion

When the visual surround is mobile, there is a large potential for misperception of self-motion, just as there is with eyes closed or in darkness. In fact, the light-equivalence classes for surround-mobile motions are closely related to the dark-equivalence classes for sustained motions, and the classification of light-equivalence is presented here by an algorithm which uses the classification of dark-equivalence already given.

Given a particular surround-mobile motion, we would like to know which other surround-mobile motions are perceptually indistinguishable from this motion. In other words, we would like to be able to compute the light-equivalence class of the given motion. The algorithm to compute this light-equivalence class is as follows:

First: specify the surround-mobile motion in question by $\mathbf{v}$, $\mathbf{a}$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, $\mathbf{a}_g$, $\mathbf{v}_{vs}$, $\boldsymbol{\omega}_{vs}$.
Second: compute the dark-equivalence class given by $\mathbf{A} = \mathbf{a} + \mathbf{a}_g$ and $\boldsymbol{\alpha}$ by consulting the classification for dark-equivalence.
Third: the light-equivalence class in question consists of all motions in the above dark-equivalence class, with visual-surround velocities appended to each motion as follows: for each sustained motion X (with $\mathbf{v}^X$, $\mathbf{a}^X$, $\boldsymbol{\omega}^X$, $\boldsymbol{\alpha}^X$, $\mathbf{a}_g^X$) in the dark-equivalence class,
let $\mathbf{v}_{vs}^X = \mathbf{v}^X - (\mathbf{v} - \mathbf{v}_{vs})$,
let $\boldsymbol{\omega}_{vs}^X = \boldsymbol{\omega}^X - (\boldsymbol{\omega} - \boldsymbol{\omega}_{vs})$,
put surround-mobile motion given by $\mathbf{v}^X$, $\mathbf{a}^X$, $\boldsymbol{\omega}^X$, $\boldsymbol{\alpha}^X$, $\mathbf{a}_g^X$, $\mathbf{v}_{vs}^X$, $\boldsymbol{\omega}_{vs}^X$ into the light-equivalence class being computed.
Conclude: after computing $\mathbf{v}_{vs}^X$, $\boldsymbol{\omega}_{vs}^X$ and the corresponding surround-mobile motions as indicated in the third step, for every motion in the dark-equivalence class above, the resulting collec-

