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Section 1: Review of the Classical Regression Model

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1 General Stuff

1.1 Announcements

• This section has a website. The address is: http://works.bepress.com/econ_240b_econometrics/
  You should sign up for the mailing list in the website. After the second week of classes we
  will use only this list to communicate with you.

• Problem Sets will be due in section the Friday following the date Powell sets in the Problem
  Set. You can bring it to section, and I absolutely don’t mind if you stay or not, or you can
  have someone else bring it to section. You can also leave it in Jeff’s mailbox (since there is
  always confusion with my mailbox). Look for the box named GREENBAUM in 612 Evans,
  until 4 p.m. of the Friday it is due.

• My office hours will be on Thursdays 5-7 in 639 Evans.
2 Review

2.1 The Classical Linear Regression Model with Normality

The Classical Linear Model has the following assumptions:

1. $E(y) = X\beta$, where $\beta$ is an unknown vector of dimension $K$.
2. $V(y) = \sigma^2 I$
3. The $(N \times K)$ matrix $X$ is non-random
4. $\text{rank}(X) = K$
5. The vector $y - X\beta$ is normally distributed

as a result, $y \sim N(X\beta, \sigma^2 I)$. We can estimate $\beta$ by OLS, which is the $\beta$ that solves:

$$\hat{\beta} = \arg\min_{\beta} (y - X\beta)'(y - X\beta)$$

which is equivalent to:

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{n} (y_i - x'_i\beta)^2$$

(You should get used to notation in both matrix and summation format, as it will show up fairly often in both ways.)

As you saw in 240A, the solution to this minimization problem is:

$$\hat{\beta} = (X'X)^{-1}X'y$$

or, in summation notation:

$$\hat{\beta} = \left(\sum_{i=1}^{n} x_i x'_i\right)^{-1} \sum_{i=1}^{n} x_i y_i$$

The solution $\hat{\beta}$ is also the coefficient that gives the orthogonal projection of $y$ onto the space generated by the columns of $X$. Can you show this? What characterizes an orthogonal projection onto a space $S$?\(^1\)

1. (projection part) It is an "Idempotent Linear Operator": $Pv = P^2v$.
2. (space $S$ part) $v \in S \Rightarrow Pv = v$.
3. (orthogonality part) $v \perp S^2 \Rightarrow Pv = 0$

\(^1\)You can read more in Ruud p. 31-34.

\(^2\) $v \perp S$ means that for all vector $u \in S$, $v'u = 0$. 

we can show that $X\hat{\beta}$ is really the orthogonal projection of $y$ onto the column space of $X$:

1. $X\hat{\beta} = X(X'X)^{-1}X'y$, so first we will show that $X(X'X)^{-1}X'$ is a projection: $(X(X'X)^{-1}X')^2 = X(X'X)^{-1}X'X(X'X)^{-1}X' = X(X'X)^{-1}X'$.

2. Let $v \in \text{Col}(X)$, then $\exists \alpha$ s.t. $v = X\alpha$. Then: $X(X'X)^{-1}X'v = X(X'X)^{-1}X'X\alpha = X\alpha = v$.

3. $v \perp \text{Col}(X) \Rightarrow \alpha'X'v = 0 \forall \alpha \Rightarrow X'v = 0 \Rightarrow X(X'X)^{-1}X'v = 0$.

2.2 Hypothesis Testing

Given the assumptions of the Classical Linear Model with Normality, we can derive the distribution of our estimator of $\beta$.

$$\hat{\beta} = (X'X)^{-1}X'y \sim (X'X)^{-1}X'N(X\beta, \sigma^2 I)$$
$$\sim N((X'X)^{-1}X'X\beta, (X'X)^{-1}X'X(X'X)^{-1})$$
$$\sim N(\beta, \sigma^2(X'X)^{-1})$$

