Homotopy analysis method for boundary layer flow and heat transfer over a permeable flat plate in a Darcian porous medium with radiation effects

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Abstract

The aim of this article is to extend recent work of Dayyan et al. [24] in the case of radiative plate. We also introduced another parameter namely Darcy number which characterizes the porous medium. The governing equations (for boundary layer flow and heat transfer over permeable plate in Darcy porous medium) are transformed into a system of nonlinear ordinary differential equations using similarity transformations available in the literature. Approximate analytical solutions for the dimensionless velocity, temperature, friction factor and heat transfer rates are obtained using homotopy analysis method (HAM). Convergence of the HAM solution is discussed in detail. The effects of important controlling parameters on the dimensionless velocity, temperature, friction factor and heat transfer rates are analyzed graphically. It is shown that the friction factor increases with Darcy number, whereas the heat transfer rate decreases with the radiation–conduction parameter and increases with Darcy number.

1. Introduction

The boundary layer flow and heat transfer of a viscous fluid along flat surfaces has been used in several technological processes. Examples include metal extrusion, hot rolling, continuous stretching of plastic films and glass-fiber, polymer extrusion, wires drawing and metal spinning [1]. Sakiadis [2] is the pioneer for investigation of boundary layer flow over a surface which is moving with a constant velocity and also developed the boundary layer equation for axisymmetric flows in two-dimensions. Advancement was made by the addition of suction and injection at a surface moving with constant velocity and its effects were investigated by Erickson et al. [3]. Many researchers have examined the physical phenomena related to stretching/shrinking sheet under various thermal conditions [4–13] among others.

Some significant applications of radiative heat transfer include MHD accelerators, high temperature plasmas, power generation systems and cooling of nuclear reactors. The phenomenon of heat transfer within a porous medium has attracted the attention of many researchers due to its wide applications in industries and reservoirs. The literature is rich for describing the important characteristics of boundary-layer flow in a porous medium [14–26].

In general, porous medium are used for transport and storage of energy. Analysis of flow through a porous medium has become the core of several scientific and engineering applications. This type of flow is important in a wide range of technical problems, such as flow through packed beds, environmental pollution, centrifugal separation of particles, and blood rheology [27]. Rashad [28] investigated the radiative effects on heat transfer from the stretching surface in a porous medium, whereas, thermal radiation effects on unsteady mixed convection flow and heat transfer over a stretching surface in porous medium were studied by Mukhopadhyay [29], while Ayub et al. [30] provided the exact solution for third grade fluid past a porous plate by means of homotopy analysis method. Porous medium has been the focus of many studies during the last two decades because of its applications in various fields [31–35].

Homotopy analysis method (HAM), proposed by Liao [36–40], is a very powerful method and has been employed by numerous researchers in various physical phenomena [41–46]. In this paper, we extend the results of the model presented by Dayyan et al. [24]
for radiative surface in the porous medium. We shall apply HAM to solve the similarity equations obtained from the governing boundary layer equations with the help of similarity transformations.

The structure of the paper is as follows: The problem formulation and quantities of physical interest are presented in Section 2. HAM solution for the proposed problem is presented in Section 3. In Section 4 we provide the convergence of the HAM solution. Results and discussion are reported in Section 5, whereas, Section 6 is reserved for concluding remarks.

2. Governing equations

We consider steady, two dimensional flow in a Darcian porous medium of permeability $K_p$, past a flat plate, with velocity $u_0$. We assume that the transport properties of the medium are independent of temperature when the difference between ambient and wall temperatures is not deemed to be significant. The wall is stretching along $x$-axis and $y$-axis is taken as perpendicular to the surface origin and is being kept fixed. The coordinate system and the flow model are shown in Fig. 1. Under these assumptions and boundary layer approximations, the boundary layer equations, in dimensional form, can be written as [22]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\mu}{K_p} u \right).$$

The hydrodynamic boundary conditions are

$$u(x) = u_w(x) = u_0 \frac{y}{L}, \quad v = v_w \text{ at } y = 0,$$

$$u \to 0 \text{ as } y \to \infty.$$  

Here $L$ is the characteristic length of the porous plate. The thermal boundary conditions are

$$T = T_w \text{ at } y = 0 \quad \text{and} \quad T \to T_\infty \text{ as } y \to \infty.$$  

