Astrophysical Mechanisms for Pulsar Spindown

Eric Addison, Utah State University

Available at: https://works.bepress.com/eric_addison/7/
Astrophysical Mechanisms for Pulsar Spindown
PhD Candidacy Exam Report

Eric Addison∗
Department of Physics, Utah State University, Logan, UT 84322

Pulsars are astrophysical sources of pulsed electromagnetic radiation. The pulses have a variety of shapes in the time-domain, and the pulse energy generally peaks in the radio spectrum. The accepted models theorize that pulsars are rapidly rotating neutron stars with strong dipolar magnetic fields. Current models predict that rotational kinetic energy is extracted from the pulsar in the form of electromagnetic and gravitational radiation, causing it to slowly lose rotational speed, or “spin down”. This spindown can be characterized by a single value $n$, known as the braking index. This report will review basic characteristics of pulsars, including short treatments of the magnetic dipole and gravitational quadrupole radiation models. Recent research involving the braking index will be examined, as well as an application to gravitational wave astronomy with LIGO.

I. INTRODUCTION

Pulsars were discovered in 1967 as astrophysical sources of pulsed radio emissions. An identification of pulsars as highly magnetized, rapidly rotating neutron stars followed shortly thereafter, since theoretical progress had already been made in this area. Despite their discovery having occurred more than forty years ago, pulsars continue to present a rich arena for scientific research. Current research involving pulsars ranges through many sub-disciplines of physics including observational astronomy [1], solid state physics [2], plasma physics [3], high-energy physics [4], and relativity theory [5]. Clearly, pulsars should be considered an interesting subject from a scientific perspective.

This report will proceed with a short summary of basic information about pulsars in section II, followed by developments of the two main models of pulsar spindown in section III. Methods for combining these models will be addressed in section IV, followed by an application to gravitational wave astronomy in section V.

II. PULSAR BASICS

A. History

Before the discovery of pulsars, radio astronomers generally employed antenna arrays with time resolutions of seconds or more in order to smooth out noise and interference. In 1967, the Mullard Radio Astronomy Observatory at Cambridge University was put into operation. This was an 81.5 Mhz radio antenna array with time constants of $\sim 0.1$ seconds, allowing for much finer time resolution. The purpose of the array was to study interplanetary scintillation, but quite unexpectedly detected clock-like, pulsed radio sources. The discovery was made at this observatory by Anthony Hewish and Jocelyn Bell, and first reported in the journal Nature in 1968 [6]. This discovery provoked a flurry of theoretical and observational papers over following years, and key details concerning the nature of pulsars were quickly uncovered.

B. Identification

Today the identification of pulsars with rapidly rotating, magnetized neutron stars is ubiquitous. Though this model was well established within a year of discovery, there was an initial outburst of theoretical work attempting to identify the origin of these signals [7–14].

Of the first 100 pulsars discovered, only two had pulsation periods smaller than 100ms [15]. This prompted researches to consider several possible mechanisms for the pulsed signals. These mechanisms included: orbital motion, stellar pulsations, and stellar rotation.

For the case of orbital motion, first appeal to Kepler’s third law of orbital motion

$$ P = 2\pi a^{3/2}(GM)^{-1/2} $$

The orbital period is inversely related to the orbital energy through the semi-major axis $a$, and so changing the period requires changing the energy of the binary. One possible mechanism for extracting energy from the binary is through emission of gravitational waves. The average rate of change of the semi-major axis due to gravitational radiation is given by [16]

$$ \langle \frac{da}{dt} \rangle = \frac{64}{5} \frac{G^3 m_1 m_2 (m_1 + m_2)}{c^5 a^{7/2} (1 - e^2)^2} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right) $$

where the angle brackets denote an average over one orbital period. We can now compute

$$ \langle \frac{dP}{dt} \rangle = \frac{d}{dt} \left( \frac{2\pi a^{3/2}}{(GM)^{1/2}} \right) = \pi (GM)^{-1/2} a^{1/2} \langle \frac{da}{dt} \rangle $$

Which is explicitly negative for elliptical orbits, predicting a decrease in orbital period over time. This is in direct conflict with the consistently observed increase in

∗Electronic address: eric.addison@aggiemail.usu.edu
pulsar periods, so we can rule out the possibility of orbital motion evolving via gravitational radiation as the mechanism for the pulses.

