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The variational iteration method is successfully extended to the case of solving fractional differential equations, and the Lagrange multiplier of the method is identified in a more accurate way. Some diffusion models with fractional derivatives are investigated analytically, and the results show the efficiency of the new Lagrange multiplier for fractional differential equations of arbitrary order.

Keywords: time-fractional diffusion equation, Captuo derivative, Riemann–Liouville derivative, variational iteration method, Laplace transform

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1. Introduction

Fractional calculus is a power tool for finding solution of nonlinear problems in various application areas where successful fractional models have been developed for metal sorption,[1] fractional optical solitons,[2] sub-diffusion processes,[3] frequency-dependent attenuation,[4] etc. Concurrently, approximate solutions of fractional differential equations (FDEs) are intensively developed, leading to the advent of several nonlinear analytical techniques such as the Adomian decomposition method (ADM),[5,6] the heat balance integral method (HBM),[7,8] the homotopy perturbation method (HPM),[9,10] and some other numerical methods.[11–13]

The variational iteration method (VIM)[14,15] has been proven to be an efficient nonlinear method for solving ordinary differential equations (ODEs). Some efforts have been made to solve the FDEs but not very successfully, owing to the difficulty in identifying the Lagrange multiplier.

In fact, the Lagrange multiplier of the method can be explicitly identified using Laplace transform. For clarity of the further expression, we will briefly pay the attention to the methodology of the VIM of solving the following FDE:

$$ {}_0^{\mathrm{C}}D_t^{\alpha} u(t) + R[u(t)] + N[u(t)] = k(t), \quad \alpha > 0, \quad (1)$$

where ${}_0^{\mathrm{C}}D_t^{\alpha}$ is the left Caputo derivative,[16,17] R is the linear operator, N is the nonlinear operator, and $k(t)$ is a given continuous function. Then, we propose some new variational iteration formulas and investigate some fractional diffusion equations describing the flow through porous medium.

2. Preliminaries

Definition 1 The Caputo derivative is defined as

$$ {}_0^{\mathrm{C}}D_t^{\alpha} u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} \frac{\mathrm{d}^m}{\mathrm{d}\tau^m} u(\tau)\,\mathrm{d}\tau, $$
$$ 0 < \alpha,\ m = [\alpha]+1,\ 0 < t, \quad (2)$$

where $u(t) \in C^m[0,T]$ and Γ is the Gamma function, with $C^m[0,T]$ being a space of functions, which are m times continuously differentiable on the interval $[0,T]$.

Definition 2 The α-th Riemann–Liouville (R–L) derivative of function $u(t)$ is defined as

$$ {}_0^{\mathrm{RL}}D_t^{\alpha} u = \frac{1}{\Gamma(m-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \times \int_0^t \frac{1}{(t-\tau)^{\alpha-m+1}} u(\tau)\,\mathrm{d}\tau, $$
$$ 0 < \alpha,\ m = [\alpha]+1,\ 0 < t, \quad (3)$$

Definition 3 The R–L integration of α order is

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defined as

$$ {}_0I_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)\,\mathrm{d}\tau, \quad 0<\alpha. \quad (4) $$

The Laplace transform of an original function $u(t)$ of a real variable t, is defined by the integral (if there exists) as

$$ \bar{u}(s) = L[u(t)] = \int_0^{\infty} \mathrm{e}^{-st}u(t)\,\mathrm{d}t, \quad t>0, $$

where the parameter s is a complex number.

Proposition 1 The Laplace transform of the term ${}_0^{\mathrm{C}}D_t^{\alpha}u$ holds

$$ L[{}_0^{\mathrm{C}}D_t^{\alpha}u] = s^{\alpha}\bar{u}(s) - \sum_{k=0}^{m-1} u^{(k)}(0^+)s^{\alpha-1-k}, \\ m-1<\alpha\le m. \quad (5) $$

Proposition 2 The Laplace transform of the R–L derivative is

$$ L[{}^{\mathrm{RL}}_{\ 0}D_t^{\alpha}u] = s^{\alpha}\bar{u}(s) - \sum_{k=0}^{m-1} u^{(\alpha-k-1)}(0^+)s^{k}, \\ m-1<\alpha\le m. $$

The inverse Laplace transform is defined by the complex integral as

$$ u(t) = L^{-1}[u(s)] = \frac{1}{2\pi\mathrm{i}}\lim_{T\to\infty}\int_{\gamma-\mathrm{i}T}^{\gamma+\mathrm{i}T} \mathrm{e}^{st}\bar{u}(s)\,\mathrm{d}s, $$

where the integration is performed along the ordinate axis $\mathrm{Re}(s)=\gamma$ in the complex plane such that γ is greater than the real part of all singularities of $\bar{u}(s)$.

