Nontrivial solutions of inclusions involving perturbed maximal monotone operators

Dhruba Adhikari, Kennesaw State University
NONTRIVIAL SOLUTIONS OF INCLUSIONS INVOLVING PERTURBED MAXIMAL MONOTONE OPERATORS

DHRUBA R. ADHIKARI

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Abstract. Let $X$ be a real reflexive Banach space and $X^*$ its dual space. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, and $T : X \supset D(T) \to 2^{X^*}$, $0 \in D(T)$ and $0 \in T(0)$, be strongly quasibounded maximal monotone and positively homogeneous of degree 1. Also, let $C : X \supset D(C) \to X^*$ be bounded, demicontinuous and of type $(S_\omega)$ w.r.t. to $D(L)$. The existence of nonzero solutions of \(Lx + Tx + Cx \ni 0\) is established in the set $G_1 \setminus G_2$, where $G_2 \subset G_1$ with $\overline{G_2} \subset G_1$, $G_1, G_2$ are open sets in $X$, $0 \in G_2$, and $G_1$ is bounded. In the special case when $L = 0$, a mapping $G : G_1 \to X^*$ of class $(P)$ introduced by Hu and Papageorgiou is also incorporated and the existence of nonzero solutions of $Tx + Cx + Gx \ni 0$, where $T$ is only maximal monotone and positively homogeneous of degree $\alpha \in (0, 1]$, is obtained. Applications to elliptic partial differential equations involving $p$-Laplacian with $p \in (1, 2]$ and time-dependent parabolic partial differential equations on cylindrical domains are presented.

1. Introduction and preliminaries

Let $X$ be a real reflexive Banach space with its dual space $X^*$. The norms of $X, X^*$ will be denoted by $\| \cdot \|_X$ and $\| \cdot \|_{X^*}$, respectively. We denote by $\langle x^*, x \rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. The symbols $\partial D, \hat{D}, \bar{D}$, denote the strong boundary, interior and closure of the set $D$, respectively. The symbol $B_Y(0, R)$ denotes the open ball of radius $R$ with center at 0 in a Banach space $Y$.

If $\{x_n\}$ is a sequence in $X$, we denote its strong convergence to $x_0$ in $X$ by $x_n \to x_0$ and its weak convergence to $x_0$ in $X$ by $x_n \rightharpoonup x_0$. An operator $T : X \supset D(T) \to Y$ is said to be “bounded” if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of $Y$. The operator $T$ is said to be “compact” if it maps bounded subsets of $D(T)$ onto relatively compact subsets in $Y$. It is said to be “demicontinuous” if it is strong-weak continuous on $D(T)$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote $(-\infty, \infty)$ and $[0, \infty)$, respectively. The normalized duality mapping $J : X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2, \|x^*\| = \|x\|\}, \quad x \in X.$$
The Hahn-Banach theorem ensures that $D(J) = X$, and therefore $J : X \to 2^{X^*}$ is a multivalued mapping defined on the whole space $X$.

By a well-known renorming theorem due to Trojanski [27], one can always renorm the reflexive Banach space $X$ with an equivalent norm with respect to which both $X$ and $X^*$ become locally uniformly convex (therefore strictly convex). Henceforth, we assume that $X$ is a locally uniformly convex reflexive Banach space. With this setting, the normalized duality mapping $J$ is single-valued homeomorphism from $X$ onto $X^*$ and satisfies

$$J(\alpha x) = \alpha J(x), \quad (\alpha, x) \in \mathbb{R}_+ \times X.$$ 

For a multivalued operator $T$ from $X$ to $X^*$, we write $T : X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X : Tx \neq \emptyset\}$ is the effective domain of $T$. We denote by $Gr(T)$ the graph of $T$, i.e., $Gr(T) = \{(x, y) : x \in D(T), y \in Tx\}$.

An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be “monotone” if for every $x, y \in D(T)$ and every $u \in Tx, v \in Ty$ we have

$$(u - v, x - y) \geq 0.$$ 

A monotone operator $T$ is said to be “maximal monotone” if $Gr(T)$ is maximal in $X \times X^*$, when $X \times X^*$ is partially ordered by the set inclusion. In our setting, a monotone operator $T$ is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0, \infty)$.