Stellar pulsations can be ruled out by a detailed analysis of the normal modes of oscillation for white dwarfs and neutron stars. A rough estimate of pulsation period can be found by calculating the time it takes for a sound wave to cross the diameter of a star with radius $R$ and constant density $\rho$. The adiabatic sound speed in a gas is [17]

$$v_s = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma P}{\rho}} \tag{4}$$

where $B$ is the bulk modulus of the gas, $P$ is the pressure, and $\gamma$ is the ratio of specific heats. Using the equations of hydrostatic equilibrium, taking simple state variables $\gamma_{WD} = 4/3$ and $\rho_{WD} = 10^{10}$ kg/m$^3$ for a white dwarf, and $\gamma_{NS} = 1$ and $\rho_{NS} = 10^{16}$ kg/m$^3$ for a neutron star [18], we can find estimates of the crossing time to be

$$P_{WD} \approx 7 \text{ s} , \quad P_{NS} \approx 3 \text{ ms}$$

This estimate reveals that white dwarfs have pulsation periods that are too high to be pulsars, while neutron stars have pulsation periods that are too low. So we discount the possibility of stellar pulsations as a mechanism for the pulses.

The third option, rotation, can be checked as follows. The angular momentum of the sun is approximately $J_\odot \approx 1.16 \times 10^{42}$ kg m$^2$s$^{-1}$. If we assume (unrealistically) that the sun collapses to a neutron star and angular momentum is conserved, we find that the neutron star will be left with a rotational period of $P_{NS} \approx 6.3 \times 10^{-4}$ s. This period is much smaller than any observed pulsars, but if we allow for angular momentum to be lost in a supernova explosion, then this could certainly account for the range of observed pulsar periods.

White dwarfs, on the other hand, will reach breakup velocity when a rotational speed of $P_{WD} \approx 7.4$ s, implying that rotating white dwarfs cannot be the origin of the radio pulses.

The surviving possibility is that of a rotating neutron star. By 1969, continued observations and theoretical work supported this model, and it quickly gained general acceptance.

### C. Emission

The shape and spectrum of pulsar pulses present incredibly difficult and interesting problems. The time-domain shape of pulses can vary widely, possibly containing some number of sub-pulses. The spectrum of the pulses can range from 10’s of Mhz in the radio band well into the 10 GeV $\gamma$-ray range [19]. Discontinuities in the spectrum pose difficulties for developing an all encompassing model. As it stands, there is no generally accepted, detailed model for the emission mechanism of pulsars, and research in this area is ongoing.

The basic idea of emission held by many pulsar researchers is that charged particles stream out of the pulsar along magnetic field lines close to the magnetic poles [20]. These field lines do not close on themselves and allow the streaming particles to reach ultra-relativistic speeds. Accordingly, it is generally accepted that the emission is due to some combination of cyclotron, synchrotron, and curvature radiation, as well as inverse Compton scattering.

### III. PULSAR SPINDOWN MODELS

Early accurate observations of pulsars revealed an interesting characteristic: pulsar rotation periods tend to increase. This increase in period is often referred to as pulsar spindown. If we assume that the energy in the pulses is derived from rotational energy, then spindown is a natural consequence. Deceleration models of pulsars can often be described as a power law [21], i.e.

$$\dot{\Omega} = -k\Omega^n \tag{5}$$

where $\Omega$ is the angular frequency of rotation, an over-dot represents differentiation with respect to time, and $k$ is a model dependent constant. A few example power laws are plotted in figure 1.

