Relations Relevant to the One Body Reduction of the Gravitational Two Body Problem

Eric Addison, Utah State University

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Equivalent One Body Formulation

The reduction of the gravitational two body problem begins with the standard Lagrangian for the system:

$$L = T - V = \frac{m_1}{2} \dot{r}_1^2 + \frac{m_2}{2} \dot{r}_2^2 - V(\vec{r})$$  \hspace{1cm} (1)

where the vector $\vec{r}$ is the displacement vector between the two bodies:

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$  \hspace{1cm} (2)

Combine the two kinetic energy terms and make the following multiplication:

$$L = \frac{m_1}{2} \dot{r}_1^2 + \frac{m_2}{2} \dot{r}_2^2 \times \frac{m_1 + m_2}{m_1 + m_2} - V(\vec{r})$$  \hspace{1cm} (3)

Multiply out the numerator:

$$L = \frac{m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + m_1 m_2 \dot{r}_1 \dot{r}_2}{2(m_1 + m_2)} - V(\vec{r})$$  \hspace{1cm} (4)

Now define a new coordinate, $\vec{R}$, the center of mass coordinate:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$  \hspace{1cm} (5)

For which the time derivative is:

$$\dot{\vec{R}} = \frac{m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2}{m_1 + m_2}$$  \hspace{1cm} (6)

And, incidentally,

$$\dot{\vec{R}}^2 = \frac{m_1^2 \dot{r}_1^2 + m_1 m_2 \dot{r}_1 \dot{r}_2 + m_2^2 \dot{r}_2^2}{(m_1 + m_2)^2}$$  \hspace{1cm} (7)
We know rearrange the Lagrangian:

\[
L = \frac{m_1 + m_2}{2} \frac{\dot{\vec{r}}_1^2 + \dot{\vec{r}}_2^2 + \dot{\vec{r}}_1 \dot{\vec{r}}_2}{(m_1 + m_2)^2} + \frac{m_1 m_2}{2(m_1 + m_2)} V(\vec{r})
\]

(8)

Where now we can recognize the previously defined quantities:

\[
L = \frac{m_1 + m_2}{2} \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{\vec{r}}^2 - V(\vec{r})
\]

(9)

This is what we’re after, as now the Lagrangian has been separated into two independent variables \(\vec{R}\) and \(\vec{r}\), which will produce separate and independent equations of motion. Also, because the quantity \(\dot{\vec{R}}\) is a cyclic coordinate, \(\dot{\vec{R}}\) is a conserved quantity. This means that the center of mass is either stationary or moving with constant velocity. Hence in the barycenter frame, with the center of mass at the origin, the motion of the binary can be completely determined by the dynamics of \(\vec{r}\). We consider the one body problem of a body with mass \(\mu = \frac{m_1 m_2}{m_1 + m_2}\), known as the reduced mass, subject to the potential \(V(\vec{r})\).

**Immediate Relations**

Again, we will assume we are working in the Barycenter frame, so that the center of mass is at the origin. Assume from here on that all other coordinates have been transformed to meet this assumption.

By the definition of \(\vec{R}\), we can conclude:

\[
\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0
\]

(10)

\[
\Rightarrow m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0
\]

(11)

\[
\Rightarrow \vec{r}_2 = -\frac{m_1}{m_2} \vec{r}_1
\]

(12)

From this expression, we can quickly find the relation between \(\vec{r}\) and \(\vec{r}_1\):

\[
\vec{r} = \vec{r}_2 - \vec{r}_1 = -\frac{m_1}{m_2} \vec{r}_1 - \vec{r}_1 = -\frac{m_1 + m_2}{m_2} \vec{r}_1 = -\frac{m_1}{\mu} \vec{r}_1
\]

(13)

And similarly,

\[
\vec{r} = \frac{m_2}{\mu} \vec{r}_2
\]

(14)

2
So we can write the following relations, which will allow us to go back and forth from the one-body dynamics to the barycenter dynamics:

\[
\vec{r}_1 = -\frac{\mu}{m_1}\vec{r}, \quad \vec{r}_2 = \frac{\mu}{m_2}\vec{r}
\] (15)

From here we can make an important conclusion. Because the reduced mass orbits the origin, then the two separate bodies ALSO orbit the origin in the barycenter frame, the only difference being the size of the ellipse. Because the origin of the barycenter frame is the location of the center of mass,

**THE BODIES ORBIT ABOUT THEIR COMMON CENTER OF MASS**

**More Relations**

Armed with the previous relations, we can now easily move between reduced mass parameters and barycenter parameters.

**The Ellipse**

From standard analysis of the one-body problem, the reduced mass is found to follow a trajectory given by a conic section with eccentricity \(e\). We are specifically interested in the case where \(e < 1\), corresponding to an ellipse. In this case, the magnitude of \(\vec{r}\) can be described in terms of the true anomaly, \(\theta\), by:

\[
r(\theta) = \frac{a(1 - e^2)}{1 + e \cos(\theta)}
\] (16)

This is the general equation for an ellipse, and the numerator is a property of the ellipse called the *semi-latus rectum*. We can immediately see from here that the two bodies will follow elliptical paths with the same eccentricity, but scaled by the factor \(\frac{\mu}{m_i}\), i.e.

\[
r_1(\theta) = \frac{\mu}{m_1} \frac{a(1 - e^2)}{1 + e \cos(\theta)}
\] (17)

This equation (and the corresponding equation for \(r_2\)) tells us something very important. Because the two bodies follow elliptical paths with one focus at the origin, and the origin is the location of the center of mass:

**THE CENTER OF MASS IS LOCATED AT THE FOCUS OF EACH BODY’S ELLIPTICAL PATH**
Semi-Major Axis, Periapse, and Apoapse

From equation (17), we might as well define new semi-major axes for the elliptical paths followed by the real bodies:

\[ a_1 = \frac{\mu}{m_1} a, \quad a_2 = \frac{\mu}{m_2} a \]  

(18)

The pericenter and apocenter distances for each body are now easily found:

\[ r_{p,i} = a_i(1 - e), \quad r_{a,i} = a_i(1 + e) \]  

(19)

or, if we prefer, perhaps if we already know the reduced mass pericenter or apocenter distance:

\[ r_{p,i} = \frac{\mu}{m_i} r_p, \quad \frac{\mu}{m_i} r_a \]  

(20)

Consider the magnitude of the vector \( \vec{r} \):

\[
\begin{align*}
|\vec{r}| &= |\vec{r}_2 - \vec{r}_1| \\
&= (|\vec{r}_2 - \vec{r}_1| \cdot (\vec{r}_2 - \vec{r}_1))^{1/2} \\
&= (r_2^2 - 2r_1r_2 \cos(\phi) + r_1^2)^{1/2} \\
&= (r_2^2 + 2r_1r_2 + r_1^2)^{1/2} \\
&= (r_2 + r_1) \\
\end{align*}
\]

(21)

(22)

(23)

(24)

(25)

Where \( \phi \) is the angle between \( \vec{r}_1 \) and \( \vec{r}_2 \), which we know to be equal to \( \pi \) at all times (\( \vec{r}_1 \) and \( \vec{r}_2 \) are anti-parallel).

Because \( \theta \) is common to the two bodies, they will reach pericenter and apocenter at the same time, and we can easily conclude the minimum possible separation, and therefore the pericenter distance of \( \vec{r} \), will be:

\[ r_{\text{min}} = r_{p,2} + r_{p,1} = r_p \]  

(26)