For nondimensionalized forms of momentum and energy equations, we introduce dimensionless variables [24]:

$$\eta = \frac{y}{\sqrt{K_p}}, \quad \psi = \frac{u}{L} \sqrt{K_p f(\eta)}, \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}.$$  

Using Eq. (9), the momentum and energy equations can be reduced to ordinary differential equations:

$$f'''' + Re Da \left( f'''' - f''' \right) - f'' = 0,$$

$$\left( 1 + \frac{4}{3} R \right) f'''' + Pe Da \ f'' = 0.$$  

The transformed boundary conditions are:

$$f(0) = f_w, \quad f'(0) = 1, \quad f''(0) = 0, \quad \theta(0) = 1, \quad \theta(\infty) = 0.$$  

where $f_w = -Lu_0/L\sqrt{K_p}$ is the suction/injection parameter, $Pr = v/\alpha$ is Prandtl number, $Re = u_wL/\nu$ is Reynolds number, $Da = K_pL^2$ is Darcy number for Darcian porous media, $Pe = Re Pr$ is Péclet number, and $R = 4\alpha T_w^3/k is the radiation–conduction parameter.

Quantities of physical interest are the local friction factor, $C_{f_x}$ and the local Nusselt number, $Nu_x$. Physically, $C_{f_x}$ represents the dimensionless wall shear stress, $Nu_x$ defines the dimensionless heat transfer rates.

These quantities can be determined from the following relations:

$$C_{f_x} = \frac{2\mu}{\rho u_{w}^2} \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad Nu_x = \frac{-x}{f_w - f_\infty} \left( \frac{3T}{\nu} \right)_{y=0}.$$
Using Eq. (9) to obtain the dimensionless forms of skin-friction coefficient and Nusselt number,
\[
\frac{1}{2\text{Re}_x}C_h\frac{\partial u}{\partial y} = f'(0), \quad Nu_x\frac{\partial u}{\partial y} = \theta'(0),
\] (13)
where $\text{Da}_x = K_x/\alpha_x^2$ is the local Darcy number for Darcian porous media and $\text{Re}_x = u_x\alpha_x/\nu$ is the local Reynolds number.

The analytical solution of the semi-coupled nonlinear system consisting of Eqs. (10) and (11) will be obtained by employing HAM in the next section.

3. Solution via homotopy analysis method

The dimensionless velocity $f(\eta)$ and temperature $\theta(\eta)$ can be expressed by the set of base functions
\[
(n^\eta\exp(-n\eta))[k \geq 0, n \geq 0],
\] (14)
in the form of following series
\[
\begin{aligned}
&f(\eta) = \frac{\partial \alpha_0}{\partial \eta} + \sum_{n=0}^{\infty} \sum_{k=0}^{N_{\infty}} a_{nk} \eta^k \exp(-n\eta) \\
&\theta(\eta) = \sum_{k=0}^{N_{\infty}} b_{k} \eta^k \exp(-n\eta)
\end{aligned}
\] (15)
where $a_{nk}, b_{k}$ are the coefficients. Following [37], we choose the initial guesses $f_0(\eta), \theta(\eta)$ based on boundary condition (11) and linear operators $L_1$ and $L_2$ in the following way
\[
f_0(\eta) = 1 + \frac{f_w}{e^\eta}, \quad \theta_0(\eta) = e^{-\eta},
\] (16)
\[
L_1(f) = \frac{d^2 f}{d\eta^2} - \frac{d f}{d\eta}, \quad L_2(\theta) = \frac{d^2 \theta}{d\eta^2}.
\] (17)
The operators $L_1, L_2$ have the following properties:
\[
L_1(C_1 + C_2 e^{-\eta} + C_3 e^\eta) = 0, \quad L_2(C_4 e^{-\eta} + C_5 e^\eta) = 0,
\] (18)
where $C_i (i = 1-5)$ are arbitrary constants. Let $q \in [0,1]$ represents an embedding parameter and $h \neq 0$ be the auxiliary parameter to adjust the convergence rate of the perturbation series. We construct the following zeroth order deformation of the problem
\[
(1 - q)L_1(f_0(\eta) - f(\eta)) = q h L_1(f_0(\eta)),
\] (19)
\[
(1 - q)L_2[\theta_0(\eta) - \theta(\eta)] = q h L_2(\theta_0(\eta)),
\] (20)
subject to the conditions
\[
f(0; q) = f_w, \quad \bar{f}'(0; q) = 1, \quad \bar{f}'(\infty; q) = 0, \quad \bar{\theta}(0; q) = 1, \quad \bar{\theta}(\infty; q) = 0,
\] (21)
where the non-linear operators are defined as
\[
N_1 = \frac{d^2 f(\eta; q)}{d\eta^2} + Re Da \left[ \frac{d^2 f(\eta; q)}{d\eta^2} \left( \frac{\partial f(\eta; q)}{\partial \eta} \right)^2 \right] - \frac{\partial f(\eta; q)}{\partial \eta},
\] (23)
\[
N_2 = \left( 1 + \frac{4R}{3} \right) \frac{d^2 \theta(\eta; q)}{d\eta^2} + Pe Da \left[ \frac{d \theta(\eta; q)}{d\eta} \right]
\] (24)