![Log-Log Power Law Plots](image)

**FIG. 1:** Log-Log power law plots for $n = 3, 5, 2.5$

By taking a derivative of Eq. (5), we find

$$\ddot{\Omega} = -kn\dot{\Omega}\Omega^{n-1} = \frac{n\dot{\Omega}^2}{\Omega} \tag{6}$$

which, upon solving for $n$, gives:

$$n = \frac{\Omega\dot{\Omega}}{\dot{\Omega}^2} \tag{7}$$
The number \( n \) is referred to as the braking index, since it’s role as the exponent in the power law characterizes the deceleration. In principle, the braking index is measurable from observations of the pulsar frequency and it’s derivatives.

In this section, I will present two models which lead to deceleration power laws, and hence predict theoretical braking indices. These models are the magnetic dipole model and the gravitational quadrupole model.

### A. Magnetic Dipole Model

The dominant model for the current mode of energy dissipation from a pulsar is the magnetic dipole model. In this model, the pulsar is presumed to possess a purely dipolar magnetic field. The axis of the dipole need not (and should not) line up with the spin axis. The spinning motion then changes the direction of the magnetic pole, creating a time dependent magnetic field, and hence emits electromagnetic radiation. Figure 2 shows a cartoon of the pulsar and dipole.

Convenient coordinates used for pulsar calculations can be defined by \( \mathbf{e}_\parallel \), a unit vector parallel to the spin axis, and two orthonormal unit vectors \( \mathbf{e}_\perp \) and \( \mathbf{e}_\perp' \) set in the plane of the pulsar equator. Figure 3 displays this coordinate system, where \( R \) is the radius of the neutron star, \( \vec{m} \) is the magnetic dipole axis and \( \alpha \) is the misalignment between \( \vec{m} \) and \( \mathbf{e}_\parallel \).

It can be shown \([22]\) from the Larmor formula for the power radiated by an accelerated point charge, that the power lost by a radiating electric dipole is

\[
P_{ed} = \frac{\mu_0 \dot{p}^2}{6\pi c} \quad (8)
\]

where \( E \) is the energy of the system and \( \vec{p} \) is the electric dipole moment. Then, with the replacement \( \dot{p} \to \vec{m}/c \) \([23]\), we have the power radiated from a magnetic dipole

\[
P_{md} = \frac{\mu_0 \ddot{m}^2}{6\pi c^3} \quad (SI) \rightarrow \frac{2\ddot{m}^2}{3c^3} \quad (cgs) \quad (9)
\]

It can also be shown, from the magnetic field of a pure dipole

\[
\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \left[ 3\hat{n}(\hat{n} \cdot \vec{m}) - \vec{m} \right] \quad (10)
\]

where \( \hat{n} \) is a unit vector in the direction of \( \vec{x} \), that the magnitude of the dipole can be given by

\[
m = \frac{B_p R^3}{2} \quad (cgs) \quad (11)
\]

where \( B_p \) is the magnetic field strength at the magnetic pole, and \( R \) is the pulsar radius.

If we write out \( \vec{m} \) in the pulsar coordinates of figure 3, we find

\[
\vec{m} = \frac{1}{2} B_p R^3 \left( \sin \alpha \cos \Omega t \mathbf{e}_\perp + \sin \alpha \sin \Omega t \mathbf{e}_\perp' + \cos \alpha \mathbf{e}_\parallel \right) \quad (12)
\]

Taking two derivatives and the magnitude, we see that

\[
\ddot{m} = \frac{1}{2} B_p R^3 \sin^2 \alpha \quad (13)
\]

Substituting Eq. (13) into Eq. (9), we find the power radiated by the spinning dipole is

\[
P = -\dot{E} = \frac{B_p^2 R^6 \Omega^4 \sin^2 \alpha}{6c^3} \quad (14)
\]

where \( E \) is the energy of the pulsar, or

\[
\dot{E} = -\frac{2m^2 \Omega^4}{3c^3} \quad (15)
\]
where \( m_\perp = m \sin \alpha \) is the component of the dipole moment perpendicular to the spin axis.

