Derivation of the Binary Mass Function

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The binary mass function, $f(m_1)$ is a piece of information constructed from observables of a spectroscopic binary system. Assuming we have values for the orbital period $P_{\text{orb}}$ and the maximum radial velocity of one of the stars, say $V_2$, we can compute the value for the quantity $m_1 \sin^3 i$ through the mass function:

$$f(m_1) \equiv \frac{m_1 \sin^3 i}{(1 + q)} = \frac{P_{\text{orb}} V_2^2}{2\pi G}$$

Here $m_1$ is the mass of the first component of the binary, $i$ is the inclination angle of the orbit, $q \equiv m_2/m_1$ is the mass ratio of the system, and $G$ is the gravitational constant.

If the inclination angle and mass ratio can be determined by some observational method, then the value of $m_1$ can be determined explicitly. Otherwise, $m_1$ and $\sin i$ cannot be separated. This is known as the mass-inclination degeneracy.

To derive $f(m_1)$, begin with Kepler’s third law:

$$GM = \left(\frac{2\pi}{P_{\text{orb}}}ight)^2 a^3 \tag{2}$$

Make the substitutions

$$M = m_1 + m_2 = m_1(1 + m_2/m_1) = m_1(1 + q) \tag{3}$$

and

$$a^3 = (a_1 + a_2)^3 = a_2^3(1 + a_1/a_2)^3 = a_2(1 + m_2/m_1)^3 = a_2^3(1 + q)^3, \tag{4}$$

where the fact that $m_1a_1 = m_2a_2$ has been used.

Now

$$Gm_1(1 + q) = \left(\frac{2\pi}{P_{\text{orb}}}ight)^2 a_2^3(1 + q)^3 \tag{5}$$

Now we want to eliminate $a_2$ in exchange for something we can measure, i.e. the maximum radial velocity of $m_2$: $V_2 = v_2 \sin i$, where $v_2$ is the orbital velocity of $m_2$. We can get at this exchange by noting:

$$P_{\text{orb}} = \frac{2\pi}{\omega} = \frac{2\pi a_2}{v_2} = \frac{2\pi a_2 \sin i}{V_2} \tag{6}$$

or, by rearranging,

$$a_2 = \frac{P_{\text{orb}} V_2}{2\pi \sin i} \tag{7}$$

substituting Eq. 7 into Eq. 5, we get
\[ Gm_1 = \left(\frac{2\pi}{P_{\text{orb}}}\right)^2 \left(\frac{P_{\text{orb}}V_2}{2\pi \sin i}\right)^3 (1 + q)^2 \]  

(8)

And, by rearranging once again, we find

\[ \frac{m_1 \sin^3 i}{(1 + q)^2} = \frac{P_{\text{orb}}V_2^2}{2\pi G} \]

(9)