If $T$ is maximal monotone, then the operator $T_\lambda \equiv (T^{-1} + \lambda J^{-1})^{-1} : X \to X^*$ called the Yosida approximant is bounded, demicontinuous, maximal monotone and such that $T_\lambda x \to T^{(0)}x$ as $t \to 0^+$ for every $x \in D(T)$, where $T^{(0)}x$ denotes the element $y^* \in Tx$ of minimum norm, i.e., $\|T^{(0)}x\| = \inf\{\|y^*\| : y^* \in Tx\}$. In our setting, this infimum is always attained and $D(T^{(0)}) = D(T)$. Also, $T_\lambda x \in TJ_\lambda x$, where $J_\lambda \equiv I - tJ^{-1}T_\lambda : X \to X$ and satisfies $\lim_{t \to 0} J_\lambda x = x$ for all $x \in co D(T)$, where co $A$ denotes the convex hull of the set $A$. In addition, $x \in D(T)$ and $t_0 > 0$ imply $\lim_{t \to t_0} T_\lambda x = T_{t_0}x$. The operators $T_\lambda, J_\lambda$ were introduced by Brézis, Crandall and Pazy in [9]. For their basic properties, we refer the reader to [9] as well as Pascale and Sburlan [23] pp. 128-130).

We need the following lemmas about maximal monotone operators.

**Lemma 1.1 ([28] p. 915).** Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true:

(i) $\{x_n\} \subset D(T)$, $x_n \to x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.

(ii) $\{x_n\} \subset D(T)$, $x_n \to x_0$ and $Tx_n \ni y_n \to y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.

The next lemma is essentially due to Brézis, Crandall and Pazy [9], and its proof can be found in [3].

**Lemma 1.2.** Assume that the operators $T : X \supset D(T) \to 2^{X^*}$ and $S : X \supset D(S) \to 2^{X^*}$ are maximal monotone, with $0 \in D(T) \cap D(S)$ and $0 \in S(0) \cap T(0)$. Assume, further, that $T + S$ is maximal monotone and that there is a sequence $\{t_n\} \subset (0, \infty)$ such that $t_n \downarrow 0$, and a sequence $\{x_n\} \subset D(S)$ such that $x_n \to x_0 \in X$ and $T_{t_n}x_n + w_n^* \to y_n^* \in X^*$, where $w_n^* \in Sx_n$. Then the following are true.

(i) The inequality

$$\lim_{n \to \infty} \langle T_{t_n}x_n + w_n^*, x_n - x_0 \rangle < 0 \quad (1.1)$$

(ii) $y_n^* \to 0$.

(iii) $\lim_{n \to \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle = 0$.
is impossible.

(ii) If
\[
\lim_{n \to \infty} \langle T_{i_n} x_n + w_i^*, x_n - x_0 \rangle = 0,
\]
then \(x_0 \in D(T + S)\) and \(y_0^* \in (T + S)x_0\).

**Definition 1.3.** An operator \(T : X \supset D(T) \to 2^{X^*}\) is said to be “strongly quasi-bounded” if for every \(S > 0\) there exists \(K(S) > 0\) such that
\[
\|x\| \leq S, \quad \langle x^*, x \rangle \leq S, \quad \text{for some } x^* \in Tx,
\]
implies \(\|x^*\| \leq K(S)\).

Browder and Hess have shown in [13] that a monotone operator \(T\) with \(0 \in D(T)\) is strongly quasi-bounded. The proof of the following lemma, which is due to Browder and Hess [13], can also be found in [17, Lemma D].

**Lemma 1.4.** Let \(T : X \supset D(T) \to 2^{X^*}\) be a strongly quasibounded maximal monotone operator such that \(0 \in T(0)\). Let \(\{t_n\} \subset (0, \infty)\) and \(\{u_n\} \subset X\) be such that
\[
\|u_n\| \leq S, \quad \langle T_{t_n} u_n, u_n \rangle \leq S, \quad \text{for all } n,
\]
where \(S\) is a positive constant. Then there exists a number \(K = K(S) > 0\) such that \(\|T_{t_n} u_n\| \leq K\) for all \(n\).

**Definition 1.5.** An operator \(G : X \supset D(G) \to 2^{X^*}\) is said to belong to class \((P)\) if it maps bounded sets to relatively compact sets, for every \(x \in D(G), G(x)\) is closed and convex subsets of \(X^*\) and \(G(\cdot)\) is upper-semicontinuous \((usc)\), i.e., for every closed set \(F \subset X^*\), the set \(G^-(F) = \{x \in D(G) : G(x) \cap F \neq \emptyset\}\) is closed in \(X\).

An important fact about a compact-set valued upper-semicontinuous operator \(G\) is that it is closed. Furthermore, for every sequence \(\{[x_n, y_n]\} \subset Gr(G)\) such that \(x_n \rightharpoonup x \in D(G)\), the sequence \(\{y_n\}\) has a cluster point in \(G(x)\).