We now make the standard assumption that the energy lost by the pulsar is lost directly from the rotational kinetic energy, i.e.

\[
E_{\text{rot}} = \frac{1}{2} I \Omega^2 \Rightarrow \dot{E} = I \dot{\Omega}^2
\]  

(16)

where \( I \) is the moment of inertia of the pulsar. From here we make the identification

\[
I \dot{\Omega} = -\frac{2m_\perp^2 \Omega^4}{3c^3}
\]  

(17)

\[
\Rightarrow \dot{\Omega} = -\frac{2m_\perp^2}{3Ic^3} \Omega^3
\]  

(18)

This can be rewritten to look exactly like the power law in Eq. (5) if we set \( n = 3 \) and \( k = (2m_\perp^2)/(3Ic^3) \).

Archetypal values for pulsar quantities include \( R = 12 \) km, \( M = 1.4M_\odot \), and \( I = 1.4 \times 10^{45} \) g cm\(^2\) [21]. Using these values, \( B_p \) is calculated to be \( B_p \approx 5.2 \times 10^{12} \) G. It is reasonable to ask whether this strong magnetic field is plausible in an astrophysical sense. Intermediate and high-mass stars can achieve magnetic field strengths ranging from 100 G to 10 kG [24], and stellar radii on the order of \( R_\odot \approx 10^{11} \) cm. In highly conductive medium and high density plasmas, i.e. stellar interiors, magnetic field lines become “frozen-in” in the plasma [25]. This implies conservation of magnetic flux through an area moving with the plasma, i.e. a patch of stellar surface. If we imagine a star with surface magnetic field strength \( B_p = 100 \) G, then the amount of magnetic flux through a patch of surface \( dA \) will be

\[
d\Phi = B_i dA = B_i R_i^2 d\Omega
\]  

(19)

where \( R_i \) is the initial radius of the star. Then if we let the star collapse to a new radius \( R_f \), we have, through conservation of flux

\[
B_i R_i^2 d\Omega = B_f R_f^2 d\Omega \Rightarrow B_f = B_i \left( \frac{R_i}{R_f} \right)^2
\]  

(20)

For the values above, this implies a new surface field strength of \( B_f \approx 7 \times 10^{11} \) G. Since this estimate was at the low end of typical \( B \) field strengths, we see that pulsar fields on the order of \( B_p \sim 10^{12} \) G are realistic.

A important point to make about the energy loss \( \dot{E} \) is that this is electromagnetic energy being radiated at frequency \( \Omega \). The fastest pulsars observed have periods on the order of milliseconds, which corresponds to an EM frequency of \( f \sim kHz \), much lower than the observed frequencies in the actual pulses. The general consensus is that this energy is absorbed by charged particles, which are accelerated to high speeds and emit more energetic radiation as cyclotron, synchrotron, and curvature radiation.

### B. Gravitational Quadrupole Model

A second important model of pulsar energy loss is the gravitational quadrupole model in which rotational energy is carried away by gravitational waves.

It is well known that by adding a small perturbation to a flat space-time metric

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}
\]  

(21)

with \( |h_{\alpha\beta}| \ll 1 \), the Einstein equations can be written as the weak field Einstein equations

\[
\Box h_{\alpha\beta} = -16\pi T_{\alpha\beta}
\]  

(22)

where \( T_{\alpha\beta} \) is the stress energy tensor, \( h_{\alpha\beta} \) is the trace reversed version of \( h_{\alpha\beta} \), and \( \Box \) is the D’Alembertian operator. In addition, the equations are written in the Lorentz gauge imposing the divergence free condition \( h_{\alpha\beta,\nu} = 0 \). This is analogous to the gauge freedom for the vector potential in electromagnetism.