Observe that although the distribution we derived is well known (we are now supposing that we know $\sigma^2$), we don’t know the values of the distribution $N(\beta, \sigma^2(X'X)^{-1})$. If $\beta$ is a scalar, we could do:

$$\frac{\hat{\beta} - \beta}{\sigma \sqrt{(X'X)^{-1}}} \sim N(0, 1) \quad (1)$$

for which we have tables. In that case, if we wanted to test the hypothesis: $\beta = \beta_0$, we know that if the hypothesis is true, from (1), $\frac{\hat{\beta} - \beta_0}{\sigma \sqrt{(X'X)^{-1}}} \sim N(0, 1)$. Under these circumstances, we would expect to find that the value $\frac{\hat{\beta} - \beta_0}{\sigma \sqrt{(X'X)^{-1}}}$ > 1.96 about 5% of the time, and since we are willing to reject the null hypothesis when it is true about 5% of the time, we can set up the test statistics:

$$\left| \frac{\hat{\beta} - \beta_0}{\sigma \sqrt{(X'X)^{-1}}} \right|$$

and we reject the null hypothesis: $\beta = \beta_0$ if:

$$\left| \frac{\hat{\beta} - \beta_0}{\sigma \sqrt{(X'X)^{-1}}} \right| > 1.96$$

When $\beta$ is not a scalar, we can’t do the trick above. If we have a joint null hypothesis that can be expressed as: $R\beta = \theta$, then:
\( R\hat{\beta} \sim N(R\beta, \sigma^2 R(X'X)^{-1}R') \)

if \( R \) is a \((1 \times K)\) vector, then:

\[
\frac{R(\hat{\beta} - \beta)}{\sigma \sqrt{R(X'X)^{-1}R'}} \sim N(0, 1)
\]  

(2)

and we can set up a hypothesis test, where we reject the null hypothesis: \( R\beta = \theta \) if:

\[
\left| \frac{R(\hat{\beta} - \beta)}{\sigma \sqrt{R(X'X)^{-1}R'}} \right| > 1.96
\]

If \( R \) is a \((p \times K)\) vector, we van use an interesting approach. If \((v_1, v_2, ..., v_p)\) is a vector of \(p\) independent standard normal variables, then \( \sum_{i=1}^{p} v_i^2 \sim \chi^2_p \). We will use exactly that property.

\[
\frac{(R(X'X)^{-1}R')^{-1/2}R(\hat{\beta} - \beta)}{\sigma} \sim N(0, I_p)
\]

\[
\Rightarrow \frac{(\hat{\beta} - \beta)'R'(R(X'X)^{-1}R')^{-1}R(\hat{\beta} - \beta)}{\sigma^2} \sim \chi^2_p
\]

In particular, if we are testing all coefficients at the same time, \( R = I_K \), so:

\[
\frac{(\hat{\beta} - \beta)'(X'X)(\hat{\beta} - \beta)}{\sigma^2} \sim \chi^2_K
\]  

(3)

and, since the \( \chi^2 \) distribution is well known, we can look up critical values in tables, and set up a test easily.

### 2.3 Exercises

**Exercise 1:** In the Classical Linear Model with Normality, suppose that you know that \( \sigma^2 = 2 \), and that \( X'X = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \). In a sample of 400 observations, the LS coefficients are: \( \beta_1 = 2 \), and \( \beta_2 = 2 \).

a) Test at the 5% significance level the joint null hypothesis that \( \beta_1 = \beta_2 = 3 \).

We will use (3), and the test statistics in this case will be equal to 3.5, which we will compare to the 5% critical value of a \( \chi^2_2 \), which is: 5.99, so we don’t reject the null hypothesis.

b) State the alternative hypothesis against which you are testing.

We are testing against the alternative hypothesis that \( \beta \neq (3, 3)' \). Observe that we may have one of the coefficients equal to 3 in the alternative hypothesis, but not both.
c) What if we wanted to test only whether $\beta_1 = 3$?

Now, the null hypothesis is equivalent to: $(1, 0)\beta = 3$, so $R = (1, 0)$, and $\theta = 3$. We use (2). First we calculate $(X'X)^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$ and the test statistic value is: $\frac{1}{5} = 0.2$, which is smaller than 3.84, so we don’t reject the null hypothesis at the 5% significance level.