For $q = 0$ and $q = 1$ we have
\[
\hat{f}(q; \eta; 0) = f_0(q; \eta), \quad \hat{f}(q; \eta; 1) = f(q; \eta).
\] (25)
\[
\hat{\theta}(q; \eta; 0) = \theta_0(q; \eta), \quad \hat{\theta}(q; \eta; 1) = \theta(q; \eta).
\] (26)

Defining
\[
(27) f_m(\eta) = \frac{1}{m} \frac{d^m f(\eta; q)}{d\eta^m} \bigg|_{q=0}, \quad \theta_m(\eta) = \frac{1}{m} \frac{d^m \theta(\eta; q)}{d\eta^m} \bigg|_{q=0}
\] and expanding $f(\eta; q), \theta(\eta; q)$ by means of Taylor’s theorem with respect to $q$, we obtain
\[
\hat{f}(q; \eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) q^m,
\] (28)
\[
\hat{\theta}(q; \eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) q^m.
\] (29)

The auxiliary parameters are properly chosen so that series (28) and (29) converge at $q=1$ and thus
\[
f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta),
\] (30)
\[
\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta).
\] (31)

The resulting problems at the $m$th-order deformation are
\[
L_1(f_m(\eta)) - \chi_m f_{m-1}(\eta) = h f'_m(\eta),
\] (32)
\[
L_2(\theta_m(\eta)) - \chi_m \theta_{m-1}(\eta) = h \theta'_m(\eta),
\] (33)
\[
f_m(0) = 0, \quad f'_m(0) = 0, \quad f'_m(\infty) = 0, \quad \theta_m(0)
\] = 0, \quad \theta_m(\infty) = 0,
\] (34)
\[
R_m = \int_{m-1}^{m} + Re Da \int_{m-1}^{m} \left( \sum_{k=0}^{m-1} f_k' f_{m-1-k}' \right) - f_m.
\] (35)
\[
R_m' = \left( 1 + \frac{4R}{3} \right) \int_{m-1}^{m} + Pe Da \int_{m-1}^{m} f_k' \theta_m(\eta; q) \right],
\] (36)
\[
\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}
\] (37)
The general solution of Eqs. (32) and (33) is
\[
f_m(\eta) = f'_m(\eta) + C_1 + C_2 e^{-\eta} + C_3 e^\eta,
\] (38)
\[
\theta_m(\eta) = \theta'_m(\eta) + C_4 e^{-\eta} + C_5 e^\eta,
\] (39)
where $f'_m(\eta)$ and $\theta'_m(\eta)$ are the particular solution and the constants are to be determined by the boundary condition Eq. (34).
The residual error for the HAM 25th-order approximation solution are defined as

\[
\text{Res}_f = \frac{d^2 f}{dh^2} + Re \cdot Da \left( f \frac{d^2 f}{dh^2} \right) - \frac{df}{dh},
\]

\[ (40) \]

\[
\text{Res}_\theta = \left( 1 + \frac{4}{3} h \right) \frac{d^2 \theta}{dh^2} + Pe \cdot Da \left( f \frac{d^2 \theta}{dh^2} \right).
\]

\[ (41) \]

4. Convergence of homotopy solution

Liao [36–38] has mentioned that the convergence rate of approximation of the HAM solution is strongly dependent upon the values of non-zero auxiliary parameter \( \eta \). As Eqs. (38) and (39) involve \( h_f \) and \( h_0 \), so we can adjust the convergence of our HAM solution. To compute the range of admissible values of \( h_f \) and \( h_0 \), we display the \( h \)-curves of the function \( f'(0) \) and \( \theta'(0) \) for different orders of approximations. Fig. 2 depicts that range for the admissible values of \( h_f \) and \( h_0 \) is \(-1.6 < h_f < -0.3 \), and \(-1.5 < h_0 < -0.3 \). Fig. 2 shows the \( h \)-curves for the dimensionless velocity and temperature. Convergence of the series solution up to 50th order of approximations is presented in Table 1. It is found from Table 1 that the convergence is achieved up to 25th order of approximation.