For the vacuum condition \( T_{\alpha\beta} = 0 \), these equations admit plane wave solutions of the form

\[
h_{\alpha\beta} = A_{\alpha\beta} \exp(ik_\alpha x^\alpha)
\]  

(23)

with the condition

\[
k_\alpha A^\alpha_{\beta} = 0
\]  

(24)

Further development of the linearized wave theory shows that the power radiated by a massive system through gravitational waves is

\[
\dot{E} = \frac{G}{5c^5} \left\langle \left( \tilde{T}_{ij} \tilde{T}^{ij} \right) \right\rangle
\]  

(25)

where \( \tilde{T}_{ij} \) is the trace-free mass quadrupole tensor, and the indices \( i, j \) run over the spatial dimensions. This says that the energy lost by gravitational radiation is related to the third time derivative of the mass quadrupole tensor. Systems with mass quadrupoles that do not change in time (e.g. axisymmetric rotating systems) do not radiate gravitational waves.

In order to assess the gravitational radiation of a pulsar, we must identify the source of asymmetry. A spinning ball of fluid will, in general, distort so that it becomes oblate, but this alone will not cause the system to emit gravitational radiation because the shape is still axisymmetric. With the addition of a dipole magnetic field, however, it can be shown [26–28] that an additional distortion will develop in the equatorial plane, characterized by a dimensionless quantity known as the equatorial ellipticity \( \epsilon_e \). The exact form of \( \epsilon_e \) depends on the details of the derivation, however the dependence on field strength is \( \epsilon_e \sim B^2_p \).

This equatorial ellipticity generates the asymmetry about the axis of rotation we need to have a time dependent quadrupole moment. Note here that the trace free quadrupole moment, defined by

\[
\tilde{F}_{ij} = \int \rho (x; x_j) - \frac{1}{3} \delta_{ij} r^2) d^3 x
\]  

(26)
and the moment of inertia tensor

\[ I_{ij} = \int \rho(\delta_{ij}r^2 - x_i x_j) d^3x \]  

are related by

\[ I_{ij} = -I_{ij} + \frac{1}{3} \delta_{ij} \text{Tr}(I) \]

Because of this, and because the trace \( \text{Tr}(I) \) is a constant, the \( I_{ij} \) in Eq. (25) can be replaced with \( I_{ij} \), essentially simplifying the calculation. The non-zero components of the inertia tensor in the body-frame of a spinning ellipsoid are

\[
\begin{align*}
I'_{xx} &= \frac{\mathfrak{m}}{5}(a_3^2 + a_2^2) \\
I'_{yy} &= \frac{\mathfrak{m}}{5}(a_1^2 + a_3^2) \\
I'_{zz} &= \frac{\mathfrak{m}}{5}(a_2^2 + a_1^2)
\end{align*}
\]

where \( \mathfrak{m} \) is the total mass of the ellipsoid, and \( a_i \) are the semi-axes. The calculation proceeds by defining a rigid coordinate system in which the inertia tensor is labelled \( I_{ij} \), and a coordinate system fixed to the ellipsoid with inertia tensor labelled \( I'_{ij} \). The two inertia tensors are related by

\[ I = R^T I' R \]

where \( R \) is a standard rotation matrix representing a rotation of the ellipsoid about the \( \vec{e}_3 \) axis by an angle \( \Omega t \). It is by this relation that the previous claim of \( \text{Tr}(I) \) being constant can be justified; simply note that \( \text{Tr}(I) = \text{Tr}(I') \) = constant. From here, straightforward calculations show that the only values of \( I_{ij} \) changing in time are \( I_{xx}, I_{xy} = I_{yx}, \) and \( I_{yy} \). The resulting expression for the energy loss is

\[ \dot{\mathcal{E}} = -\frac{32}{5} G \frac{\mathfrak{m}}{c^5} \Omega^5 (I_1 - I_2)^2 \]

where \( I_1 = I'_{xx} \) and \( I_2 = I'_{yy} \). For small ellipticities, \( a_1 \approx a_2, (I_1 - I_2)^2 \) can be rewritten as

\[
(I_1 - I_2)^2 \approx I_3^2 \left( \frac{a_1 - a_2}{(a_1 + a_2)/2} \right)^2
\]