**Definition 1.6.** Let \(L : X \supset D(L) \to X^*\) be a densely defined linear maximal monotone operator and \(C : X \supset D(C) \to X^*\) be bounded and demicontinuous. We say that \(C : X \supset D(C) \to X^*\) is of type \((S+)\) w.r.t. \(D(L)\) if for every sequence \(\{x_n\} \subset D(L) \cap D(C)\) with \(x_n \rightharpoonup x_0\) in \(X\), \(Lx_n \rightharpoonup Lx_0\) in \(X^*\) and
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0,
\]
we have \(x_n \to x_0\) in \(X\). In this case, if \(L = 0\), then \(C\) is of class \((S+)\).

**Definition 1.7.** The family \(C(t) : X \supset D \to X^*, t \in [0, 1]\), of operators is said to be a “homotopy of type \((S+)\) w.r.t. \(D(L)\)” if for any sequences \(\{x_n\} \subset D(L) \cap D\) with \(x_n \rightharpoonup x_0\) in \(X\) and \(Lx_n \rightharpoonup Lx_0\) in \(X^*\), \(\{t_n\} \subset [0, 1]\) with \(t_n \to t_0\) and
\[
\limsup_{n \to \infty} \langle C(t_n)x_n, x_n - x_0 \rangle \leq 0,
\]
we have \(x_n \to x_0\) in \(X, x_0 \in D\) and \(C(t_n)x_n \rightharpoonup C(t_0)x_0\) in \(X^*\). In this case, if \(L = 0\), then \(C(t)\) is a homotopy of type \((S+)\). A homotopy of type \((S+)\) w.r.t. \(D(L)\) is “bounded” if the set
\[
\{C(t)x : t \in [0, 1], x \in D\}
\]
is bounded.
Let $G$ be an open and bounded subset of $X$. Let $L : X \supset D(L) \to X^*$ be densely defined linear maximal monotone, $T : X \supset D(T) \to 2^{X^*}$ maximal monotone, and $C(s) : X \supset \overline{G} \to X^*$, $s \in [0,1]$, a bounded homotopy of type $(S_+)$ w.r.t. $D(L)$. Since the graph $Gr(L)$ of $L$ is closed in $X \times X^*$, the space $Y = D(L)$ associated with the graph norm
\[ \|x\|_Y = \|x\|_X + \|Lx\|_{X^*}, \quad x \in Y, \]
becomes a real reflexive Banach space. We may now assume that $Y$ and its dual $Y^*$ are locally uniformly convex.

Let $j : Y \to X$ be the natural embedding and $j^* : X^* \to Y^*$ its adjoint. Note that since $j : Y \to X$ is continuous, we have $D(j^*) = X^*$, which implies that $j^*$ is also continuous. Since $j^{-1}$ is not necessarily bounded, we have, in general, $j^*(X^*) \neq Y^*$. Moreover, $j^{-1}(\overline{G}) = \overline{G} \cap D(L)$ is closed and $j^{-1}(G) = G \cap D(L)$ is open, and
\[ \overline{j^{-1}(G)} \subset j^{-1}(\overline{G}), \quad \partial(j^{-1}(G)) \subset j^{-1}(\partial G). \]

We define $M : Y \to Y^*$ by
\[ (Mx, y) = (Ly, J^{-1}(Lx)), \quad x, y \in D(L). \]
Here, the duality pair $(\cdot, \cdot)$ is in $Y^* \times Y$ and $J^{-1}$ is the inverse of the duality map $J : X \to X^*$ and is identified with the duality map from $X^*$ to $X^{**} = X$. Also, for every $x \in Y$ such that $Mx \in j^*(X^*)$, we have $J^{-1}(Lx) \in D(L^*)$ and
\[ Mx = j^* \circ L^* \circ J^{-1}(Lx), \quad (1.3) \]
\[ (Mx - My, x - y) = (Lx - Ly, J^{-1}(Lx) - J^{-1}(Ly)) \geq 0 \quad (1.4) \]
for all $y \in Y$ such that $My \in j^*(X^*)$.

We now define $\hat{L} : Y \to Y^*$ and $\hat{C}(s) : j^{-1}(\overline{G}) \to Y^*$ by
\[ \hat{L} = j^* \circ L \circ j \quad \text{and} \quad \hat{C}(s) = j^* \circ C(s) \circ j \]
respectively, and for every $t > 0$, we also define $\hat{T}_t : Y \to Y^*$ by
\[ \hat{T}_t = j^* \circ T_t \circ j, \]
where $T_t$ is the Yosida approximant of $T$.

