Quantization of maximally charged slowly moving black holes

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I. INTRODUCTION

The study of moduli space of a system of maximally charged black holes has recently attracted a lot of attention [1]. It was first discussed by Ferrell and Eardley [2] in four spacetime dimensions. It was subsequently extended to five dimensions [1]. In the near-horizon limit, an $SL(2,\mathbb{R})$ conformal symmetry was discovered that generalized the case of two black holes. Consequently, the general system inherited the pathologies of the system of two black holes: the Hamiltonian possessed no well-defined vacuum state.

This problem was studied a long time ago by de Alfaro, Fubini and Furlan (DFF) [3]. The simplest quantum mechanical system with conformal symmetry is described by the Hamiltonian

$$H = \frac{p^2}{2} + \frac{g}{2\chi^2}. \quad (1)$$

This Hamiltonian represents a single-particle rational Calogero-Moser system, which is equivalent to a free particle system [4]. It possesses a continuous spectrum down to zero energy and there is no well-defined ground state. DFF suggested a solution to this problem. They proposed the redefinition of the Hamiltonian by the addition of a harmonic oscillator potential which is also the generator of special conformal transformations:

$$K = \frac{\chi^2}{2}. \quad (2)$$

The new Hamiltonian is defined by

$$H' = \frac{1}{\omega} (H + \omega^2 K) \quad (3)$$

where we introduced a scale parameter $\omega$ (infrared cutoff). $H'$ has a well-defined vacuum and a discrete spectrum, which can actually be computed exactly.

$$E_n = 2n + 1 + \sqrt{g + 1}/4. \quad (4)$$

Notice that the spectrum is independent of the arbitrary scale parameter $\omega$. The supersymmetric case can be dealt with in the same way.

In the case of two slowly moving maximally charged black holes [5–9], the near-horizon geometry is $AdS_2 \times S^2$. The isometries of the $AdS_2$ space are conformal symmetries. As a result, the moduli (spatial distance between the two black holes) has dynamics governed by (super)conformal non-relativistic quantum mechanics of the form discussed above. The DFF redefinition of the Hamiltonian has a nice interpretation in this case as a redefinition of the time coordinate. The DFF Hamiltonian corresponds to a globally defined time coordinate whereas the conformally invariant definition does not. Thus, the DFF trick appears plausible on physical grounds.

The multiple black hole moduli space possesses similar properties [1]. There is a conformal $SL(2,\mathbb{R})$ symmetry and the quantization of the system leads to a Hamiltonian with no well-defined ground state. The DFF trick of replacing the Hamiltonian $H$ by $H + K$, where $K$ is the generator of special conformal transformations, is applicable in this general case. However, it lacks an obvious physical interpretation in terms of a redefinition of the time coordinate.

In Ref. [10], we presented an alternative derivation of the DFF procedure in the case of a particle moving in the background of an extreme Reissner-Nordström black hole. This is equivalent to a system of two black holes. We showed that the redefinition of the Hamiltonian amounted to a different choice of gauge. In the conformally invariant case, we identified an obstruction to the standard gauge-fixing procedure that led to a modification of the usual quantization rules. This obstruction came from the boundary of spacetime and was rooted in the fact that the time coordinate was not defined at the boundary. On the other hand, there was no obstruction in the choice of gauge leading to the DFF Hamiltonian. We
concluded that the DFF Hamiltonian corresponded to a good gauge choice, whereas the conformally invariant Hamiltonian did not. Our discussion was based on the standard Faddeev-Popov quantization procedure and was therefore applicable to more general systems, as long as the system had an underlying gauge invariance.

Here, we extend the procedure discussed in [10] to the case of multiple black hole scattering [1]. We show how the gauge can be fixed systematically without encountering obstructions from the singularities of moduli space. The resultant Hamiltonian is modified by the addition of the potential prescribed by the DFF trick. Thus, we show that the DFF trick is a consequence of a standard gauge-fixing procedure in the case of multiple black hole scattering.

Our discussion is organized as follows. In Sec. II, we apply the Faddeev-Popov procedure to a particle moving in a fixed background of curved spacetime as well as an external electromagnetic field. We also show how the procedure is equivalent to the commutation rules one obtains from Dirac brackets. We discuss a subtlety that arises when spacetime possesses boundaries. In Sec. III, we specialize to the case of an extreme Reissner-Nordström black hole in five spacetime dimensions. We show that the DFF trick is equivalent to a choice of gauge. In Sec. IV, we discuss multiple black hole scattering and show how the system of black holes can be quantized leading to a modification of the Hamiltonian in the manner of DFF. In Sec. V, we discuss the case of four spacetime dimensions. Finally, in Sec. VI, we summarize our conclusions.

II. CHARGED PARTICLE IN CURVED SPACETIME

In this section, we discuss the quantization of a charged particle moving in a fixed spacetime background and electromagnetic field. We introduce the path integral in curved spacetime and apply the Faddeev-Popov procedure to fix the gauge. We also show that this is equivalent to the canonical quantization through commutation relations coming from Dirac brackets. We discuss a subtlety that arises in the quantization procedure when spacetime has boundaries. This is a review of Ref. [10]. Consider a particle of mass $m$ and charge $q$ moving along a trajectory described by coordinates $x^\mu(\tau)$ ($\mu = 0,1,\ldots,D-1$) where $\tau$ is the proper time of the particle. The action is

$$S = \int d\tau \left[ L - \frac{1}{2} \dot{x}^\mu \dot{x}_\mu - \frac{1}{2} m \dot{x}^2 + q A_\mu \dot{x}^\mu \right]$$

where we raise and lower indices with the background metric $g_{\mu\nu}$. $A_\mu$ is the background electromagnetic vector potential. Varying $\eta$, we obtain the constraint

$$\eta^2 = -\dot{x}^\mu \dot{x}_\mu / m^2.$$  \hspace{1cm} (6)