Proposition 3 Assuming $\bar{f}(s) = L[f(t)]$ and $\bar{g}(s) = L[g(t)]$, the convolution theorem is

$$ f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau \quad (6) $$

and

$$ \bar{f}(s)\bar{g}(s) = L[f(t)*g(t)]. \quad (7) $$

The detail properies of fractional calculus and Laplace transform can be found in Refs. [16]–[19].

3. Generalized Lagrange multipliers

The variational iteration method was ever extended to FDEs. Since the similar integration by parts from calculus cannot hold, the extension mainly employed the Lagrange multipliers in ordinary differential equations. For example, the so-called Lagrange multiplier $\lambda(t,\tau) = -1$ was used and the variational iteration formula of Eq. (1) was given as

$$ \begin{cases} u_{n+1} = u_n + \displaystyle\int_0^t \lambda(t,\tau)({}_0^{\mathrm{C}}D_{\tau}^{\alpha}u_n + R[u_n] \\ \qquad + N[u_n] - k(\tau))\,\mathrm{d}\tau, \\ \qquad 0<\alpha<1, \\ \lambda(t,\tau) = -1. \end{cases} \quad (8) $$

One can check that the convergence of the above iteration formula is poor for a linear FDE. Then we propose the following iteration formula:

$$ u_{n+1} = u_n + \int_0^t \lambda(t,\tau)({}_0^{\mathrm{C}}D_{\tau}^{\alpha}u_n + R[u_n] \\ +N[u_n] - k(\tau))\,\mathrm{d}\tau, \quad \alpha>0, \quad (9) $$

where $\lambda(t,\tau) = (-1)^{\alpha}(\tau-t)^{\alpha-1}/\Gamma(\alpha)$ is a Lagrange multiplier or a weighted function for any fractional order $\alpha>0$.

In fact, we can construct a correction functional through the R–L integration

$$ u_{n+1} = u_n + {}_0I_t^{\alpha}\lambda(t,\tau)[{}_0^{\mathrm{C}}D_{\tau}^{\alpha}u_n \\ + R[u_n] + N[u_n] - k(\tau)]. \quad (10) $$

Perform Laplace transform on both sides of Eq. (10)

$$ \bar{u}_{n+1}(s) = \bar{u}_n(s) + L[{}_0I_t^{\alpha}\lambda(t,\tau)({}_0^{\mathrm{C}}D_{\tau}^{\alpha}u_n \\ + R[u_n] + N[u_n] - k(\tau)]. \quad (11) $$

Setting $R[u_n]$ and $N[u_n]$ as restricted variations, respectively, we only need to consider the term

$$ {}_0I_t^{\alpha}\lambda{}_0^{\mathrm{C}}D_t^{\alpha}u_n \\ = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\lambda(t,\tau){}_0^{\mathrm{C}}D_{\tau}^{\alpha}u_n(\tau)\,\mathrm{d}\tau. \quad (12) $$

Assuming the Lagrange multiplier $\lambda(t,\tau) = \lambda(X)/_{X=t-\tau}$, equation (12) becomes the convolution of the function $a(t) = \lambda(t)t^{\alpha-1}/\Gamma(\alpha)$ and the term ${}_0^{\mathrm{C}}D_t^{\alpha}u_n(t)$.

From Laplace transform (5) and making correction functional (11) stationary, we can obtain

$$ \delta\bar{u}_{n+1}(s) = \delta\bar{u}_n(s) + \delta[\bar{a}(s)s^{\alpha}\bar{u}_n(s) \\ - \sum_{k=0}^{m-1} u^{(k)}(0^+)s^{\alpha-1-k}] \\ = (1+\bar{a}(s)s^{\alpha})\delta\bar{u}_n(s). \quad (13) $$

Here δ is the variational operator in the classical variational theory.

The extreme condition $\delta\bar{u}_{n+1}(s)/\delta\bar{u}_n(s) = 0$ requires $1+\bar{a}(s)s^\alpha = 0$. With the inverse Laplace transform, we have

$$a(t) = L^{-1}[a(s)] = -\frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0. \tag{14}$$

Taking into account $a(t) = \lambda(t)t^{\alpha-1}/\Gamma(\alpha)$, the Lagrange multiplier in Eq. (12) can be explicitly identified as

$$\lambda(t,\tau) = -1. \tag{15}$$

As a result, iteration formula (10) is given as

$$\begin{aligned} u_{u+1} &= u_n - {}_0I_t^\alpha[{}_0^{\mathrm{C}}D_\tau^\alpha u_n + R[u_n] + N[u_n] - k(\tau)] \\ &= u_n + \int_0^t \frac{(-1)^\alpha(\tau-t)^{\alpha-1}}{\Gamma(\alpha)}({}_0^{\mathrm{C}}D_\tau^\alpha u_n + R[u_n] \\ &\quad + N[u_n] - k(\tau))\,\mathrm{d}\tau, \quad \alpha > 0. \end{aligned} \tag{16}$$

This completes the proof.