Defining the quantity \( \epsilon = (a_1 - a_2)/[(a_1 + a_2)/2] \), we can write the final energy loss formula as

\[ \dot{\mathcal{E}} = -\frac{32}{5} G \frac{\mathfrak{m}}{c^5} \Omega^5 I_3^2 \epsilon^2 \]

where \( I \) have made the substitution \( I_3 = I \) for the moment of inertia of the pulsar about the spin axis. Note that the \( \epsilon \) in Eq. (32) is the same physical quantity as \( \epsilon_x \) previously discussed.

Once again, by writing \( \dot{\mathcal{E}} = \dot{H} \Omega \dot{\Omega} \), this corresponds to a deceleration power law of the form

\[ \dot{\Omega} = -k \Omega^3 \]

with \( n = 5 \) and \( k = 32 G I c^2/(5 c^5) \).

In this section we have computed the theoretical braking index for gravitational radiation to be \( n = 5 \), which is different from the magnetic dipole value of \( n = 3 \). In the next section we will combine these models in two different ways in order to form a more complete picture of pulsar spindown.

IV. COMBINED MODELS

It is reasonable to suspect that the total energy loss of a pulsar is due to a combination of electromagnetic and gravitational radiation. By combining the two models to reflect this notion, new properties of pulsars can be examined.

A. Separate Braking Indices and Pulsar Age Estimates

It would be useful if we could develop a way to estimate the age of a pulsar based on quantities we know or can observe. This can be done by combining the two spindown models. A good initial method for combining the two models is simply to add them:

\[ \dot{\Omega} = -k_m \Omega^3 - k_g \Omega^2 \]

where the constants

\[ k_m = \frac{2 m_0^2}{3 I c^3}, \quad k_g = \frac{32}{5} G I c^2 \]

With this combined model, we can make predictions about the age of a pulsar. Define the quantities

\[ \tau_m = k_m^{-1} \Omega_0^{-3}, \quad \tau_g = k_g^{-1} \Omega_0^{-4}, \quad x = \frac{\Omega_0}{\Omega}, \quad \eta = \frac{\tau_g}{\tau_m} \]

where \( \Omega_0 \) is the rotational frequency at some time which we will define at \( t = 0 \), and \( \Omega = \Omega(t) \). The physical interpretations of \( \tau_m \) and \( \tau_g \) are as characteristic times of the form

\[ \tau_m = \frac{\Omega_0 m}{\Omega_0 m}, \quad \tau_g = \frac{\Omega_0 g}{\Omega_0 g}, \]

i.e. the characteristic time if the spindown is due entirely to EM or gravitational radiation. The quantity \( x \) can be rewritten as \( x = P/P_0 \), a normalized rotational period. The equation of motion for \( x^2 \) is found to be

\[ \frac{d}{dt}(x^2) = \frac{2}{\tau_m} \left( 1 + \frac{1}{\eta x^2} \right) \]