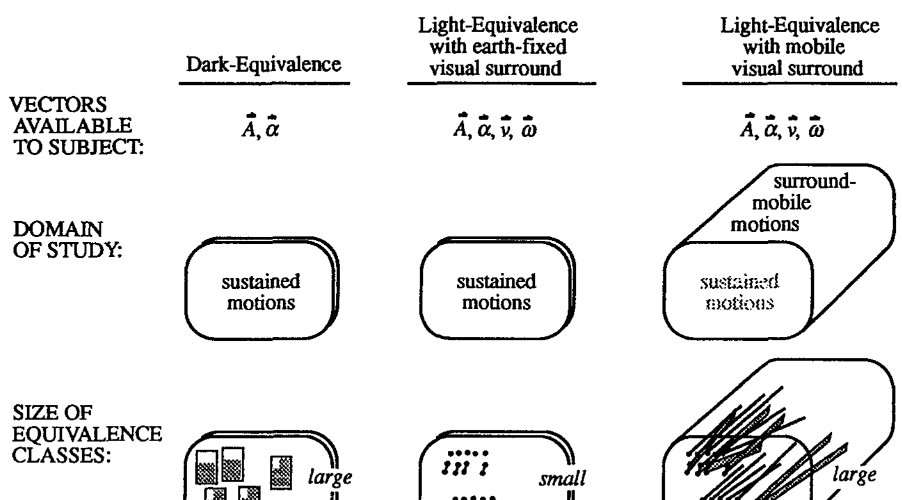

Fig. 14. The domains of study, vectors available to the subject and relative sizes of equivalence classes for the three realms of perceptual equivalence investigated in this paper. As shown in the "vectors available to subject" row, the term "dark-equivalence" means that the subject can detect $\mathbf{A}$ and $\boldsymbol{\alpha}$, while the term "light-equivalence" means that the subject can detect $\mathbf{A}$, $\boldsymbol{\alpha}$, $\mathbf{v}$ and $\boldsymbol{\omega}$. We study the effects of dark-equivalence and light-equivalence on certain domains of study. Sustained motions, in which the subject can move relative to the earth, are investigated for dark-equivalence and for light-equivalence, the latter study being called "light-equivalence with earth-fixed visual surround". Surround-mobile motions, in which both the subject and the visual surround can move, are investigated for light-equivalence, this study being called "light-equivalence with mobile visual surround". (An investigation of surround-mobile motions for dark-equivalence would be frivolous since a moving visual surround has no relevance to a subject's motion in the dark.) The third row illustrates the sizes of equivalence classes in each of the three situations. Sample schematic equivalence classes are shown; those for dark-equivalence and light-equivalence of sustained motions are displayed in more detail in Figs 9 and 12, respectively. As shown here, the light-equivalence classes of surround-mobile motions are extensions of the light-equivalence classes of sustained motions. By extending into the full domain of surround-mobile motions, these classes become large. In fact, they are the same size as the dark-equivalence classes, but because they project into the domain of surround-mobile motions, they contain more complicated motions than the dark-equivalence classes.

tion of surround-mobile motions is the desired light-equivalence class.

Justification for this algorithm follows by direct analysis: the dark-equivalence class computed in the second step consists exactly of those sustained motions that would be perceptually equivalent to the present motion if it were to take place in the dark. The next step is to compute the necessary movement of the visual surround to induce the correct optokinetic input, and the exact computation takes place under "for each sustained motion X . . .". The values for $\mathbf{v}_{\mathrm{vs}}^X$ and $\boldsymbol{\omega}_{\mathrm{vs}}^X$ are derived by noting that the original motion has apparent velocities of $\mathbf{v} - \mathbf{v}_{\mathrm{vs}}$ and $\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathrm{vs}}$ detected with vision, and a motion X that is light-equivalent to the original motion must have these same values of apparent velocities. In other words,

$$\mathbf{v}^X - \mathbf{v}_{\mathrm{vs}}^X = \mathbf{v} - \mathbf{v}_{\mathrm{vs}} \text{ and}$$

$$\boldsymbol{\omega}^X - \boldsymbol{\omega}_{\mathrm{vs}}^X = \boldsymbol{\omega} - \boldsymbol{\omega}_{\mathrm{vs}}.$$

The equations for $\mathbf{v}_{\mathrm{vs}}^X$ and $\boldsymbol{\omega}_{\mathrm{vs}}^X$ in the algorithm follow from the equations here.

Examples of light-equivalence with mobile surround

Earth gravity. In order to compare with dark-equivalence and the earlier case of light-equivalence, we return once again to the example of being upright and stationary. As demonstrated previously, this "motion" has no perceptual ambiguity when vision is available and the visual surround is fixed relative to the earth; the light-equivalence class of sustained motions contains exactly one element. In sharp contrast, however, we demonstrate that the light-equivalence class of surround-mobile motions is quite large, and many motions with moving visual surround are perceptually indistinguishable from being upright and stationary.

The algorithm for computing light-equivalence classes leads to computing the relevant dark-equivalence class, and appending appropriate visual surround velocities to the motions. The dark-equivalence class in question at present, that with $\mathbf{A} = \mathbf{k}$ and $\boldsymbol{\alpha} = \mathbf{0}$, has already been computed under the heading "Examples in the dark", and the visual surround velocities necessary here are best described as "entire visual surround moves in the same direction and with the same velocity as the subject". We use the existing summary of the relevant dark-equivalence class to form a description of the present light-equivalence class of surround-mobile motions as follows.

A person who sits apparently upright and stationary with eyes open may actually be:

(i) moving at a non-zero velocity along with a moving visual surround;
(ii) accelerating toward the earth while oriented upside down, possibly at an angle, with the visual surround doing the same;
(iii) upright and rotating, surrounded by a visual reference rotating at the same velocity;
(iv) upright and rotating while also moving up or down, with the visual surround rotating and moving up or down at a matching velocity;
(v) rotating while accelerating downward, upside down, with the visual surround rotating and accelerating downward at a matching acceleration and rotation.

Circular vection. A common example[5] of induced self-motion perception is that of "circular vection", a sensation of rotation when stationary but surrounded by a rotating drum. (Sometimes a rotating pattern is simply projected onto a cylindrical surround.) For this example, we take the case of a drum rotating counterclockwise at 60°/s about a vertical axis through the head, and describe the entire class of surround mobile motions that are light-equivalant to this situation.