The Hamiltonian is

$$H = \dot{x}^\mu P_\mu - L = m \eta \chi, \quad \chi = \frac{1}{2m} \pi_\mu \pi^\mu + \frac{1}{2} m,$$

$$\pi_\mu = P_\mu - q A_\mu.$$  \hspace{1cm} (8)

In the canonical formalism, the action reads

$$S = \int d\tau (\dot{x}^\mu P_\mu - m \eta \chi).$$  \hspace{1cm} (9)

Therefore, $\eta$ is a Lagrange multiplier enforcing the constraint

$$\chi = \frac{1}{2m} \pi_\mu \pi^\mu + \frac{1}{2} m = 0$$  \hspace{1cm} (10)

which is the mass-shell condition in the presence of an external vector potential.

The orbits of the gauge transformations ($\tau$ reparametrizations) are the trajectories of the equations of motion (Lorentz force law in curved spacetime)

$$\ddot{x}^\mu = \frac{1}{m} \pi^\mu, \quad \pi_\mu = \frac{1}{m} \Gamma_{\nu\lambda\mu} \pi^\nu \pi^\lambda - \frac{q}{m} F_{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$  \hspace{1cm} (11)

where $\Gamma_{\nu\lambda\mu}$ are the Christoffel symbols,

$$\Gamma_{\nu\lambda\mu} = \frac{1}{2} \left( \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\lambda\mu} + \partial_\mu g_{\lambda\nu} \right)$$  \hspace{1cm} (12)

or purely in terms of the coordinates $x^\mu$,

$$\ddot{x}^\mu + \Gamma_{\nu\lambda\mu} \dot{x}^\nu \dot{x}^\lambda = \frac{q}{m} \dot{x}^\nu F_{\mu\nu}.$$  \hspace{1cm} (13)

To quantize the system, consider the path integral,

$$Z = \mathcal{N} \int D\chi D\pi e^{iS} = \mathcal{N} \int D\chi D\pi \delta(\chi) e^{ij\pi \chi} \pi_\mu.$$  \hspace{1cm} (14)

To define it, we need to fix the gauge by imposing the gauge-fixing condition

$$h(x^\mu) = \tau$$  \hspace{1cm} (15)

which defines a hyper-surface that cuts each orbit precisely once. Physically, this amounts to choosing $h(x^\mu)$ as the time coordinate. Then its conjugate momentum, $\mathcal{H}$, is the Hamiltonian of the reduced system. Following the standard Faddeev-Popov procedure, we insert

$$1 = \text{det}(h, \chi) \int D\epsilon \delta(h - \{h, \chi\}) \epsilon - \tau$$  \hspace{1cm} (16)

where $\{ , \}$ denotes Poisson brackets, into the path integral and perform a reparametrization to obtain
$Z = N \int D\xi D\phi \det\{h, \chi\} \delta(h - \tau) \delta(\chi) e^{iD\xi \phi \mu}$. (17)

We may integrate over the $\delta$-functions to reduce the dimension of phase space. The reduced system will be described by coordinates $\bar{x}^i$ and conjugate momenta $\bar{P}_i$. The Faddeev-Popov determinant is canceled by the integration over $\delta(\chi)$. The momentum conjugate to $h$ (which is identified with time) plays the role of the Hamiltonian $\mathcal{H}$ of the reduced system. The path integral becomes

$$Z = N \int D\bar{x} D\bar{P} e^{iD\xi \bar{x}^i \bar{P}_i - \mathcal{H}}. (18)$$

Equivalently, we may quantize the system in the operator formalism. To this end, we need to calculate Dirac brackets,

$$\{A, B\}_D = \{A, B\} - \{A, \chi_i\} \{\chi_i, B\}^{-1} \{\chi_j, B\} (19)$$

where $i, j = 1, 2$, $\chi_1 = \chi$, $\chi_2 = h$, and promote them to commutators.

As an example, consider the special case $h(x^\mu) = x^0$ (i.e., identify $x^0$ with time), and set $q = 0$. The reduced system is described by the coordinates $\bar{x}^i = x^i$ and the Hamiltonian is

$$\mathcal{H} = -P_0 = \sqrt{P_1^2 + m^2}. (20)$$

The commutation relations we obtain from the Dirac brackets are

$$[P_i, x^j] = -i \delta^i_j, \quad [\mathcal{H}, x^i] = -i \frac{P_i}{\mathcal{H}} (21)$$

which are appropriate for $\mathcal{H}$ given by Eq. (20).