Kartsatos and the author developed a new degree theory in [2] for the triplet $L + T + C$, where $L$ is densely defined linear maximal monotone, $T$ is (possibly nonlinear) maximal monotone and strongly quasibounded, and $C$ is bounded, demicontinuous and of type $(S_+)$ w.r.t. the set $D(L)$. This degree theory extends the degree theory of Berkovits and Mustonen [8] who considered the case $T = 0$. As in [8], the construction of the degree mapping in [2] uses the graph norm topology of the space $Y = D(L)$ and is based on the Skrypnik degree and its invariance under homotopies of type $(S_+)$. In fact, it is shown that the mapping
\[ H(t, x) := \hat{L} + \hat{T}_t + \hat{C} + tMx, \quad (t, x) \in (0, \infty) \times j^{-1}(\overline{G}), \quad (1.5) \]
has the Skrypnik degree, $d_\mathcal{S}(H(t, \cdot), \overline{G}, 0)$, under the usual boundary condition on the boundary of an open and bounded set $\overline{G} \subset Y$, which remains fixed for all sufficiently small $t \in (0, \infty)$. Then the degree is defined by
\[ d(L + T + C, G, 0) = \lim_{t \downarrow 0} d_\mathcal{S}(\hat{L} + \hat{T}_t + \hat{C} + tM, \overline{G}, 0), \quad (1.6) \]
where $G$ is an open bounded subset of $X$ related to $\tilde{G}$. The operator $C$ above satisfies the $(S_+)$-condition w.r.t. $Y = D(L)$ and $T$ is strongly quasibounded and maximal monotone with $0 \in T(0)$. In order to show that the degree $d_3$ is fixed as above, it can be shown, in addition, that the family of mappings $f^t := H(t, \cdot)$ is a homotopy of class $(S_+)$ in the sense of Browder \cite{10} Definition 3, p. 69 on every interval $[t_1, t_2] \subset (0, t_0]$, where $t_0$ is an appropriate fixed positive number. The approach discussed here is that of Berkovits and Mustonen in \cite{8} and Addou and Memri in \cite{1}.

In Section 2, we establish the existence of nonzero solutions of the inclusion $Lx + Tx + Cx \ni 0$, where $L$, $C$ are as above and $T$ is a strongly quasibounded maximal monotone operator and positively homogeneous of degree 1. This result is in the spirit of similar results in \cite{3} for operators of the form $T + C$, where $T$ is single-valued maximal monotone, $0 = T(0)$, and $C$ bounded demicontinuous and of type $(S_+)$. Mild and natural boundary conditions are considered in order to establish the result by utilizing the graph norm topology on $D(L)$ and relevant topological degree theory. The theory is applicable to parabolic partial differential equations in divergence form on cylindrical domains.

In Section 3, the existence of nonzero solutions of $Tx + Cx + Gx \ni 0$ is established by utilizing the topological degree theories developed by Browder \cite{13} and Skrypnik \cite{26}. In this case, $T$ is only maximal monotone with $0 \in T(0)$ and positively homogeneous of degree $\alpha \in (0, 1]$, and $C$ is bounded demicontinuous of type $(S_+)$. This result extends and generalizes a similar result in \cite{9} for $\alpha = 1$ and $G = 0$ and has applications to elliptic boundary value problems involving $p$-Laplacian.

For additional facts and various topological degree theories related to the subject of this paper, the reader is referred to Kartsatos and the author \cite{1}, Kartsatos and Lin \cite{16}, and Kartsatos and Skrypnik \cite{20, 18}. For information on various concepts and ideas of Nonlinear Analysis used herein, the reader is referred to Barbu \cite{7}, Browder \cite{11}, Pascali and Sburlan \cite{23}, Simons \cite{24}, Skrypnik \cite{25, 26}, and Zeidler \cite{28}.

The following lemma from \cite{5} about the boundedness of the solutions of a homotopy equation will be needed in the sequel.

**Lemma 1.8.** Let $G \subset X$ be open and bounded. Assume the following:

(A1) $L : X \supset D(L) \to X^*$ is linear, maximal monotone with $D(L)$ dense in $X$;
(A2) $T : X \supset D(T) \to 2^{X^*}$ is strongly quasibounded, maximal monotone with $0 \in T(0)$;
(A3) $C(t) : X \supset \overline{G} \to X^*$ is a bounded homotopy of type $(S_+)$ w.r.t. $D(L)$.

Then, for a continuous curve $f(s), 0 \leq s \leq 1$, in $X^*$, the set

$$K = \{ x \in j^{-1}(G) : \hat{L} + \hat{T}_t + \hat{C}(s) + tMx = j^*f(s), \text{ for some } t > 0, s \in [0, 1] \}$$

is bounded in $Y$. Thus, there exists $R > 0$ such that $K \subset B_Y(R)$, where $B_Y(R)$ is the open ball of $Y$ of radius $R$.