Remark 1

(I) If the correction functional (10) is constructed through the integration of integer order,

$$\begin{aligned} u_{n+1} &= u_n + \int_0^t \lambda(t,\tau)({}_0^{\mathrm{C}}D_\tau^\alpha u_n + R[u_n] \\ &\quad + N[u_n] - k(\tau))\,\mathrm{d}\tau, \ \ \alpha > 0, \end{aligned} \tag{17}$$

one can derive $\lambda(t,\tau) = (-1)^\alpha(\tau-t)^{\alpha-1}/\Gamma(\alpha)$ instead $\lambda(t,\tau) = -1$ in Eq. (8). The above iteration formula is also valid for FDEs in the sense of R–L derivative. One can also find the analysis of the iteration formula in Ref. [19].

(II) On the other hand, we only derive the simplest Lagrange multiplier here. A more explicit one can be identified if more term in $R[u_n]$ (if there exists) is used. For example, consider the following FDE:

$$\begin{aligned} &{}_0^{\mathrm{C}}D_t^\alpha u + \omega^\alpha u + f(t,u) = 0, \quad 0 < \alpha \ \ \text{or} \\ &{}_0^{\mathrm{RL}}D_t^\alpha u + \omega^\alpha u + f(t,u) = 0, \quad 0 < \alpha, \end{aligned}$$

where $f(t,u)$ is the nonlinear term.

Similarly, the variational iteration formula can be obtained as

$$\begin{cases} u_{n+1} = u_n + \displaystyle\int_0^t \lambda(t,\tau)({}_0^{\mathrm{C}}D_\tau^\alpha u_n + \omega^\alpha u_n \\ \qquad + f(\tau,u_n))\,\mathrm{d}\tau, \quad 0 < \alpha, \\ \lambda = -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha), \end{cases}$$

or

$$\begin{cases} u_{n+1} = u_n + \displaystyle\int_0^t \lambda(t,\tau)({}_0^{\mathrm{RL}}D_\tau^\alpha u_n + \omega^\alpha u_n \\ \qquad + f(\tau,u_n))\,\mathrm{d}\tau, \quad 0 < \alpha, \\ \lambda = -(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha), \end{cases}$$

where $E_{\alpha,\alpha}(-\omega^\alpha(t-\tau)^\alpha)$ is the Mittag–Leffler function with two parameters.

4. Approximate solutions of time-fractional diffusion equations

The diffusion process has been discussed in many real physical systems such as highly ramified medium in porous system, anomalous diffusion in fractal medium, and heat transfer close to equilibrium. The flow through porous medium can be better described by the fractional model than by the classical one since it includes inherently memory effects caused by obstacles in the structure. Readers are referred to the recent publications.[20–27] In this section, three types of time fractional diffusion equations are analytically investigated by the VIM.

Example 1 Consider the linear time-fractional diffusion equation

$$\begin{aligned} &{}_0^{\mathrm{C}}D_t^\alpha u(x,t) = u_{xx}(x,t), \\ &u(x,0) = \sin(x), \quad 0 < \alpha \le 1. \end{aligned} \tag{18}$$

Equation (18) can be used to describe the flow through porous medium or a transient heat-conduction. For $1 < \alpha \le 2$, equation (18) reduces to the wave model recently solved by the VIM.[28,29]

From iteration formula (16), we have

$$u_{n+1} = u_n - {}_0I_t^\alpha({}_0^{\mathrm{C}}D_\tau^\alpha u_n - u_{n,xx}). \tag{19}$$

Starting from the initial iteration $u_0 = u(x,0) = \sin(x)$, the successive approximate solutions can be obtained as

$$\begin{aligned} u_1 &= u_0 - {}_0I_t^\alpha({}_0^{\mathrm{C}}D_\tau^\alpha u_0 - u_{0,xx}) \\ &= \sin(x) - \sin(x)\frac{t^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{20a}$$

$$\begin{aligned} u_2 &= u_1 - {}_0I_t^\alpha\left({}_0^{\mathrm{C}}D_\tau^\alpha u_1 - u_{1,xx}\right) \\ &= \sin(x) - \sin x\frac{t^\alpha}{\Gamma(1+\alpha)} + \sin x\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \end{aligned} \tag{20b}$$

$$u_n = \sin(x)\sum_{k=0}^{n}\frac{(-t)^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{20c}$$