Having set up the quantization procedure, we now wish to discuss a subtlety which arises when the spacetime possesses boundaries [11]. Let us follow the Faddeev-Popov gauge-fixing procedure a little more carefully. We need to insert Eq. (16) into the path integral and then perform an inverse gauge transformation to eliminate the gauge parameter. In doing so, we encounter an obstruction at the boundary of spacetime. Under a gauge transformation, the change in the action is

$$\delta S = \int d\tau \frac{d}{d\tau} (\delta x^\mu P_\mu) - \int d\tau \epsilon \dot{\chi}. (22)$$

Since $\dot{\chi} = \{x^\mu, \chi\} = 0$, we conclude that the action changes by a total derivative,

$$\delta S = \int d\tau \frac{d}{d\tau} (\delta x^\mu P_\mu). (23)$$

We have

$$\delta x^\mu = \{x^\mu, \chi\} \epsilon = \frac{\partial \chi}{\partial P_\mu} \epsilon (24)$$

therefore,

$$\delta S = P_\mu \frac{\partial \chi}{\partial P_\mu} \epsilon \bigg|_{\tau = 0}. (25)$$

Notice that, if the generator of gauge transformations, $\chi$, is quadratic in the momenta (as in the neutral particle case), then the boundary contribution vanishes after imposing the constraint $\chi = 0$. In general, $\chi$ [Eq. (10)] is not quadratic in the momenta, due to the presence of the vector potential. Therefore $\delta S \neq 0$ and we cannot in general get rid of the gauge parameter on the boundary of spacetime. Thus, we obtain a boundary contribution to the path integral,

$$\int d\tau d^4x d^4p \{h, \chi\} \delta(h - \tau) \delta(\chi) \exp \left(i P_\mu \frac{\partial \chi}{\partial P_\mu} \epsilon \right). (26)$$

This obstruction is absent when at the boundary,

$$\{h, \chi\} \big|_{\tau = 0} = 0. (27)$$

Physically, this condition implies that the boundary of spacetime is invariant under transformations generated by $h$, which is the time coordinate after gauge-fixing ($h = \tau$). In other words, the boundary is fixed under time translations. Thus the time coordinate $x^0$ is not a good global coordinate and leads to an obstruction in the gauge invariance of the theory. Integrating over the gauge parameter, we obtain an additional constraint at the boundary,

$$P_\mu \frac{\partial \chi}{\partial P_\mu} \bigg|_{\tau = 0} = 0. (28)$$

This alters the standard commutation relations and the eigenvalue problem for the Hamiltonian. We have not carried out an explicit computation. This would involve the introduction of a regulator which would break gauge invariance. Nevertheless, the resulting system should be equivalent to the one obtained through other choices of gauge due to the gauge invariance of the theory.

To summarize, the identification of the time coordinate (15) in general leads to an obstruction in the gauge-fixing procedure for the path integral. If this obstruction is accounted for by an appropriate modification of the commutation relations, this choice of the time coordinate leads to a well-defined Hamiltonian problem.

III. EXTREME REISSNER-NORDSTRÖM BLACK HOLE

We are now ready to discuss the quantization of a particle moving near an extreme Reissner-Nordström black hole [5–9]. In this section, we will discuss five spacetime dimensions. The results in the four-dimensional case are similar and will be taken up in Sec. V.

Consider a maximally charged black hole sitting at the origin of spacetime. We shall work with units in which Newton’s constant $G = 1$. Then

$$M = Q. (29)$$
for this black hole. The particle moving near the black hole will also be taken as maximally charged, so

\[ m = q. \tag{30} \]

The black hole creates a metric

\[ ds^2 = -\frac{1}{\psi^2} dt^2 + \psi^2 dx^2, \quad \psi = 1 + \frac{M}{x^2} \tag{31} \]

and a vector potential

\[ A_0 = \frac{1}{\psi}, \quad \vec{A} = 0 \tag{32} \]

where the vectors live in a four-dimensional Euclidean space. Near the horizon, \( \psi = M/x^2 \). Using polar coordinates and switching variables to \( \psi \), we obtain

\[ ds^2 = -\frac{1}{\psi^2} d\tau^2 + \psi^2 d\Omega^2 \tag{33} \]

Defining

\[ x^\pm = \tau \pm \sqrt{\frac{M}{2} \psi} \tag{34} \]

the metric becomes

\[ ds^2 = -\frac{1}{\psi^2} d\tau^2 + \psi^2 dx^+ dx^- + M d\Omega^2 \tag{35} \]

and

\[ \psi = \frac{x^+ - x^-}{\sqrt{M}}. \tag{36} \]

The vector potential has non-vanishing components

\[ A_+ = A_- = \frac{1}{2} \frac{c}{\psi}. \tag{37} \]

Thus, spacetime factorizes into a product AdS$_2 \times$S$^1$. Henceforth, we shall work with AdS$_2$. The only non-vanishing connection coefficients are \( \Gamma_{\pm \pm \pm} = \partial_\pm \ln |g_{\pm \pm}| \). Therefore, the geodesic equations for \( x^\pm \) are (setting \( m = q \) [Eq. (13)])

\[ \ddot{x}^\pm (\ln |g_{\pm \pm}|)'(\dot{x}^\mp)^2 = \pm \ddot{x}^\mp F_{\pm \pm} \tag{38} \]

where \( A = A_+ = A_- \), and \( (\ln |g_{\pm \pm}|)' = \partial_\pm \ln |g_{\pm \pm}| \)

\[ = - \partial_\pm \ln |g_{\pm \pm}|. \]

Using \( \psi (dA/d\psi) = -A \), \( \psi (dg_{\pm \pm}/d\psi) = -2g_{\pm \pm}, F_{\pm \pm} = 2 \partial_\pm A \), it is straightforward to show that the following quantities are gauge-invariant (constant along geodesics):

\[ H = -P_+ - P_-, \quad D = 2x^+ P_+ + 2x^- P_-, \]

\[ K = -(x^+)^2 P_+ - (x^-)^2 P_+ + \frac{1}{2} m M \psi. \tag{39} \]