The vectors describing the present motion are $\mathbf{v} = \mathbf{0}$, $\mathbf{a} = \mathbf{0}$, $\boldsymbol{\omega} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{a}_g = \mathbf{k}$, $\mathbf{v}_{\mathrm{vs}} = \mathbf{0}$, $\boldsymbol{\omega}_{\mathrm{vs}} = 60\mathbf{k}$ (with °/s the unit of measure for $\boldsymbol{\omega}_{\mathrm{vs}}$), so the algorithm for computing the light-equivalence class here involves the computation of the dark-equivalence class given by $\mathbf{A} = \mathbf{a} + \mathbf{a}_g = \mathbf{k}$ and $\boldsymbol{\alpha} = \mathbf{0}$, which is the same dark-equivalence class used in the "Earth gravity" example above. By determining the necessary visual surround velocities, we obtain the following description of the present light-equivalence class.

A person who sits upright and stationary at the center of a visible drum rotating counterclockwise at 60°/s may actually be:

(i) moving at a non-zero velocity while the surrounding drum moves at the same velocity but also rotates counterclockwise at 60°/s;
(ii) accelerating toward the earth while oriented upside down, possibly at an angle, with the surrounding drum doing the same but also rotating 60°/s about the head's z-axis;
(iii) (a) upright and rotating at a velocity of, say, d°/s, counterclockwise while the surrounding drum rotates at $60 + d$°/s counterclockwise; (b) upright and rotating at a velocity of d°/s clockwise while the surrounding drum rotates at $60 - d$°/s counterclockwise (which gives clockwise drum rotation if $d > 60$°/s);
(iv) rotating under the same circumstances as described in (iii), but with everything (subject and drum) also moving up or down at a constant velocity.

Notice that one motion described under (iiib) is "rotating at 60°/s clockwise while the surrounding drum is stationary (rotates at $60 - 60 = 0$°/s)". During circular vection, self-rotation of this kind is (mis-)perceived.

DISCUSSION

We have identified many classes of subject motion that are perceptually indistinguishable to the subject, under three sets of conditions: no vision, with vision and earth-fixed visual surround, and with vision during possible movement of the visual surround. Under conditions of no vision, there are large "dark-equivalence classes" of sustained motions that are perceptually indistinguishable due to having the same accelerations and forces detected by the vestibular and somatosensory systems. These dark-equivalence classes are found to be infinite in size; some of them include motions of several different kinds, as demonstrated in the striking case of a subject who feels upright and stationary: the subject may actually be moving, accelerating while upside-down, or rotating and possibly moving and/or accelerating while upside-down. However, a dark-equivalence class cannot include all kinds, or categories, of sustained motion—as is evident from Fig. 9, there is a clear division of the dark-equivalence classes into two groups: those that contain only motions without angular acceleration and those that contain only motions with angular acceleration. In contrast to the large dark-equivalence classes, the light-equivalence classes with vision and velocity information available are quite small when the visual surround is earth-fixed. Each such light-equivalence class contains only one or two motions, confirming our intuition that misperception of self-motion is rare under these circumstances. On the other hand, the light-equivalence classes with mobile visual surround are as large as the dark-equivalence classes. Our classification of perceptually indistinguishable motions with mobile surround agrees with experimental results (many of which are reviewed in Dichgans and Brandt[5]) on circular and linear vection, and predicts new classes of perceptually indistinguishable motions.

The main result of the present paper is a classification within which experimentalists, clinicians, modelers, theorists and other scientists can look up the sustained motions that cause equivalent responses from the receptors involved in self-motion perception, in any given situation. The experimentalist, for example, commonly uses devices such as a centrifuge to simulate certain motions, especially motions in aircraft and spacecraft.[7,8] The present classification expands the range of possible simulators by showing a full range of sustained motions that would feel the same as the motion to be simulated. Even for aircraft and spacecraft motions or pathological conditions that are impossible to simulate, the classification shows the motions of a healthy earth-bound subject that are perceptually equivalent to the unusual conditions, possibly predicting the subject's interpretation and report of self-motion under the circumstances. In addition, the techniques developed here can be used to investigate perception when a subject is adapted to a new environment such as micro-gravity. Astronauts adapted to zero-G misperceive self-motion upon return to the earth, one such misperception being the misinterpretation of roll as translation.[15] The unified approach presented here can be used to determine the new "space-adapted" equivalence classes that cause misperception upon return to a one-G environment.