Lemma 1.9 below taken from Kartsatos and Skrypnik \cite{19} will be used in the proof of Theorem 2.2

**Lemma 1.9.** Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and such that $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t, x) \to T_t x$ is continuous on the set $(0, \infty) \times X$. 


Definition 1.10. An operator \( T : X \supset D(T) \rightarrow 2^{X^*} \) is said to be positively homogeneous of degree \( \alpha > 0 \) if, for a fixed \( \alpha > 0 \), \( x \in D(T) \) implies \( tx \in D(T) \) for all \( t \in \mathbb{R}_+ \) and \( T(tx) = t^\alpha Tx \).

The following lemma, which plays an important role in the existence theorems of Section 2 and Section 3, shows in particular that the Yosida approximants of a positively homogeneous maximal monotone operator of degree \( \alpha \) are also positively homogeneous only when \( \alpha = 1 \).

Lemma 1.11. Let \( T : X \supset D(T) \rightarrow 2^{X^*} \) be maximal monotone and positively homogeneous of degree \( \alpha > 0 \). Then, for each \( t > 0 \), the Yosida approximant \( T_t \) satisfies
\[
T_t(sx) = s^\alpha T_{ts^{\alpha - 1}}(x) \quad \text{for all } (s,x) \in (0, +\infty) \times X. \tag{1.7}
\]
Proof. Let \( y = T_t(sx) = (T^{-1} + tJ^{-1})^{-1}(sx) \), for \( t, s > 0 \), \( x \in X \). The homogeneity of the duality mapping \( J \) implies
\[
y \in T(-tJ^{-1}y + sx) = T\left(s\left(-\frac{t}{s}J^{-1}y + x\right)\right) = s^\alpha T\left(-\frac{t}{s^\alpha}J^{-1}\left(\frac{y}{s^\alpha}\right) + x\right).
\]
This is equivalent to
\[
x \in T^{-1}\left(\frac{y}{s^\alpha}\right) + ts^{\alpha - 1}J^{-1}\left(\frac{y}{s^\alpha}\right)
\]
and
\[
y = s^\alpha(T^{-1} + ts^{\alpha - 1}J^{-1})^{-1}x = s^\alpha T_{ts^{\alpha - 1}}(x). \]
\( \square \)

2. Nonzero solutions of \( Lx + Tx + Cx \ni 0 \)

Guo and Lakshmikantham have shown in \([14]\) the following result for compact operators defined on a cone in a Banach space. The operator \( T \) satisfies non-contractive and non-expansive type of conditions only on the boundary of the subsets \( G_1, G_2 \) of \( X \) for the existence of a nonzero fixed of \( T \).

Theorem 2.1. Let \( X \) be a Banach space and \( K \) a positive cone in \( X \) which induces a partial ordering “\( \preceq \)” in \( X \). Let \( G_1, G_2 \subset X \) be open, \( 0 \in G_2, \overline{G_2} \subset G_1, G_1 \) bounded, and \( T : K \cap \overline{G_1} \rightarrow K \) compact with \( T(0) = 0 \). Suppose that one of the following two conditions holds.

1. \( Tx \not\preceq x \) for \( x \in K \cap \partial G_1 \), and \( Tx \not\preceq x \) for \( x \in K \cap \partial G_2 \);
2. \( Tx \not\preceq x \) for \( x \in K \cap \partial G_1 \), and \( Tx \not\preceq x \) for \( x \in K \cap \partial G_2 \).

Then there exists a fixed point of \( T \) in \( K \cap (G_1 \setminus G_2) \).

By imposing certain conditions only on the boundary of sets \( G_1, G_2 \), the author and Kartsatos \([3]\) established the existence of nonzero solutions of \( Tx + Cx = 0 \), where \( T \) is positively homogeneous of degree 1 and single-valued maximal monotone, and \( C \) is a bounded demicontinuous of type \((S_+)\). The following result is obtained in the spirit of \([3]\) Theorem 6, p.1246 in the context of the Berkovits-Mustonen theory in \([8]\).
Theorem 2.2. Assume that $G_1, G_2 \subset X$ are open, bounded with $0 \in G_2$ and $G_2 \subset G_1$. Let $L : X \ni D(L) \to X^*$ be linear maximal monotone with $D(L) = X$, and $T : X \ni D(T) \to 2^{X^*}$ strongly quasibounded, maximal monotone and positively homogeneous of degree 1. Also, let $C : \overline{G_1} \to X^*$ be bounded, demicontinuous and of type $(S_+)$ w.r.t. to $D(L)$. Moreover, assume the following:

(H1) there exists $v^* \in X^* \setminus \{0\}$ such that $Lx + Tx + Cx \not\ni \lambda v^*$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(T) \cap \partial G_1)$, and

(H2) $Lx + Tx + Cx + \lambda Jx \not\ni 0$ for all $(\lambda, x) \in \mathbb{R}_+ \times (D(L) \cap D(T) \cap \partial G_2)$.