They obey an \( SL(2, \mathbb{R}) \) algebra

\[ \{H, D\} = -2H, \quad \{H, K\} = -D, \quad \{K, D\} = 2K \tag{40} \]

reflecting the symmetry of the AdS$_2$ spacetime. \( H, D \), and \( K \) generate time translations, dilatations, and special conformal transformations, respectively. The brackets may be Poisson or Dirac, so this is also an algebra of the gauge-fixed system, as expected. The constraint (generator of gauge transformations) \( \chi = (1/2m) \pi_\mu \pi^\mu + \frac{1}{2} m = 0 \) reads

\[ 2m \chi = -\psi^2 P_+ P_+ + \frac{1}{2} m \psi (P_+ P_-) + \frac{L^2}{M} = 0 \tag{41} \]

where \( L^2 = \hat{r}^2 P_i P_i \) is the square of the angular momentum operator. The simplest gauge-fixing condition to impose is

\[ h(x^+, x^-) = \frac{1}{2} (x^+ + x^-) = \tau. \tag{42} \]

In this case, the Hamiltonian is

\[ H = -P_+ - P_- \tag{43} \]

Using the constraint (41), we obtain

\[ H = \frac{1}{\psi} (-m + \sqrt{m^2 + 4(\psi^2 P_+^2 + L^2)/M}). \tag{44} \]

The other two operators in the \( SL(2, \mathbb{R}) \) algebra can be written as

\[ D = -2\tau H + 2 \psi P_\psi, \]

\[ K = -\tau^2 H - \tau D + \frac{1}{4} m \psi^2 H + \frac{1}{2} m M \psi. \tag{45} \]

In the non-relativistic limit and for large \( \psi \) (near the horizon),

\[ H = \frac{2 \psi P_\psi}{m \psi}, \quad D = 2 \psi P_\psi, \quad K = \frac{1}{2} m M \psi. \tag{46} \]

where we also used the fact that these are conserved quantities to set \( \tau = 0 \). Switching back to \( \tilde{x} \), we may write these operators in terms of the coordinate \( \tilde{x} \) and its conjugate momentum \( \tilde{P} \) as

\[ H = \frac{\tilde{x}^2 \tilde{P}^2}{2m \tilde{P}^2}, \quad D = -\tilde{x} \cdot \tilde{P}, \quad K = \frac{m \tilde{M}^2}{2 \tilde{x}^2}. \tag{47} \]

In terms of a variable \( u \), defined by

\[ \psi = \frac{u^2}{M} \tag{48} \]

we obtain a simple representation,
where \( P \) is the momentum conjugate to \( u \). This system does not have a well-defined vacuum. The question then arises whether the underlying theory is inherently sick. One may apply the DFF trick to produce a Hamiltonian system with a well-defined ground state. The DFF trick can be understood in this case as a different choice of time coordinate leading to a different Hamiltonian. From our point of view, any two choices of time coordinates should be equivalent to each other, for they merely correspond to different gauge choices. Since the underlying theory is gauge-invariant, all gauge choices should be equivalent to each other.

Before we discuss the vacuum problem in conjunction with the gauge-fixing procedure, we shall introduce a class of gauges that lead to a Hamiltonian system with a well-defined ground state. Let the gauge-fixing condition be

\[
  h(x^+, x^-) = \arctan \left( \frac{\omega x^+ + \omega x^-}{1 - \omega^2 x^+ x^-} \right) = \tau
\]

where \( \omega \) is an arbitrary scale. Differentiating with respect to \( \tau \), we obtain

\[
  \partial_+ h \dot{x}^+ + \partial_- h \dot{x}^-= 1, \quad \partial_+ h = \frac{\omega}{1 + \omega^2 (x^+)^2}.
\]

To find the Hamiltonian, we start from the Lagrangian,

\[
  L = \dot{x}^+ P_+ + \dot{x}^- P_- + \Lambda
\]

where we added the time derivative of a function to be specified shortly. This does not alter the dynamics, provided there is no boundary contribution. Alternatively, it can be viewed as a gauge transformation \((A \rightarrow A + d\Lambda)\). By introducing the coordinate

\[
  \zeta = \arctan \left( \frac{\omega x^+ - \omega x^-}{1 + \omega^2 x^+ x^-} \right)
\]

we can write the Lagrangian as

\[
  L = \zeta P_\zeta - \dot{\mathcal{H}}
\]

where

\[
  \mathcal{H} = -\frac{1}{2} \left( \frac{P_+}{\partial_+ h} + \frac{P_-}{\partial_- h} \right) - \partial_\Lambda \Lambda = \frac{1}{2\omega} \left( H + \omega^2 K' \right),
\]

\[
  K' = (x^+)^2 P_+ - (x^-)^2 P_- - \frac{2}{\omega} \partial_\Lambda \Lambda
\]

is the momentum conjugate to \( h \) and \( P_\zeta \) is conjugate to \( \zeta \). Since \( \dot{h} = 1 \), it follows that \( \mathcal{H} \) plays the role of the Hamiltonian. We will choose \( \Lambda \) so that \( K' = K \) [Eq. (39)]. This ensures that the constraint \( \chi \) [Eq. (41)] and therefore the Hamiltonian will have no explicit time dependence, because

\[
  \partial_\Lambda \chi = \{ \chi, K \} = 0
\]

due to the conservation of the charge \( K \). It is easy to see that

\[
  \Lambda = m \sqrt{\frac{M}{4 \ln \frac{\partial_+ h}{\partial_- h}}} = -m \sqrt{\frac{M}{4 \ln \frac{1 + \omega^2 (x^+)^2}{1 + \omega^2 (x^-)^2}}}
\]

The non-relativistic limit can be easily deduced from Eq. (55).

\[
  \mathcal{H} = \frac{1}{2\omega} (H + \omega^2 K), \quad K = \frac{2}{\omega} \partial_\Lambda \Lambda = \frac{1}{2} mM \psi
\]

which is the Hamiltonian in the naive gauge (43) corrected by the potential \( K \) [cf. Eq. (47)]. Hence the origin of the non-relativistic potential \( K \) is the total time derivative \( \dot{A} \) that we added to the Lagrangian [Eq. (52)]. Thus, even though the total time derivative does not alter the dynamics, it is very useful in the calculation of the Hamiltonian in the non-relativistic limit.