Most other theoretical research on self-motion perception has consisted of modeling studies.[1,14,17,19] These studies usually consider an input consisting of direct stimulus to some or all of the vestibular endorgans and sometimes input to the visual and/or other systems, then develop a model based upon engineering techniques that produces an output indicating the motion perceived by a subject during such input. Our theoretical research, which takes a different approach, relates to modeling studies in various ways. First, while modeling studies have often concentrated on resolving sensory conflict between inputs to the different sensory systems, we have shown that even when there is no sensory conflict, there can still be a large degree of perceptual ambiguity between the actual motions that a subject may experience. The present research demonstrates the need to separate the idea of "stimulus" into its two parts: actual motion of the subject (upward, downward, rotating, etc.) and direct stimulus to the sensors (acceleration on vestibular hairs, etc.). Two subjects with the same "sensory stimuli" may have completely different "motion stimuli". Another point is that, in contrast with modeling studies, we do not state which motion a subject will report during the perception. Rather, we leave this question open, simply listing all possible motions that could cause a given sensation. Computer models typically choose the simplest motion possible, a choice that has strong foundations in experimental research. We generally agree with this choice, but list the entire range of possibilities in order to keep the view that cognitive influence and individual differences may affect the choice of perceived motion. A subject's perception could even be "I don't know what my motion is. I'm disoriented."

The role of cognition in self-motion perception should be addressed in relation to the equivalence classes presented here. One could argue that two motions within the same equivalence class could be distinguished by prior knowledge of the different motions. However, our claim is that two motions cannot be distinguished when placed within the same sensory context. To illustrate this point, we imagine that a bellhop turns the lights off on an elevator at the 20th floor of a 40-story building, and explains to a guest that the elevator will undergo a few random accelerations before proceeding upward at constant velocity. Once the elevator reaches constant velocity and the guest's perceptual transients have died away, the guest can report correctly that the elevator is moving upward. Another time, the bellhop explains that the elevator's final motion will be downward; then after constant velocity is reached, the guest can report correctly that the elevator is moving downward. However, if the bellhop performs this same experiment by telling the guest that the elevator's final motion will be upward when, in fact, the elevator's final motion is downward, then the guest will just as easily report a final motion of upward, even though it is incorrect. The point is that the two constant-velocity motions of "upward" and "downward" are still perceptually indistinguishable in the given sensory context. In fact, subjects often have an inaccurate perception of self-motion despite correct information, such as when a pilot disbelieves flight instruments.[16]

A notable shortcoming in the present dark-equivalence classification is that fast spins may actually be distinguishable from other motions in their dark-equivalence classes. The indistinguishability of dark-equivalent motions relies upon the approximation that linear acceleration is the same at all points on the subject's body. This equality of linear acceleration across the body is true during linear motion, and is a good approximation if angular velocity is not too large. However, linear acceleration detectors such as the otolith organs can theoretically detect differences in linear acceleration across the head, especially during fast on-axis spins. Proprioception of the limbs is an even more likely candidate for detecting outward centrifugal forces. One solution to this problem would be to set up an equivalence-class scheme in which accelerations of all points on the body are specified. Such a solution would be the most rigorous, but a more practical solution is to simply take fast spins out of their current dark-equivalence classes and put them into their own, new equivalence classes. The current dark-equivalence classification is then used in the usual way except that, instead of ignoring fast spins, we acknowledge their existence by giving them their own equivalence classes.

A subtle issue arises in connection with detection of angular velocity in the dark. Although the issue does not affect the present results, it has implications for the companion paper,[12] so we address it here. There are two slightly different types of equivalence involved in the topic of spin detection in the dark. The main type of equivalence is that of equivalence due to the laws of physics. For example, the laws of physics state that the attitude vector and the linear acceleration vector cannot be distinguished by any accelerometer such as the inner ear. Most of the perceptual equivalence investigated in the present paper is due to the laws of physics. However, the laws of physics do not preclude the detection of angular velocity by the comparison of linear acceleration at different points on the body. When two different angular velocities are not distinguished by a subject in the dark, the indistinguishability is due to attributes of the sensors or the nervous system. Even during slow on-axis rotation, the laws of physics show that the right and left otolith organs have oppositely-directed accelerations imposed upon them; however, the vestibular system apparently does not pick up the difference, or the nervous system does not make use of this difference. A quick calculation, using a y-axis linear acceleration perception threshold of $0.2\ \mathrm{m/s^2}$ (a rough minimum from various studies of constant acceleration along the y-axis),[7] and noting that the left and right otolith organs are approximately 0.07 m apart,[4] shows that an on-axis spin would have to exceed $130°/\mathrm{s}$ before the otolith organs would be exposed to linear accelerations reaching the threshold of linear acceleration perception.