Then the inclusion $Lx + Tx + Cx \ni 0$ has a solution $x \in D(L) \cap D(T) \cap (G_1 \setminus G_2)$.

Proof. To solve the inclusion

$$Lx + Tx + Cx \ni 0, \quad x \in \overline{G_1}, \quad (2.1)$$

let us consider the associated equation

$$\hat{L}x + \hat{T}x + \hat{C}x + tMx = 0, \quad t \in (0, +\infty), \quad x \in j^{-1}(\overline{G_1}). \quad (2.2)$$

One can show as in [2] that there exists $R > 0$ such that the open ball $B_Y(0, R) = \{y \in Y : \|y\|_Y < R\}$ contains all solutions of (2.2). We shall prove that (2.2) has a solution $x_\tau \in j^{-1}(G_1 \setminus G_2)$ for all sufficiently small $t$. We first claim that there exist $\tau_0 > 0, t_0 > 0$ such that

$$\hat{L}x + \hat{T}x + \hat{C}x + tMx = \tau j^* v^* \quad (2.3)$$

has no solution in $G_1^1(Y) := j^{-1}(G_1) \cap B_Y(0, R)$ for all $t \in (0, t_0]$ and all $\tau \in [\tau_0, +\infty)$. Assume the contrary and let $\{\tau_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, 1)$ and $\{x_n\} \subset G_1^1(Y)$ such that $\tau_n \to \infty$, $t_n \downarrow 0$ and

$$Lx_n + \hat{T}t_n x_n + \hat{C}x_n + t_n M x_n = \tau_n j^* v^*. \quad (2.4)$$

We note that $j^*$ is one-to-one because $j(Y) = Y$ which is dense in $X$. This implies that $j^* v^*$ is nonzero, and therefore $\|\tau_n j^* v^*\|_{Y^*} \to +\infty$. Also, the sequence $\{x_n\}$ is bounded in $Y$ and so we may assume that $x_n \rightharpoonup x_0$ in $X$ and $Lx_n \to Lx_0$ in $X^*$. In particular, $\{Lx_n\}$ is bounded in $X^*$. Since $Mx_n \in j^*(X^*)$, we have $J^{-1}(Lu) \in D(L^*)$ and

$$Mx_n = j^* L^* J^{-1}(Lx_n).$$

Since $j^*, L^*, J^{-1}$ are bounded, we obtain the boundedness of $\{M(x_n)\}$. It is clear that $\hat{C}x_n$ is bounded in $Y^*$, and therefore (2.4) implies that $\|Lx_n + \hat{T}t_n x_n\|_{Y^*} \to +\infty$. Define

$$\alpha_n = \frac{1}{\|Lx_n + \hat{T}t_n x_n\|_{Y^*}} \quad \text{and} \quad u_n = \alpha_n x_n.$$ 

It is obvious that $u_n \to 0$ in $Y$.

Since $T$ is positively homogeneous of degree 1, $T_t$ is also positively homogeneous of degree 1 by Lemma 1.11. From (2.4), we obtain

$$(\hat{L} + \hat{T}t_n)(\alpha_n x_n) + \alpha_n \hat{C}x_n + t_n \alpha_n M x_n = \tau_n \alpha_n j^* v^*. \quad (2.5)$$

Since $\|\hat{L} + \hat{T}t_n\|_{Y^*} = 1$, (2.5) implies

$$\tau_n \alpha_n \to \frac{1}{\|j^* v^*\|_{Y^*}},$$

and therefore

$$(\hat{L} + \hat{T}t_n)(u_n) = (\hat{L} + \hat{T}t_n)(\alpha_n x_n) \to y_0.$$
where

\[ y_0 = \frac{j^*v_s}{\|j^*v_s\|_Y}. \]

Since \( u_n \to 0 \), we have

\[ \lim_{n \to \infty} \langle (\hat{L} + \hat{T}_{t_n})u_n, u_n \rangle = \langle y_0, 0 \rangle = 0. \]

Since \( \hat{L}, \hat{T}_{t_n}, \) and \( \hat{L} + \hat{T}_{t_n} \) are maximal monotone, by Lemma 1.2 (ii), we have

\[ y_0 = (\hat{L} + \hat{T})(0) = 0, \]

which is a contradiction to \( \|y_0\|_{Y^*} = 1 \).