In the set of gauges (50), there is no boundary contribution, because the Faddeev-Popov determinant vanishes there. Indeed,

\[
  \{ \h, \chi \} = \h - \frac{\omega x^+}{1 + \omega^2 (x^+)^2} + \frac{\omega x^-}{1 + \omega^2 (x^-)^2}
\]

which vanishes as \( x^\pm \rightarrow \infty \) for finite velocities \( x^\pm \). Therefore, one obtains standard commutation relations in this gauge. It should also be pointed out that, since we insist \( \dot{h} = 1 \), the region near the boundary does not contribute in the non-relativistic limit.

On the other hand, for the gauge (42), we obtain a boundary contribution to the path integral,

\[
  \int d\mathcal{H} d^D x d^D P \{ \h, \chi \} \delta(\h - \tau) \delta(\chi) \exp \left[ i P^\mu \partial_{\mu} \epsilon \right].
\]

Thus, the naive identification of the time coordinate (42) leads to an obstruction in the gauge-fixing procedure for the path integral. If this obstruction is accounted for by an appropriate modification of the commutation relations, this choice of the time coordinate leads to a well-defined Hamiltonian problem. The Hamiltonian system thus obtained is equivalent to applying the DFF trick [3], or identifying the time coordinate as in Eq. (50) [8]. The latter is merely a different gauge choice in a gauge-invariant theory.

**IV. SLOWLY MOVING REISSNER-NORDSTRÖM BLACK HOLES**

Having understood the quantization of a particle in a background created by a Reissner-Nordström black hole, we now turn to a discussion of the quantization of a system of dynamical Reissner-Nordström black holes. Again, we consider five spacetime dimensions. The results are similar in the four-dimensional case (see Sec. V).

The action may be written as

\[
  S = S_{\text{fields}} + S_{\text{source}}
\]
where the action for the fields is

\[ S_{\text{fields}} = \frac{1}{12\pi^2} \int d^5x \sqrt{g} \left( R - \frac{3}{4} F^2 \right) + \frac{1}{12\pi^2} \int A \wedge F \wedge F \]  

(62)

in terms of a dynamical metric field \( g_{\mu \nu} \) and electromagnetic vector potential \( A_\mu \), both functions of the coordinates \( x^\mu \) \((\mu = 0, 1, \ldots, 4)\). The sources are described by coordinates \( X^I_\mu \), where the index \( I \) labels the source and conjugate momenta \( P_I \). The action is

\[ S_{\text{source}} = \sum_I \int dX^I_\mu P_I^\mu \]  

(63)

together with the constraints

\[ \chi^I = \frac{1}{2M_I} \pi I_\mu \pi^I_\mu + \frac{1}{2} M_I = 0, \quad \pi I_\mu = P_I - Q I A_\mu, \]

\[ Q I = M_I. \]  

(64)

There is also a fermionic contribution which we omit because it is of no relevance to our discussion.

Using the ansatz

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = -\frac{1}{\psi^2} dt^2 + \psi^2 d\vec{x}^2 + 2 \vec{N} \cdot d\vec{x} \]  

(65)

\[ A = \frac{1}{\psi} dt + \vec{A} \cdot d\vec{x} \]  

(66)

and keeping only terms quadratic in the potentials \( \vec{N}, \vec{A} \) and discarding total derivatives, the action for the fields becomes [1]

\[ S_{\text{fields}} = \frac{1}{12\pi^2} \int d^5x \left( 3 \partial_i \vec{P} \cdot \partial_j \vec{P} - \frac{3}{4\psi^2} F^2 - \frac{3}{4\psi^2} FG \right) \]

\[ - \frac{1}{2\psi^2} G^2 - 3 \psi (\partial_i \psi)^2 - \frac{3}{4\psi^2} F \vec{F} \]

\[ + \frac{3}{4\psi^2} \vec{F} \vec{G} - \frac{1}{4\psi^2} \vec{G} \vec{G} \]  

(67)

where we introduced the convenient (gauge-invariant) combinations

\[ \vec{P} = \vec{A} + \psi \vec{N}, \quad \vec{R} = \psi^2 \vec{N} \]  

(68)

whose field strengths respectively are

\[ F_{ij} = \partial_i P_j - \partial_j P_i, \quad G_{ij} = \partial_i R_j - \partial_j R_i \]  

(69)

and their duals: \( \vec{F}^{ij} = \epsilon^{ijkl} F_{kl}, \quad \vec{G}^{ij} = \epsilon^{ijkl} G_{kl} \).