In order to choose the optimal value of auxiliary parameter \( \eta \), the average residual error is introduced as (see Ref. [25], for more details):

\[
\Delta f_m = \frac{1}{K} \sum_{j=0}^{K} \text{Res}_f \left( \sum_{j=0}^{m} f_j(i\Delta x) \right)^2.
\]

\[ (42) \]

\[
\Delta \theta_m = \frac{1}{K} \sum_{j=0}^{K} \text{Res}_\theta \left( \sum_{j=0}^{m} \theta_j(i\Delta x) \right)^2.
\]

\[ (43) \]

where \( \Delta x = 10/K \) and \( K = 0 \). For the given order of approximation \( m \), the optimal value of \( \eta \) is given by the minimum values of \( \Delta f_m \) and \( \Delta \theta_m \) corresponding to nonlinear algebraic equations

\[
\frac{d\Delta f_m}{dh} = 0, \quad \frac{d\Delta \theta_m}{dh} = 0.
\]

\[ (44) \]

For example, in order to find the optimal values of \( h \), the residual error which is displayed in Eqs. (40) and (41), for the HAM 25th-order approximation solutions is presented in Fig. 3.

5. Results and discussion

In order to analyze the physical problem, HAM computations were carried out for various values of controlling parameters such as the suction/injection parameter \( f_w \), radiation–conduction parameter \( R \), Reynolds number \( Re \), Péclet number \( Pe \) and Darcy number \( Da \). Figs. 4–14 are plotted to illustrate the effect of controlling parameters on the flow field and heat transfer characteristics. Tables 2 and 3 are presented to show the effect of Darcy number and radiation–conduction parameter on skin friction factor and Nusselt number. In Table 4, we compared the HAM solution with the numerical results in [7–9] and an excellent agreement is observed.

**Table 1**

<table>
<thead>
<tr>
<th>Order of approximations</th>
<th>( f'(0) )</th>
<th>( \theta'(0) )</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>-1.30600</td>
<td>-0.91760</td>
</tr>
<tr>
<td>5</td>
<td>-1.48781</td>
<td>-0.85442</td>
</tr>
<tr>
<td>10</td>
<td>-1.49218</td>
<td>-0.84623</td>
</tr>
<tr>
<td>15</td>
<td>-1.49222</td>
<td>-0.84537</td>
</tr>
<tr>
<td>20</td>
<td>-1.49222</td>
<td>-0.84526</td>
</tr>
<tr>
<td>25</td>
<td>-1.49222</td>
<td>-0.84524</td>
</tr>
<tr>
<td>30</td>
<td>-1.49222</td>
<td>-0.84524</td>
</tr>
<tr>
<td>40</td>
<td>-1.49222</td>
<td>-0.84524</td>
</tr>
<tr>
<td>50</td>
<td>-1.49222</td>
<td>-0.84524</td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>Re</th>
<th>( f_w )</th>
<th>Runge–Kutta [24]</th>
<th>Present results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3</td>
<td>1.2721</td>
<td>1.27214</td>
</tr>
<tr>
<td>0</td>
<td>-0.3</td>
<td>1.4242</td>
<td>1.41425</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5721</td>
<td>1.57213</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>1.5811</td>
<td>1.58112</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2.4494</td>
<td>2.44946</td>
<td></td>
</tr>
</tbody>
</table>

**Fig. 2.** Combined \( h \)-curves for \( f'(0) \) and \( \theta'(0) \) at 12th order of approximations.
We now discuss the results obtained by using HAM method. In Figs. 4 and 5, the effects of Darcy number on the dimensionless velocity and temperature is displayed respectively. With an increase in Darcy number, the dimensionless velocity decreases while the dimensionless temperature increases with the Darcy number. The influences of suction/injection parameter \( f_w \) on the dimensionless velocity are presented in Fig. 6. The dimensionless velocity decreases with an increase in the transverse distance \( \eta \). It is further found that the increasing value of suction/injection parameter, the dimensionless velocity is in decreasing order. The dimensionless velocity for different values of Reynolds number \( Re \) is displayed in Fig. 7. It is noticed that the hydrodynamic boundary layer thickness decreases with the increase in Reynolds number.