We now consider the homotopy \( H : [0, 1] \times Y \to Y^* \) defined by

\[ H(s, x) = \hat{L}x + \hat{T}_tx + \hat{C}x + tMX - s\tau_0j^*v^*, \quad s \in [0, 1], \quad x \in j^{-1}(G_1), \quad \text{(2.6)} \]

where \( t \in (0, t_0] \) is fixed. It can be easily seen that \( C - s\tau_0v^* \) is bounded demicontinuous on \( G_1 \) and of type \((S_+)^*\) w.r.t. \( D(L) \).

We now show that the equation \( H(s, x) = 0 \) has no solution on the boundary \( \partial G_1(Y) \). Here, the number \( R > 0 \) is increased if necessary so that the ball \( BY(0, R) \) now also contains all solutions \( x \) of \( H(s, x) = 0 \). To this end, assume the contrary so that there exist \( \{t_n\} \subset (0, t_0], \{s_n\} \subset [0, 1], \) and \( \{x_n\} \subset \partial G_1(Y) \) such that \( t_n \to 0, s_n \to s_0, x_n \to x_0 \) in \( Y, T_{t_n}x_n \to w^* \) in \( X^* \) and \( Cx_n \to c^* \) and

\[ \hat{L}x_n + \hat{T}_{t_n}x_n + \hat{C}x_n + t_nMx_n = s_n\tau_0j^*v^*. \quad \text{(2.7)} \]

Here, the boundedness of \( \{T_{t_n}\} \) follows as in Step I of [3] Prop. 1. Since \( x_n \to x_0 \) in \( Y \), we have \( x_n \to x_0 \) in \( X \) and \( Lx_n \to Lx_0 \) in \( X^* \). Also, since \( x_n \in BY(0, R) \) and \( \partial(j^{-1}(G_1) \cap BY(0, R)) \subset \partial(j^{-1}(G_1)) \cup \partial BY(0, R) \subset j^{-1}(\partial G_1) \cup \partial BY(0, R), \) we have \( x_n \in j^{-1}(\partial G_1) = \partial G_1 \cap Y \subset \partial G_1. \) From (2.7) we obtain

\[ \langle Lx_n + T_{t_n}x_n + Cx_n + t_nL^*J^{-1}(Lx_n), x_n - x_0 \rangle = s_n\tau_0\langle v^*, x_n - x_0 \rangle. \quad \text{(2.8)} \]

If we assume

\[ \limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle > 0, \]

we easily get a contradiction using a standard argument in relation to Lemma 1.2 (i). This is because \( L + T \) is maximal monotone because \( T \) is strongly quasibounded (cf. Pascali and Sburlan [23] Proposition, p. 142)). Consequently,

\[ \limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0. \]

Since \( C \) is demicontinuous and of type \((S_+)^*\) w.r.t. \( D(L) \), we obtain \( x_n \to x_0 \) and \( Cx_n \to c^* = Cx_0 \). From (2.8), we obtain

\[ \lim_{n \to \infty} \langle Lx_n + T_{t_n}x_n, x_n - x_0 \rangle = 0. \]

Using Lemma 1.2 (ii), we obtain \( x_0 \in D(T) \) and \( w^* \in Tx_0 \). Then, in view of (2.8), it follows that

\[ \langle Lx_0 + w^* + Cx_0 - s_0\tau_0v^*, u \rangle = 0 \]

for all \( u \in Y \). Since \( Y \) is dense in \( X \), we have

\[ Lx_0 + Tx_0 + Cx_0 \ni s_0\tau_0v^*, \]

which contradicts the hypothesis (H1) because \( x_0 \in D(L) \cap D(T) \cap \partial G_1. \)
We shrink $t_0$ if necessary so that
\[ H(s, x) = 0, \quad s \in [0, 1], \quad x \in G^1_{R}(Y) \]
has no solution on the boundary $\partial G^1_{R}(Y)$ for all $t \in (0, t_0]$ and all $s \in [0, 1]$. The mapping $H(s, x)$ is an admissible homotopy for the Skrypnik’s degree. The Skrypnik’s degree, $d_s(H(s, \cdot), G^1_{R}(Y), 0)$, is well-defined and remains constant for all $s \in [0, 1]$. Also, the degree, $d(L + T + C, G_1, 0)$, developed in [2] is defined as
\[ d(L + T + C - \tau_0 v^*, G_1, 0) = \lim_{t \to 0^+} d_s(H(1, \cdot), G^1_{R}(Y), 0). \]

By shrinking $t_0$ further if necessary, we have
\[ d(L + T + C - \tau_0 v^*, G_1, 0) = d_s(H(1, \cdot), G^1_{R}(Y), 0), \quad \text{for all } t \in (0, t_0]. \]