For $n \to \infty$, $u(t,x) = \lim_{n\to\infty} u_n = \sin(x)E_\alpha(-t^\alpha)$ is an exact solution of Eq. (18). Here $E_\alpha(-t^\alpha)$ is the famous Mittag–Leffler function

$$E_\alpha(-t^\alpha) = \sum_{k=0}^{\infty}\frac{(-t^\alpha)^k}{\Gamma(1+k\alpha)}. \tag{21}$$

Example 2 As the second example, the time-fractional diffusion equation with a source term

$$
\begin{aligned}
&{}_0^{\mathrm{C}}D_t^{\alpha}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{\partial(xu(x,t))}{\partial x},\\
&u(x,0)=x^2,\quad 0<\alpha\leq 1,
\end{aligned}
\tag{22}
$$

was solved approximately for a semi-derivative.[30]

Now from Eq. (16), we have the following iteration formula:

$$
\begin{aligned}
u_{n+1}=&\ u_n-{}_0I_t^{\alpha}\big({}_0^{\mathrm{C}}D_\tau^{\alpha}u_n(x,\tau)\\
&-\frac{\partial^2 u_n(x,\tau)}{\partial x^2}-\frac{\partial(xu_n(x,\tau))}{\partial x}\big),\\
u_0=&\ x^2.
\end{aligned}
\tag{23}
$$

As a result, the successive approximation can be derived as

$$
\begin{aligned}
u_0(t)&=x^2,\\
u_1(t)&=x^2+\frac{(2+3x^2)t^{\alpha}}{\Gamma(1+\alpha)},\\
u_2(t)&=x^2+\frac{(2+3x^2)t^{\alpha}}{\Gamma(1+\alpha)}+\frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)},\\
u_3(t)&=x^2+\frac{(2+3x^2)t^{\alpha}}{\Gamma(1+\alpha)}+\frac{(8+9x^2)t^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\quad+\frac{(26+27x^2)t^{3\alpha}}{\Gamma(1+3\alpha)},\\
&\quad\ldots.
\end{aligned}
\tag{24}
$$

The exact solution of Eq. (22) is

$$
\begin{aligned}
u(x,t)&=\lim_{n\to\infty}u_n(x,t)\\
&=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{c^i t^{i\alpha}}{\Gamma(1+\mathrm{i}\alpha)}=E_\alpha(ct^\alpha),
\end{aligned}
\tag{25}
$$

where $c^i=x^2+(1+x^2)(3^i-1)$.

Example 3 The third example is the linear time-fractional wave equation

$$
\begin{aligned}
&{}_0^{\mathrm{C}}D_t^{\alpha}u(x,t)=\frac{1}{2}x^2u_{xx}(x,t),\\
&u(x,0)=x,\quad \frac{\partial u(x,0)}{\partial t}=x^2,\quad 1<\alpha\leq 2.
\end{aligned}
\tag{26}
$$

The iteration formula for Eq. (26) is

$$
u_{n+1}=u_n-{}_0I_t^{\alpha}\left({}_0^{\mathrm{C}}D_\tau^{\alpha}u_n-\frac{1}{2}x^2u_{n,xx}\right),\tag{27a}
$$

$$
u_0=x+x^2t,\quad 1<\alpha\leq 2.\tag{27b}
$$

As a result, the successive approximate solutions are

$$
\begin{aligned}
u_0&=x+x^2t,\\
u_1&=x+x^2\left(t+\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\right),\\
u_2&=x+x^2\left(t+\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}+\frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)}\right),\\
&\ldots.
\end{aligned}
\tag{28}
$$

The solution can be given in a compact form as follows:

$$
u_n=x+x^2t\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(2+k\alpha)},\tag{29}
$$

and the limit $u=\lim_{n\to\infty}u_n=x+x^2tE_{\alpha,2}(t^\alpha)$ is an exact solution of Eq. (26). The same approximate solutions mentioned above can be obtained by the HPM and the ADM.

5. Conclusion

The extended variational iteration method in this work allows the easy development of very good approximate analytical solutions of fractional models. Three examples are successfully illustrated. For new applications of the method in other fields, readers are referred to the fractal initial value problems in Ref. [29], q-difference equations in Refs. [31] and [32], fuzzy differential equations in Refs. [33]–[35], and differential equations in Refs. [36]–[38].

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