In Fig. 8, we depict the effects of Reynolds number and radiation–conduction parameter on the dimensionless temperature for the fixed values of \( \text{Pr}, Da \) and \( f_w \). It is noticed that, with the increasing values of \( Re \), the thermal boundary layer thickness increases. The dimensionless temperature increases with an increase in the radiation parameter. In Fig. 9, we present and highlight the influence of Péclet number \( Pe \) and suction/injection parameter \( f_w \) on the dimensionless temperature. In the absence of suction, the dimensionless temperature is found to be higher with larger thermal boundary layer thickness and decreases with increasing suction parameter. Moreover, an increase in Péclet number decreases the dimensionless temperature monotonically. As a result, the thermal boundary layer thickness decreases and heat transfer rate increases. In Fig. 10, we depict the influences of suction/injection parameter along with Reynolds and Darcy numbers on the dimensionless velocity gradient. It is observed that, in the beginning, the velocity gradient shows a decrease but after some transverse distance \( \eta \), it shows an increase.

Fig. 11 is displayed to show the variation of skin friction with Reynolds number \( Re \) for different values of suction/injection parameter. It is found that skin friction monotonically increases with an increase in \( Re \) and the similar character of skin-friction factor can be observed for suction/injection parameter at the boundary.

The variation of the Nusselt number with Reynolds number, radiation, and suction parameters is presented in Figs. 12 and 13. Fig. 12 presents the behavior of heat transfer rates against radiation parameter for different values of Reynolds numbers and suction parameter. The heat transfer rates decrease with an increase in Reynolds number and radiation–conduction parameter. In the absence of suction, the heat transfer rate at the surface is found to be smaller and increases with suction. Fig. 13 exhibits the

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Darcy effect on Skin-friction factor (-( f' )) and heat transfer rate (-( \theta' )) for ( Pr = 1.5, Re = 2, f_w = 0.3, R = 0.2 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Da</td>
<td>( f' ) (0)</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.2</td>
<td>1.24472</td>
</tr>
<tr>
<td>0.4</td>
<td>1.46698</td>
</tr>
<tr>
<td>0.5</td>
<td>1.57213</td>
</tr>
<tr>
<td>0.8</td>
<td>1.87022</td>
</tr>
<tr>
<td>1</td>
<td>2.05783</td>
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</table>

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Local Nusselt number (-( \theta' )) for various values of ( Pr ) at ( Re = 1, R = 0, Da = 1 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.58199</td>
</tr>
<tr>
<td>3</td>
<td>1.16523</td>
</tr>
<tr>
<td>10</td>
<td>2.30796</td>
</tr>
</tbody>
</table>

Fig. 4. The effects of Reynolds and Darcy numbers on dimensionless velocity.

Fig. 5. Effects of Darcy and Péclet numbers on dimensionless temperature.

Fig. 6. Effects of suction/injection parameter and Darcy number on dimensionless velocity.
Fig. 7. Effects of Reynolds number and suction on dimensionless velocity.

Fig. 8. Effects of Reynolds number and radiation parameter on dimensionless temperature.

Fig. 9. Effects of Péclet number and suction parameter on dimensionless temperature.

Fig. 10. Effects of suction/injection parameter on dimensionless velocity gradient.

Fig. 11. Variation of skin friction coefficient with Reynolds number and suction/injection.

Fig. 12. Variation of heat transfer rates with radiation, Reynolds number and suction parameter.
methods. The present solutions have shown that the velocity decreases and temperature increases with both Reynolds and Darcy numbers. Increasing Péclet number Pe reduces the boundary layer thickness and cool the surface. The absolute value of friction factor increases with Darcy number for both radiating and non-radiating solid plate. The rate of heat transfer decreases with radiation–conduction parameter for both clear and Darcy porous media.

References


6. Conclusion

We have examined theoretically the boundary layer flow and heat transfer along a permeable flat plate in a porous medium with radiation and suction/injection. The transformed two-point nonlinear boundary value problem has been solved with semi-analytical homotopy analysis method (HAM). Good agreement of HAM solution is observed with those obtained by numerical

![Fig. 13. Variation of heat transfer rates with suction/injection, Péclet number and radiation parameter.](image-url)

![Fig. 14. Effects of radiation and Darcy number on dimensionless temperature gradient.](image-url)


