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The readily available information for two body orbits generally refers to the one dimensional, reduced mass case. Information about initial conditions in the barycenter frame is not easily found. In order to derive the initial velocities required for a two-body orbit in the barycenter frame, the momentum of the reduced mass can be used.

Reduced Mass Momentum

We start from the following equation for eccentricity:

$$e = \sqrt{1 + \frac{2El^2}{mk^2}} \tag{1}$$

Where l is angular momentum, $l = mr^2\dot{\theta} = mrv_\theta$ and $k = Gm_1m_2$, and the bodies are assumed to be at pericenter so that the only component of velocity is v_θ.

Now we derive the linear momentum of the reduced mass, which can be easily extended to the barycenter frame. Start by solving equation (1) for energy:

$$E = (e^2 - 1)\frac{mk^2}{2l^2} = (e^2 - 1)\frac{k^2}{2(mr_0^2 v_\theta^2)} \tag{2}$$

And use the kinetic energy relation:

$$T = E - V = (e^2 - 1)\frac{mk^2}{2(m^2 r_0^2 v_\theta^2)} + \frac{k}{r_0} = \frac{p_0^2}{2m} \tag{3}$$

Noticing that $m^2 v_\theta^2 = p_0^2$, we can substitute and multiply through by p_0^2:

$$(e^2 - 1)\frac{mk^2}{2r_0^2} + \frac{p_0^2 k}{r_0} = \frac{p_0^4}{2m} \tag{4}$$

Now rearrange for a quadratic in p_0^2:

$$p_0^4 - p_0^2\left(\frac{2mk}{r_0}\right) + (1-e^2)\frac{m^2k^2}{r_0^2} = 0 \tag{5}$$

Solve for p_0^2:

$$p_0^2 = \frac{mk}{r_0} \pm \frac{1}{2}\sqrt{\frac{4m^2k^2}{r_0^2} - (1-e^2)\frac{4m^2k^2}{r_0^2}} \tag{6}$$

or,

$$p_0 = \sqrt{\frac{mk}{r_0}\,(1 \pm e)} \tag{7}$$

now subbing in for k and m,

$$\boxed{p_0 = \sqrt{\frac{Gm_1m_2\mu}{r_0}\,(1 \pm e)}} \tag{8}$$

Momentum Equality

Consider the total kinetic energy of the two-body orbit in both the barycenter and the reduced mass cases:

$$T_b = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\ ,\ T_r = \frac{1}{2}\frac{m_1m_2}{(m_1+m_2)}v_\mu^2 \tag{9}$$

These must be equal, of course. Asuming that there is no center of mass movement, conervation of linear momentum implies:

$$p_1 + p_2 = 0 \Rightarrow p_1^2 = p_2^2 \tag{10}$$

Now rewrite kinetic energy in terms of momentum:

$$T_b = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = T_r = \frac{p_\mu^2}{2}\left(\frac{m_1+m_2}{m_1m_2}\right) \tag{11}$$

Now using equation (10) and multiplying through by m_1m_2,

$$p_1^2(m_1+m_2) = p_\mu^2\,(m_1+m_2) \Rightarrow \boxed{|p_1| = |p_\mu|} \tag{12}$$

So the momentum of the bodies in the barycenter frame is equal to the momentum of the reduced mass in the 1D problem, which we have found in equation (8).

Initial Velocities

Using the momentum equality $|p_\mu| = |p_1| = |p_2|$, the pericenter velocities of the two bodies in the barycenter frame are given by:

$$\boxed{|v_1| = \frac{|p_0|}{m_1} = \sqrt{\frac{Gm_2^2}{Mr_0}(1+e)}} \tag{13}$$

and

$$\boxed{|v_2| = \frac{|p_0|}{m_2} = \sqrt{\frac{Gm_1^2}{Mr_0}(1+e)}} \tag{14}$$

For elliptical orbits, the velocity at apocenter is found by replacing $(1+e)$ with $(1-e)$ and r_0 with r_{apo}.