The dark- and light-equivalence classes have succeeded in capturing all perceptual indistinguishabilities due to equality of physical stimuli between sustained motions, plus certain other indistinguishabilities due to the existence of perceptual thresholds as explained above. Other perceptual thresholds, such as for angular acceleration, can lead to additional ambiguities that do not fall into equivalence classes. Confusion can also be created by inadequate sensory information such as from less-than-adequate vision. Nevertheless, the present classes demonstrate the fundamental perceptual equivalence between motions due to the necessary nature of self-motion perception by a perceiving object or organism; any additional ambiguity lies on top of this foundation.

For the present light-equivalence classification, particular assumptions have been made about the visual input. We have assumed that vision gives all information about linear and angular velocities but no information about the direction of "downward". At high velocities, however, the visual system presumably cannot distinguish between different speeds; this is an example in which additional perceptual indistinguishability lies on top of the foundation of the light-equivalence classes. Alternative assumptions about the visual input would also be interesting to investigate. For example, a pilot flying at high altitude on a clear day may not receive many visual linear velocity cues, but may obtain angular velocity information visually and a good sense of "downward". What are the perceptual ambiguities under these circumstances? In this situation and others, a good visual estimation of distance is not available, affecting velocity information. Although there is not enough space in one paper to pursue all sensory situations, the techniques developed here can be applied toward an investigation of perceptual ambiguity under other desired assumptions about the input. For example, the pilot situation described could be pursued by setting up a "high-pilot" equivalence relation focusing on equality of resultant accelerations, angular accelerations, angular velocities and attitude vectors. From there, the question could be pursued in the same way as was light-equivalence.

Although we focus here on sustained motions, it would be possible to extend the definitions of equivalence to include other motions as well. There are, for example, physical motions that are not sustained according to our definition, but that have constant resultant acceleration and constant angular acceleration. Such motions would feel sustained in the dark.