which can be integrated up to some time \( t \), resulting in

\[ t = \frac{\tau_m}{2} \left[ x^2 - 1 - \frac{1}{\eta} \ln \left( \frac{1 + \eta x^2}{1 + \eta} \right) \right] \]
From this expression, we can estimate the time \( t_i \) when the pulsar was formed by assuming \( \Omega(t_i) \gg \Omega_0 \Rightarrow x \ll 1 \), then
\[
t_i \approx -\frac{\tau_m}{2} \left[ 1 - \frac{1}{2} \ln (1 + \eta) \right] \tag{40}
\]
We now form a true characteristic time \( \tau_0 \) by
\[
\tau_0 = -\left( \frac{\Omega}{\Omega_0} \right)_0 \tag{41}
\]
which is a directly observable quantity. From this and previous definitions, we can relate \( \tau_0 \) to \( \tau_m \) and \( \tau_g \) by
\[
\tau_0 = (\tau_m^{-1} + \tau_g^{-1})^{-1} \tag{42}
\]
Assuming that the neutron star was initially spinning much faster than current observations, we can estimate the age to be
\[
\text{age} = -t_i = \frac{\tau_0}{2} \left[ 1 + \frac{1}{2} \eta \right] \left[ 1 - \frac{1}{2} \eta \ln (1 + \eta) \right] \tag{43}
\]
That is, the age can be estimated by the two quantities \( \tau_0 \) and \( \eta \).

Current observations of the Crab pulsar PSR B0531+21 show that it has an angular frequency of \( \Omega_0 = 189.9120 \text{ s}^{-1} \) and a first derivative of \( \dot{\Omega}_0 = -2.426742 \times 10^{-9} \text{ s}^{-1} \) [15]. These numbers give a characteristic time of \( \tau_0 \approx 2482 \text{ yr} \). If we assume that the radiation is completely magnetic, then \( 1/\eta \to 0 \) and we find the age is given by \( \tau_0/2 \approx 1240 \text{ yr} \). The Crab pulsar is associated with the Crab nebula supernova remnant, which is known to have formed in the year 1054, giving it an actual age of 957 years. By setting the parameter \( \tau_g \) so that we get exactly the correct age, it is found that the parameter \( \eta \approx 5 \). Assuming typical pulsar values, we can calculate the required ellipticity of the pulsar to be on the order of \( \epsilon \sim 10^{-4} \). As will be discussed later, this is the same order of magnitude as the upper bound inferred for the ellipticity from gravitational wave observations from the Laser Interferometer Gravitational Wave Observatory (LIGO).

B. Combined Braking Index and Pulsar Bounds

There is another method for combining these models that can lead to bounds on the ellipticity of the pulsar and the electromagnetic braking index \( n_{em} \). This is a short treatment of recent work done by C. Palomba [29].

By altering the dipole radiation model, i.e. by allowing the angle \( \alpha \), the field strength \( B_p \), or the pulsar inertia \( I \) to evolve in time, a braking index other than 3 can be obtained with the general form
\[
n_{em} = 3 + \frac{k_m \Omega}{k_m \Omega_m} \tag{44}
\]
where \( \hat{\Omega}_m = k_m \Omega^{n_{em}} \). The important point here is that there could be deviations from the canonical value of \( n_{em} = 3 \). Allowing for a more general model of magnetic radiation means that the constant \( k_m \) is no longer known explicitly, and in fact evolves in time. A differential equation for the time evolution of \( \Omega \) can still be obtained, however, with no explicit dependence on \( k_m \).

It is important to note that if we measure the braking index of a pulsar through observations of the rotational frequency and its derivatives, we will obtain a single number. The Crab pulsar, for example, has a measured value of \( n = 2.51 \pm 0.01 \). Therefore it would be useful to combine the magnetic and gravitational models in such a way that results in a single value for the braking index, this as opposed to simply adding the models.