The equations of motion are the Einstein and Maxwell equations which yield\(^1\)

\[ \psi = A_0, \quad F_{ij} = 2 \psi \mathcal{F}_{ij}, \quad G_{ij} = 3 \psi^2 \mathcal{F}_{ij} \]  

(70)

where \( A_0 \) is the vector potential generated by the source current \( j^\mu \) in flat spacetime and \( \mathcal{F}_{\mu \nu} \) is its field strength:

\[ \partial_\mu \mathcal{F}_{\mu \nu} = 2 \pi^2 j^\nu, \quad \mathcal{F}_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

\[ j^\mu = \sum_I M_I \int dX^I_\mu \delta^5(x - X_I). \]  

(71)

Notice that \( j^\mu \) is a gauge (reparametrization) invariant quantity, as it should be. For the sources, we obtain the Lorentz force equation,

\[ \vec{X}_j + \Gamma_{\nu \beta} \vec{X}_j \vec{X}^\beta_\nu = \vec{X}_j i P \vec{F}. \]  

(72)

The path integral is

\[ Z = \mathcal{N} \int \mathcal{D}g \mathcal{D}A \prod_I DX_I DP_I \delta(\chi^I) e^{iS}. \]  

(73)

To calculate this, we need to fix the gauge in the sources. The simplest choice is

\[ \chi^I_0 = t \]  

(74)

for all black holes. Then the current becomes

\[ j^\mu = \sum_I M_I \psi_\nu \delta^5(x - \vec{X}_I), \quad \nu_\nu = (1, \nu_I), \quad \nu_\nu = \frac{d\vec{X}_I}{dt} \]  

(75)

where \( \nu_\nu \ll 1 \) (non-relativistic limit). The vector potential is found to be

\[ A_0 = \psi = \sum_I \frac{M_I}{(x - \vec{X}_I(t))^2}, \quad \vec{A} = \sum_I \frac{M_I \vec{v}_I}{(x - \vec{X}_I(t))^2}. \]  

(76)

After solving the constraints \( \chi^I = 0 \), in the non-relativistic limit,

\[ P_{I0} = \frac{g^I_0 \pi I_\mu \pi^I_\mu + M_I A_0}{2M_I} \]  

(77)

the action for the sources becomes

\[ S_{\text{source}} = \sum_I \int dt (\chi_0 P_{I0} - H_I), \quad H_I = - P_{I0}. \]  

(78)

After integrating over the momenta \( \vec{P}_I \) in the path integral, we obtain

\(^1\)There are corrections to these expressions for \( F_{ij} \) and \( G_{ij} \), as is evident by taking exterior derivatives of both sides of Eq. (70). For a discussion, see [12]. They do not affect our results.
where $S = S_{\text{fields}} + S_{\text{source}}$, $S_{\text{fields}}$ is given by Eq. (67) and

$$S_{\text{source}} = \sum_J M_4 \int dt \left( \frac{1}{2} \psi^2 \tilde{X}_I^2 + \tilde{\chi}_I^2 + \tilde{P}_I \right).$$

(80)

After some algebra, the action can be cast into the form [1]

$$S = S_{\text{fields}} + S_{\text{source}} = \frac{1}{2} \int dt \sum_{I \neq J} G_{IJ}(\tilde{\nu}_I - \tilde{\nu}_J)^2 + \cdots$$

(81)

where

$$G_{IJ} = G_{IJ}(\tilde{X}_I - \tilde{X}_J) = \frac{M_4 M_4(M_4 + M_J)}{\langle \tilde{X}_I - \tilde{X}_J \rangle^4}, \quad \tilde{\nu}_I = \frac{dX_I}{dt}$$

(82)

and we have represented by dots the remaining less singular terms (they involve three-point interactions). This leads to a Hamiltonian with no well-defined ground state. This pathology is shared with the case of a black hole moving in the background of another static black hole. This is expected, because the latter case is encompassed by the multi-black hole system. We need to be more careful in implementing the quantization procedure as problems arise from the singularities of moduli space (when two black holes become coincident).

Instead of collectively identifying all $X_I^j$ coordinates with time, we shall adopt a gauge similar to the one discussed in the previous section [cf. Eq. (50)],

$$h_J(X_I^j) = t.$$ (83)

To determine the function $h_J$ for the $J$th black hole, we work as follows. Consider the $J$th black hole. It moves under the influence of the (gravitational and electromagnetic) fields created by the other black holes [given by Eqs. (68)–(71)], where the sum in Eq. (71) runs over $I \neq J$. Of course, this statement is only valid in the non-relativistic limit we are considering where the Einstein-Maxwell equations are essentially linearized. As our chosen black hole approaches one of the other black holes (the $I$th one, say), the influence of the rest of the system becomes negligible. By switching to the rest frame of the $I$th black hole, the dynamics of our chosen ($J$th) black hole is governed by the action discussed in the previous section. Therefore, the gauge-fixing condition $h_I = t$ should reduce to the gauge (50) in that frame and in the limit where the $I$th and $J$th black holes become coincident. As discussed in the previous section, we expect that the net effect in the non-relativistic limit will be the addition of a potential of the form [cf. Eq. (58)]

$$K^{(J)}_I = \frac{M_4 M_J^2}{2(\tilde{X}_J - \tilde{X}_I)^2}.$$ (84)

Switching back to the center-of-mass frame from the rest frame of the $I$th black hole does not alter this conclusion, because we need only perform a Galilean transformation in the non-relativistic limit. Repeating the argument with the rest of the black holes in the system, we expect to obtain a net additional potential

$$K^{(J)} = \sum_{I \neq J} K^{(J)}_I = \frac{M_4}{2} \sum_{I \neq J} \frac{M_J^2}{(\tilde{X}_J - \tilde{X}_I)^2}.$$ (85)

To implement the above considerations in detail, let us introduce the coordinates [cf. Eq. (34)]

$$x_I^j = x_I^0 + \sum_{I \neq J} \left( \arctan \left( \frac{X_0^I}{1 + \frac{1}{2} \lambda_J x_I^j} \right) - x_I^0 \right) = t.$$ (87)