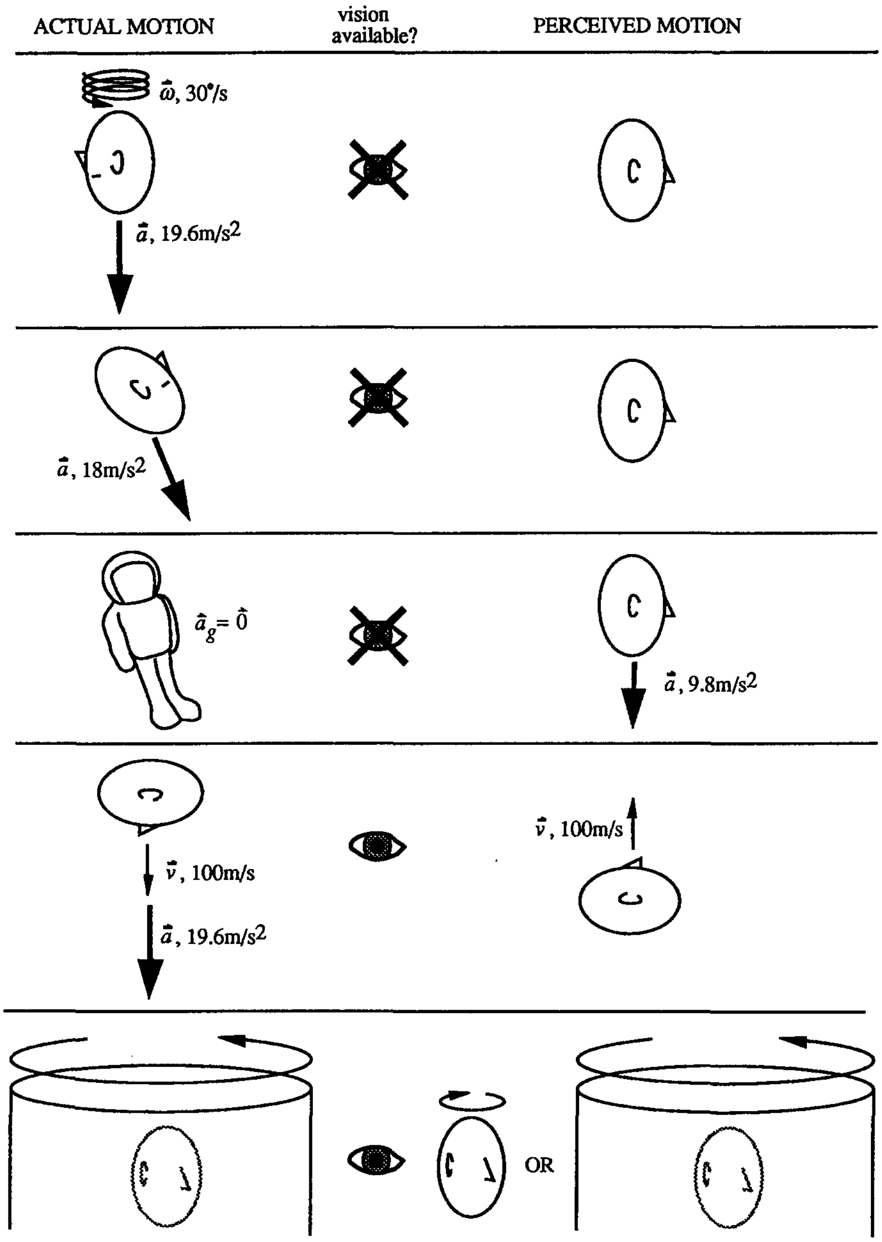

Fig. 15. Motions and the resulting perceptions of self-motion (or non-motion) if the subject reports a sensation corresponding to the simplest motion in the perceptual equivalence class of the actual motion. See text for written descriptions of the motions.

By extending the definition of "dark-equivalence" to include motions like these, even larger dark-equivalence classes would be obtained, showing that there are additional physical motions that are perceptually indistinguishable from the sustained motions classified here. However, descriptions of these motions can be quite complicated, and they do not fall into the nine categories listed in this paper. Their analysis involves more complex mathematical techniques such as the use of quaternions; such motions are a topic of current research.

CONCLUSIONS

The main results of the present research have their strength in the comparison of many different kinds of motions, giving sometimes large and varied classes of perceptually indistinguishable motions. We venture to guess that in many cases, a subject "perceives" a simple, common motion contained in the same equivalence class as the subject's actual motion, even if the actual motion is complicated. With this guess (the same guess that is implicit in modeling studies), we can immediately make several predictions, illustrated in Fig. 15, and based upon the examples computed in the paper: an upside-down subject in the dark rotating at 30°/s and accelerating with twice the acceleration of gravity straight toward the earth may have the simple sensation of being upright and stationary. A subject with occluded vision oriented 45° from upside-down and accelerating at 18 m/s^2 at an angle 22.5° from straight downward may have a sensation of being upright and stationary. (Note: in an aircraft, the sound of the engines may introduce a cognitive factor indicating forward velocity, in which case the subject may perceive a different simple motion in the same dark-equivalence class: that of being upright and traveling at some constant forward velocity.) An astronaut first experiencing zero-gravity may have the sensation of falling. With vision but no view of the ground, a subject facing downward and traveling at 100 m/s straight toward the earth at twice the acceleration of gravity may have the sensation of traveling face-upward at constant velocity of 100 m/s away from the earth. An upright subject with eyes open and surrounded by a vertical drum rotating at 60°/s may have the sensation of rotating while the drum is stationary, or may actually perceive that the drum is rotating. Both subject motion and drum motion are relatively simple, although movement of the entire visual surround as a whole is not common; a subject may have a tendency toward sensation of self-motion, or may be influenced by prior knowledge of the experimental conditions. These are just a few examples of the types of predictions that can be made by consulting the dark- and light-equivalence classifications.

