Variational iteration method for q-difference equations of second order

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Letter to the Editor

Variational Iteration Method for $q$-Difference Equations of Second Order

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Recently, Liu extended He’s variational iteration method to strongly nonlinear $q$-difference equations Liu (2010). In this study, the iteration formula and the Lagrange multiplier are given in a more accurate way. The $q$-oscillation equation of second order is approximately solved to show the new Lagrange multiplier’s validness.

1. Introduction

Generally, applying the variational iteration method (VIM) [1, 2] in differential equations follows the three steps:

(a) establishing the correction functional;
(b) identifying the Lagrange multipliers;
(c) determining the initial iteration.

Obviously, the step (b) is crucial and critical in the method.

For the strongly nonlinear $q$-difference equation,

$$\frac{d^2}{d_q t^2} x + (2 + \varepsilon x) \frac{d_q}{d_q t} x + \Omega^2 x + x^2 = 0, \quad (1.1)$$

where $d_q/d_q t$ is the $q$-derivative [3], Liu [4] used the Lagrange multiplier

$$\lambda(t,s) = s - t, \quad (1.2)$$
which results in the iteration formula (see [4, (4.10) and (4.11)]):

\[
x_{n+1} = x_n + \int_0^t (s-t) \left( \frac{d^2_q}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s.
\] (1.3)

In this paper, it is pointed out that the iteration formula (1.3) can be given in a more accurate way and a new Lagrange multiplier is explicitly identified.

2. Properties of \(q\)-Calculus

2.1. \(q\)-Calculus

Let \(f(x)\) be a real continuous function. The \(q\)-derivative is defined as

\[
\frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad 0 < q < 1,
\] (2.1)

and \((d_q/d_q x)f(x)|_{x=0} = \lim_{n \to \infty} \{(f(q^n) - f(0))/q^n\}.

The partial \(q\)-derivative with respect to \(x\) is

\[
\frac{\partial_q}{\partial_q x} f(x; y; \ldots) = \frac{f(qx; y; \ldots) - f(x; y; \ldots)}{(q-1)x}.
\] (2.2)

The corresponding \(q\)-integral [5] is

\[
\int_0^x f(t) d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x).
\] (2.3)

2.2. \(q\)-Leibniz Product Law

One has

\[
\frac{d_q}{d_q x} [g(x) f(x)] = g(qx) \frac{d_q}{d_q x} [f(x)] + f(x) \frac{d_q}{d_q x} [g(x)].
\] (2.4)

2.3. \(q\)-Integration by Parts

One has

\[
\int_a^b g(qt) \frac{d_q}{d_q t} f(t) d_q t = f(t) g(t) \big|_a^b - \int_a^b f(t) \frac{d_q}{d_q t} g(t) d_q t.
\] (2.5)

The properties above are needed in the construction of the correction functional for \(q\)-difference equations. For more results and properties in \(q\)-calculus, readers are referred to the recent monographs [5–8].
3. A $q$-Analogue of Lagrange Multiplier

In order to identify the Lagrange multipliers of the $q$-difference equations, we first establish the correctional functional for (1.1) as

$$x_{n+1} = x_n + \int_0^t \lambda(t, q^2 s) \left( \frac{d_q^2}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s. \quad (3.1)$$

The correction functional here is different from the one in ordinary calculus since the parameter $q$ “disappears” after the integration by parts (2.5) each time. As a result, we use $\lambda(t, q^2 s)$ in the above functional.

We only need to consider the leading term $(d_q^2/d_q t^2)x$ when other terms are restricted variations in (1.1)

$$x_{n+1} = x_n + \int_0^t \lambda(t, q^2 s) \left( \frac{d_q^2}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s. \quad (3.2)$$

Through the integration by parts (2.5), we can have

$$\delta x_{n+1} = \left( 1 - q \left. \frac{\partial}{\partial qs} \lambda(t, s) \right|_{s=t} \right) \delta x_n + \lambda(t, qs) \left. \delta x_n \right|_{s=t} \delta x'_n - q \int_0^t \left. \frac{\partial^2}{\partial qs^2} \lambda(t, s) \delta x_n d_q s, \quad (3.3)$$

where $\delta$ is the variation operator and “$'$” denotes the $q$-derivative with respect to $t$. As a result, the system of the Lagrange multiplier can be obtained:

- the coefficient of $\delta x_n : 1 - q \left( \partial_q / \partial qs \right) \lambda(t, s) \big|_{s=t} = 0$,
- the coefficient of $\delta x'_n : \lambda(t, qs) \big|_{s=t} = 0$,
- the coefficient of $\delta x_n$ in the $q$-integral: $q (\partial_q^2 / \partial qs^2) \lambda(t, s) = 0$,

from which we can get

$$\lambda(t, s) = q^{-1}(s - tq), \quad (3.4)$$

instead of $\lambda(t, s) = s - t$ in [4]. More introductions to the identification of various Lagrange multipliers of the VIM can be found in [9, 10].

We also can show the above $q$-analogue of Lagrange multiplier’s validness. For $0 < q < 1$, let $T_q$ be the time scale: $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$, where $\mathbb{Z}$ is the set of positive integers. For the real continuous function $u(t) : T_q \rightarrow \mathbb{R}$, a $q$-oscillator equation of second order is

$$\frac{d_q^2}{d_q t^2} u - u = 0, \quad u(0) = 1, \quad \left. \frac{d_q}{d_q t} u \right|_{t=0} = 1. \quad (3.5)$$
From (3.4), the iteration formula can be given as

\[ u_{n+1} = u_n + \int_0^t q^{-1} \left( q^2 s - t q \right) \left[ \frac{d^2}{dq^2} u_n(s) - u_n(s) \right] ds. \]  

Starting from the initial iteration \( u_0 = 1 + t/[1]_q ! \), the successive approximate solutions can be obtained as

\[ u_0 = 1 + \frac{t}{[1]_q !}, \]
\[ u_1 = 1 + \frac{t}{[1]_q !} + \frac{t^2}{[2]_q !}, \]
\[ \vdots \]
\[ u_n = \sum_{k=0}^{2n+1} \frac{t^k}{[k]_q !}. \]  

The limit \( u = \lim_{n \to \infty} u_n = e_q(t) \) is an exact solution of (3.5). Here \( e_q(t) \) is one of the \( q \)-exponential functions.

4. Conclusions

In the past ten years, the VIM has been one of the often used nonlinear methods. The \( q \)-derivative is a deformation of the classical derivative and it has played a crucial role in quantum mechanics and quantum calculus. In this study, the method is successfully extended to \( q \) difference equations of second order. A \( q \)-analogue of Lagrange multiplier is presented. Readers who feel interested in the initial value problems of the \( q \) difference equations are referred to [11–17].

References