We initiate this model by writing down a combined model similar to the previous section
\[
\hat{\Omega} = \hat{\Omega}_m + \hat{\Omega}_g \tag{45}
\]

We can now differentiate this expression, and solve for an effective braking index by
\[
n = \frac{\Omega \dot{\Omega}}{\Omega^2} = \frac{n_{em} + 5(\hat{\Omega}_g/\hat{\Omega}_m)}{1 + (\hat{\Omega}_g/\hat{\Omega}_m)} \tag{46}
\]
where we have assumed that the braking index resulting from gravitational radiation is still fixed at \( n_{gw} = 5 \). If we define a function
\[
Y(\Omega) = \frac{\dot{\Omega}_g}{\dot{\Omega}_m} = \frac{k_g}{k_m} \Omega^{5-n_{em}} \tag{47}
\]
i.e. the ratio of deceleration from gravitational and magnetic radiation, then we can write the observed braking index as
\[
n = \frac{n_{em} + 5 Y}{1 + Y} \tag{48}
\]
This expression can be easily inverted to find
\[
Y(n) = \frac{n - n_{em}}{5 - n} \tag{49}
\]

The cost of combining the two braking indices into one is that because of the dependence of \( n \) on \( \Omega \), the braking index is now time dependent. Note here that the ellipticity of the pulsar can be written
\[
\epsilon = 1.9 \times 10^5 \sqrt{P^3 I} \tag{50}
\]
where the inertia \( I \) has been taken to be \( I = 10^{45} \text{ g cm}^2 \). Rewriting this in terms of \( Y(n) \) and \( \Omega \), we find
\[
\epsilon = 7.55 \times 10^6 \sqrt{\frac{\Omega}{\Omega^5}} \frac{Y(n)}{1 + Y(n)} \tag{51}
\]

It is possible to find an expression for \( Y(\Omega) \) that is independent of \( k_m \). By taking the derivative of \( Y \), i.e.
\[
\dot{Y} = (5 - n_{em}) \frac{Y}{\Omega} \tag{52}
\]
we can integrate to find

\[ Y(\Omega) = Y_0 \left( \frac{\Omega}{\Omega_0} \right)^{5-n_{em}} \]  

(53)

where the subscript 0 refers to values at \( t = 0 \), generally taken to be the present time. Next, by writing \( \dot{\Omega} \) as

\[ \dot{\Omega} = k_m \Omega^{5} \left( 1 + \frac{Y(\Omega)}{Y(\Omega_0)} \right) \]  

(54)

we have obtained a differential equation for \( \Omega \) with no explicit dependence on the magnetic constant \( k_m \). This can be numerically integrated based on the three unknowns \( \epsilon, n_{em}, \) and initial rotational speed \( \Omega_i \). These parameters are not completely independent, Eq. (51) and the condition \( \Omega(0) = \Omega_0 \) must be satisfied. Meeting these conditions produces a range of permissible values for the parameters.

This procedure is carried out for several pulsars, and bounds on the ellipticity are found in each case. For example, the Crab pulsar, according to this method, has quantities bounded by

\[ n_{em} \geq 1.7, \quad \epsilon \leq 3 \times 10^{-4}, \quad \Omega_i \leq 394 \text{ s}^{-1} \]  

(55)

In this section, a method has been presented that combines the braking indices of the two separate models in such a way that a single braking index can be found. This led to the computation of bounds on the pulsar quantities \( \epsilon, n_{em}, \) and \( \Omega_i \).

V. DISCUSSION

Due to the predicted emission of gravitational waves, pulsars are an important possible source to be monitored by gravitational wave detectors. Data from the fifth science run of LIGO has been analyzed to search for signals from the Crab pulsar through the use of matched filtering techniques [30]. The results of this search place a direct upper bound on the gravitational wave strain from the Crab pulsar, and an upper bound on ellipticity of \( \epsilon \leq 1.8 \times 10^{-4} \) can be inferred. This bound is more restrictive than that found by Palomba, and hence could conceivably be used to tighten the bound on \( n_{em} \).

Pulsars possess a rich suite of interesting physical attributes, only the most basic of which have been presented here. This report has examined two basic models of pulsar energy loss, magnetic dipole and gravitational quadrupole radiation. The energy loss was assumed to result in a decrease in the pulsar rotational speed, which can then be characterized by a single value \( n \), known as the braking index. Methods for combining these models were then discussed, as well as a short application to gravitational wave astronomy with LIGO.