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APPENDIX

Supplement to Table 2

A One-G type dark-equivalence class ($|\mathbf{A}| = 1$ g-unit, $\boldsymbol{\alpha} = \mathbf{0}$) includes motions from Category 3 (linear acceleration). Since $\mathbf{a} + \mathbf{a}_g$ must equal $\mathbf{A}$, and $|\mathbf{A}| = |\mathbf{a}_g| = 1$ g-unit, those motions in Category 3 fitting the following description belong to the dark-equivalence class: earth-upward, and therefore, $\mathbf{a}_g$, can be any direction relative to the head except the exact direction of $\mathbf{A}$. The acceleration $\mathbf{a}$ must have a non-zero earth-downward component, and if it deviates an angle θ ($0° \leq \theta < 90°$) from straight down, $\mathbf{a}$ has magnitude given by the Law of Sines (or shown pictorially in Fig. 7):

$$\frac{|\mathbf{a}|}{\sin(180° - 2\theta)} = \frac{1 \text{ g-unit}}{\sin\theta};$$

therefore,

$$|\mathbf{a}| = \frac{\sin(180° - 2\theta)}{\sin\theta}.$$

A Hyper-G type dark-equivalence class ($|\mathbf{A}| > 1$ g-unit, $\boldsymbol{\alpha} = \mathbf{0}$) includes motions from Category 4 (angular velocity). Rotations at a non-zero radius r, given in meters, with angular velocity magnitude ω (about an earth-vertical axis) belong to the dark-equivalence class if $(\omega\pi/180)^2 r = 9.8\sqrt{(|\mathbf{A}|^2 - 1)}$ [where $(\omega\pi/180)^2 r$ is the magnitude of the centripetal acceleration $\mathbf{a}$, and 1 g-unit is taken to be 9.8 m/s^2], and if the head is oriented in such a way that $\mathbf{a} + \mathbf{a}_g$ is in the direction of $\mathbf{A}$. Since $\mathbf{a}_g$ and $\mathbf{a}$ are orthogonal here, this implies $\mathbf{a}_g + \mathbf{a} = \mathbf{A}$. Figure 8 displays the requirements pictorially.

Supplement to Table 3

A One-G Moving type light-equivalence class ($|\mathbf{A}| = 1$ g-unit, $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, $\boldsymbol{\omega} = \mathbf{0}$) with earth-fixed surround contains motions of Category 2 (linear velocity) and/or Category 3 (linear acceleration) depending on the relative directions of $\mathbf{v}$ and $\mathbf{A}$. There are two cases to consider (as shown in Fig. 11A and B).

Case $\mathbf{v} \perp \mathbf{A}$. The light-equivalence class consists of exactly one element, the member of Category 2 with $\mathbf{a}_g = \mathbf{A}$ and linear velocity $\mathbf{v}$, because $\mathbf{a}$ cannot be non-zero here while being parallel to $\mathbf{v}$.

Case $\mathbf{v} \not\perp \mathbf{A}$. The light-equivalence class consists of two elements, one being the member of Category 2 with $\mathbf{a}_g = \mathbf{A}$ and linear velocity $\mathbf{v}$. The other element is the member of Category 3 with $\mathbf{a}_g$ being the unique unit vector equal to $\mathbf{A} - c\mathbf{v}$ for some non-zero real number c, linear acceleration $\mathbf{a}$ then given by $c\mathbf{v}$, and linear velocity given by $\mathbf{v}$. This is the only way for $\mathbf{a}$ to be non-zero and parallel to $\mathbf{v}$. The mathematical fact that the vector $\mathbf{a}_g$ as described is unique is used here.

A Hyper-G Curl Type light-equivalence class ($|\mathbf{A}| > 1$ g-unit, $\boldsymbol{\alpha} = \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, $\boldsymbol{\omega} \neq \mathbf{0}$) with earth-fixed surround contains motions of either Category 4 (angular velocity), Category 5 (angular and linear velocity) or Category 6 (angular velocity and linear acceleration) depending on the relative directions of $\mathbf{v}$ and $\boldsymbol{\omega}$. Additional restrictions on the vectors for this type of class also depend on the relative directions of $\mathbf{v}$ and $\boldsymbol{\omega}$, as described below. There are three cases to consider.

Case $\mathbf{v} \parallel \boldsymbol{\omega}$. In this case, $\mathbf{A}$ is required to be parallel to both $\mathbf{v}$ and $\boldsymbol{\omega}$. The light-equivalence class consists of the two members of Category 6 with $\mathbf{a}_g$ parallel to $\mathbf{A}$. The corresponding values for (earth-vertical) $\mathbf{a}$ are $\mathbf{A} - \mathbf{a}_g$, and the linear and angular velocities are $\mathbf{v}$ and $\boldsymbol{\omega}$, respectively.