Notice that as the distance $(\tilde{X}_J - \tilde{X}_I)^2 \to 0$ for a fixed $I$, with all other distances remaining finite, the above definitions coincide with Eqs. (34) and (50), respectively. This gauge choice is guaranteed to give no boundary contribution, because $h_J \to 0$ near the boundary of moduli space. As in the previous section, in order to calculate the non-relativistic limit, we need to augment the Lagrangian by adding a total time derivative (which again gives no contribution at the boundary) ensuring that the Lagrangian will have no explicit dependence on $h_J$ (leading to a time independent Hamiltonian). Thus, we define the action for the $J$th black hole by [cf. Eq. (34)]

$$S_J = \int dt X_J^\mu P_J + \Lambda^{(J)} , \quad \Lambda^{(J)} = \sum_{I \neq J} \Lambda_I^{(J)} ,$$

$$\Lambda_I^{(J)} = - M_J \sqrt{\frac{M_4}{4}} \ln \left[ \frac{1 + \frac{1}{2} \lambda_J (x_I^j)^2}{1 + \frac{1}{2} \lambda_J (x_I^j)^2} \right].$$ (88)

It is easily verified that in the non-relativistic limit,

$$t = h_J = x_I^0 , \quad \Lambda_I^{(J)} = x_I^0 M_J M_I^2 / 8(\tilde{X}_J - \tilde{X}_I)^2 = 1 / 4 t K^{(J)}_I.$$ (89)

Therefore in the non-relativistic limit, the additional term in the action reads

$$\Lambda^{(J)} = \frac{1}{4} \sum_{I \neq J} K^{(J)}_I = \frac{1}{4} K^{(J)}$$ (90)

in agreement with our expectations (having set $\omega = 1/2$).
The above gauge-fixing procedure can be repeated with the rest of the black holes in the system. We modify the action for the sources by a total time derivative which does not alter the dynamics, but simplifies the calculation of the non-relativistic limit. Thus, we define the action for the sources by [cf. Eq. (63)]

\[ S_{\text{source}} = \sum_j S_j = \sum_j \int dt (\dot{X}_j^\mu P_{j\mu} + \lambda^{(j)}) \]  

(91)

where \( \lambda^{(j)} \) is given by Eq. (88). In the non-relativistic limit, the net effect of the gauge (87) is the addition of the potential \( \frac{1}{2} K \), where

\[ K = \sum_j K^{(j)} = \sum_j \sum_{l \neq j} \frac{M_j M_l^2}{2(\tilde{X}_j - \tilde{X}_l)^2} = \sum_j M_j M_j (M_j + M_l) \frac{1}{2(\tilde{X}_j - \tilde{X}_l)^2}. \]  

(92)

In conclusion, we have shown how to fix the gauge in a way that no obstruction occurs from the singularities of moduli space (when two black holes approach each other). The resultant Hamiltonian differs from the one obtained in the naive gauge (74) by the addition of the potential (92). This is accordance with the DFF prescription [3].

The naive gauge (74) suffers from an obstruction at the singularities of moduli space. Once the obstruction is correctly accounted for, the resulting theory is equivalent to the one in which the Hamiltonian is modified by the addition of the potential \( K \) [Eq. (92)]. This is because the underlying theory is gauge-invariant.

V. FOUR SPACETIME DIMENSIONS

The four-dimensional case was originally discussed by Ferrel and Eardley [2]. The results are similar to the five-dimensional case. Therefore, we will only summarize the major differences between the two cases.

In four dimensions, the metric due to an extreme Reissner-Nordström black hole is

\[ ds^2 = -\frac{1}{\psi^2} dt^2 + \psi^2 dx^2, \quad \psi = 1 + \frac{M}{|x|} \]  

(93)

and the vector potential is

\[ A_i = \frac{1}{\psi} - 1, \quad \tilde{A} = 0 \]  

(94)

where the vectors live in a three-dimensional Euclidean space. Near the horizon, \( \psi = M/|\tilde{x}| \). Using polar coordinates and switching variables to \( \psi \), we obtain

\[ ds^2 = -\frac{1}{\psi^2} (dt^2 - M^2 d\psi^2) + M^2 d\Omega_x^2. \]  

(95)

Defining

\[ x^\pm = t \pm M \psi \]  

(96)

the metric becomes

\[ ds^2 = -\frac{1}{\psi^2} dx^+ dx^- + M^2 d\Omega_x^2. \]  

(97)

This is of the same form as in five spacetime dimensions [Eq. (35)], apart from the scale factor in the spherical part of the metric. Working as in Sec. III, we arrive at an \( SL(2, \mathbb{R}) \) conformal algebra consisting of the operators

\[ H = \frac{|\tilde{x}|^3 \tilde{P}^2}{2m M^3}, \quad D = -2\tilde{x} \cdot \tilde{P}, \quad K = \frac{2m M^3}{|x|} \]  

(98)

in the non-relativistic limit. In the gauge

\[ h(x^+, x^-) = \arctan \left( \frac{\omega x^+ + \omega x^-}{1 - \omega^2 x^+ x^-} \right) = \tau \]  

(99)

the Hamiltonian becomes

\[ \mathcal{H} = \frac{1}{2\omega} (H + \omega^2 K) \]  

(100)

which has a well-defined vacuum state. The system reduces to a harmonic oscillator if we change variables to \( u = M \sqrt{\psi} \).