Case $\mathbf{v} \perp \boldsymbol{\omega}$. There are two requirements in this case, the first being that the earth-vertical component of $\mathbf{A}$ has magnitude 1 g-unit, since the earth-vertical component is just $\mathbf{a}_g$ in this case. The vector $\boldsymbol{\omega}$ is necessarily earth-vertical, so we use it to test the earth-vertical component of $\mathbf{A}$, and require

$$\frac{\mathbf{A} \cdot \boldsymbol{\omega}}{|\boldsymbol{\omega}|} = \pm 1.$$

Then the light-equivalence class consists of one member of Category 4 as long as the properties of rotation at a non-zero radius are also satisfied. In particular, by computing the radius of rotation,

$$r = \frac{|\mathbf{v}|}{|\boldsymbol{\omega}|}\left(\frac{180}{\pi}\right) \text{ meters,}$$

we determine that the centripetal acceleration corresponding to the given $\mathbf{v}$ and $\boldsymbol{\omega}$ must have magnitude

$$|\mathbf{v}\|\boldsymbol{\omega}|\left(\frac{\pi}{180}\right)\left(\frac{1}{9.8}\right) \text{ g-units} \tag{A1}$$

(taking 1 g-unit to be 9.8 m/s^2). Now, the vector $\mathbf{A}$ is the vector sum of $\mathbf{a}_g$ and the centripetal acceleration, so by letting

$$\mathbf{a}_g = \frac{\mathbf{A} \cdot \boldsymbol{\omega}}{|\boldsymbol{\omega}|^2}\boldsymbol{\omega} \tag{A2}$$

(this being the only possible value for $\mathbf{a}_g$ here), the second vector requirement is that $\mathbf{A} - \mathbf{a}_g$ has magnitude given by equation (A1) and be in the direction of $\boldsymbol{\omega} \times \mathbf{v}$. Then the light-equivalence class consists of the member of Category 4 that has $\mathbf{a}_g$ as given by equation (A2), $\mathbf{a} = \mathbf{A} - \mathbf{a}_g$, linear velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$.

Case $\mathbf{v} \not\perp \boldsymbol{\omega}$ *and* $\mathbf{v} \nparallel \boldsymbol{\omega}$. There are two requirements in this case, the first being that

$$\frac{\mathbf{A} \cdot \boldsymbol{\omega}}{|\boldsymbol{\omega}|} = \pm 1.$$

The light-equivalence class consists of one member of Category 5 as long as the properties of rotation at a non-zero radius with an additional, earth-vertical, component of linear velocity are satisfied. Let

$$\mathbf{v}_{\text{vert}} = \frac{\mathbf{v} \cdot \boldsymbol{\omega}}{|\boldsymbol{\omega}|^2}\boldsymbol{\omega} \text{ and } \mathbf{v}_{\text{tang}} = \mathbf{v} - \mathbf{v}_{\text{vert}}.$$

By computing the radius of rotation,

$$r = \frac{|\mathbf{v}_{\text{tang}}|}{|\boldsymbol{\omega}|}\left(\frac{180}{\pi}\right) \text{ meters,}$$

we determine that the centripetal acceleration corresponding to the given $\mathbf{v}$ and $\boldsymbol{\omega}$ must have magnitude

$$|\mathbf{v}_{\text{tang}}||\boldsymbol{\omega}|\left(\frac{\pi}{180}\right)\left(\frac{1}{9.8}\right) \text{ g-units} \tag{A3}$$

(taking 1 g-unit to be 9.8 m/s^2). The vector $\mathbf{A}$ is the vector sum of $\mathbf{a}_g$ and the centripetal acceleration, so by letting

$$\mathbf{a}_g = \frac{\mathbf{A}\cdot\boldsymbol{\omega}}{|\boldsymbol{\omega}|^2}\boldsymbol{\omega} \tag{A4}$$

(this being the only possible value for $\mathbf{a}_g$ here), the second vector requirement is that $\mathbf{A} - \mathbf{a}_g$ have magnitude given by equation (A3) and be in the direction of $\boldsymbol{\omega} \times \mathbf{v}_{\text{tang}}$. Then the light-equivalence class consists of the member of Category 5 that has $\mathbf{a}_g$ as given by equation (A4), $\mathbf{a} = \mathbf{A} - \mathbf{a}_g$, linear velocity $\mathbf{v}$ and angular acceleration $\boldsymbol{\omega}$.