As Ferrel and Eardley showed [2], in the case of two slowly-moving black holes, in the center-of-mass frame and in the near-horizon limit, the Hamiltonian becomes

\[ H = \frac{|\tilde{X}_- - \tilde{X}_+|^3 (\tilde{P}_- - \tilde{P}_+)^2}{2 \mu(1 - 2 \mu/M) M^3} = \frac{|\tilde{X}_- - \tilde{X}_+|^3 (\tilde{P}_- - \tilde{P}_+)^2}{2 M_1 M_2 (M_1^2 + M_2^2)} \]  

(101)

where \( \mu = M_1 M_2 / (M_1 + M_2) \) is the reduced mass and \( M = (M_1 + M_2) \) is the total mass of the system. This generalizes to an arbitrary number of maximally-charged slowly-moving black holes,

\[ H = \sum_{j \neq l} \frac{|\tilde{X}_j - \tilde{X}_l|^3 (\tilde{P}_j - \tilde{P}_l)^2}{2 M_j M_l (M_j^2 + M_l^2)} + \ldots \]  

(102)

where we have omitted the less singular terms. Evidently, this system has no ground state. The origin of this pathology is the same as in the five-dimensional case. To remedy this, we work as in Sec. IV. The action for the fields is now [cf. Eq. (62)]

\[ S_{\text{fields}} = \frac{1}{16\pi} \int d^4 x \sqrt{g} (R - F^2) \]  

(103)

whereas the action for the sources remains unchanged. We introduce the gauge-fixing condition (87), where [cf. Eq. (86)]

\[ x^{(j)}_\pm = x_0^\pm \pm \frac{M_j^2}{2|\tilde{X}_j - \tilde{X}_l|} \]  

(104)

and augment the Lagrangian for the sources with the total derivative [cf. Eq. (88)].
\[ \Lambda = \sum_j \Lambda^{(j)} \quad \Lambda^{(j)} = \sum_{i \neq j} \Lambda_i^{(j)} \]

\[ \Lambda_i^{(j)} = -2M_j M_l \ln \left( \frac{1 + \frac{1}{2} (x^{(j)}_{i+})^2}{1 + \frac{1}{2} (x^{(j)}_{i-})^2} \right) \]  

In the non-relativistic limit, the gauge condition \( h_{ij} \approx X^0_{ij} = t \) and the additional term in the action to \( \text{cf. Eqs. (89) and (90)} \)

\[ \Lambda \approx \frac{1}{4} \sum_{i,j} K_{ij}^{(j)} \]  

where

\[ K_{ij}^{(j)} = \frac{2M_j M_i^3}{|X_j - X_i|} \]  

Therefore, in the gauge (87), the Hamiltonian is modified by the potential \( K = \Sigma_{i<j} K_{ij} \), where \( \text{cf. Eq. (92)} \)

\[ K_{ij} = \frac{M_i M_j (M_i^2 + M_j^2)}{2 |X_j - X_i|} \]  

similar to five spacetime dimensions (Sec. IV) and in accordance with the DFF prescription [3].

VI. CONCLUSIONS

We considered the problem of quantization of a system of slowly-moving extreme Reissner-Nordström black holes. The moduli have dynamics governed by (super)conformal quantum mechanics and the Hamiltonian has no well-defined ground state. This problem is fixed by an application of the DFF trick [3]. To justify this trick on physical grounds, we approached the problem through the path integral and the Faddeev-Popov gauge-fixing procedure. We showed that the DFF trick can be understood in terms of the standard Faddeev-Popov procedure [10]. We started with a discussion of the quantization of a particle in the presence of a background metric field as well as an external vector potential. We performed the standard Faddeev-Popov procedure in the canonical formalism and showed its connection to commutation relations through Dirac brackets. We then applied the procedure to the case of an extreme Reissner-Nordström black hole [1]. We showed that the naive identification of time coordinate (which leads to a Hamiltonian system with no well-defined ground state) corresponds to a gauge which is not “good.” We found that in this gauge the Faddeev-Popov procedure encounters an obstruction at the boundary of spacetime introducing an additional constraint there. This alters the standard commutation relations and the eigenvalue problem for the attendant Hamiltonian system. We did not calculate the effects of this obstruction explicitly. This would require the introduction of a regulator which would break gauge invariance explicitly and therefore alter the commutation rules. Instead, we exhibited another set of gauges where no obstruction existed on the boundary. We showed that this set of gauges led to a Hamiltonian system with a well-defined vacuum, equivalent to the one obtained through the DFF trick [3].

We then applied our procedure to multiple black hole scattering [1]. We noted that the underlying theory is a gauge theory, so the Faddeev-Popov procedure should be applicable. We discussed a systematic implementation of the quantization procedure which correctly accounted for the singularities of moduli space. Each black hole is described by moduli (position vector) \( X^0(\tau) \) and the action is reparametrization invariant. This gauge invariance necessitated the introduction of gauge-fixing conditions equal in number to the number of black holes. By identifying \( X^0 \) with time for all black holes, one arrives at the standard Hamiltonian that possesses no well-defined ground state. This pathology comes from a subtlety in the Faddeev-Popov quantization procedure that does not take into account the singularities of moduli space. To properly account for these singularities would be tedious (entailing the introduction of a regulator) and would lead to a modification of the quantization rules. Instead, we introduced gauge-fixing conditions that did not suffer from this pathology. The resultant Hamiltonian differed from the pathological one by the addition of the potential \( \frac{1}{2} K \), where \( K \) is the generator of special conformal transformations, in accordance with the DFF prescription.

Our method is generalizable to any system of black holes and more general solutions of the Einstein-Maxwell equations. It would be interesting to apply the Faddeev-Popov procedure to these systems, such as near-extreme black holes. This would enable us to move away from the AdS limit.

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