Suppose, if possible, that
\[ d_s(H(1, \cdot), G^1_{R}(Y), 0) \neq 0 \]
for some $t_1 \in (0, t_0]$. Then there exists $x_0 \in G^1_{R}(Y)$ such that
\[ \hat{L}x + \hat{T}_t x + \hat{C}x + t_1 Mx = \tau_0 j^* v^*. \]
This contradicts the choice of $\tau_0$ as stated in (2.3). Since
\[ d_s(H(0, \cdot), G^1_{R}(Y), 0) = d_s(H(1, \cdot), G^1_{R}(Y), 0), \]
we have
\[ d_s(\hat{L} + \hat{T}_t + \hat{C} + tM, G^1_{R}(Y), 0) = d_s(H(0, \cdot), G^1_{R}(Y), 0) = 0 \quad (2.11) \]
for all $t \in (0, t_0]$. Next, we consider the homotopy $\tilde{H} : [0, 1] \times Y \to Y^*$ defined by
\[ \tilde{H}(s, x) = s(\hat{L}x + \hat{T}_t x + \hat{C}x) + tMx + (1 - s)\hat{j}x, \quad s \in [0, 1], \quad x \in j^{-1}(G^2_{R}). \]
As in [3] Step III, p.29, it can be shown that there exists $t_0 > 0$ (choose it even smaller than the one used previously if necessary) such that all the solutions
\[ \tilde{H}(s, x) = 0, \quad t \in (0, t_0], \quad s \in [0, 1] \]
are bounded in $Y$. We enlarge the previous number $R > 0$ if necessary so that all solutions of $\tilde{H}(s, x) = 0$ as above are contained in $B_Y(0, R)$ in $Y$.

We first show that there exists $t_1 \in (0, t_0]$ such that the equation $\tilde{H}(s, x) = 0$ has no solutions on $\partial G^2_{R}(Y)$ for any $t \in (0, t_1]$ and any $s \in [0, 1]$.

Here, $G^2_{R}(Y) := j^{-1}(G^2) \cap B_Y(0, R)$. Suppose that the contrary is true. Then there must exist sequences $\{t_n\} \subset (0, t_0]$, $\{s_n\} \subset [0, 1]$, $\{x_n\} \subset \partial G^2_{R}(Y)$ such that
\[ s_n(\hat{L}x_n + \hat{T}_{t_n} x_n + \hat{C}x_n) + t_n Mx_n + (1 - s_n)\hat{j}x_n = 0. \quad (2.12) \]

We may assume that $t_n \downarrow 0$, $s_n \to s_0$, $x_n \to x_0$ in $X$ and $Lx_n \to Lx_0$ in $X^*$. AS in the previous part, we can show that $x_n \in \partial G^2_2 \cap Y \subset \partial G^2_2$. If $s_n = 0$ for some $n$, then we obtain $t_n Mx_n + \hat{j}x_n = 0$. Since $M$ is monotone for such $x_n$’s by [1.3], [1.4], and $\hat{j}$ is strictly monotone, we obtain $x_n = 0$ which is a contradiction to $0 \in G^2_2$. We may now assume that $s_n \in (0, 1]$. Suppose $s_0 = 0$. Dividing both sides of (2.12), we obtain
\[ \hat{L}x_n + \hat{T}_{t_n} x_n + \hat{C}x_n + \frac{t_n}{s_n} Mx_n = -\frac{1}{s_n} s_n \hat{j}x_n, \quad (2.13) \]
which implies
\[ \langle CX_n, x_n \rangle \leq -\frac{(1-s_n)}{s_n} \|x_n\|_X^2. \]
Since \( x_n \in \partial G \), the sequence \( \{\|x_n\|_X\} \) is bounded away from zero. This leads to a contradiction to the boundedness of \( \{CX_n, x_n\} \) because \( (1-s_n)/s_n \to \infty \).

Assume that \( s_0 = 1 \). Now, by Lemma 1.4, the strong quasiboundedness of \( T \) implies that the sequence \( \{T_{t_n}x_n\} \) is bounded, and so we may assume that \( T_{t_n}x_n \to w^* \) for some \( w^* \in X^* \). From (2.12), we obtain
\[ \lim_{n \to \infty} \langle LX_n + T_{t_n}x_n + CX_n, x_n - x_0 \rangle = 0. \] (2.14)
If (2.9) is true, we obtain a contradiction to (i) of Lemma 1.2. Therefore (2.10) must hold true. With (2.14), this implies \( x_n \to x_0 \in \partial G \), and therefore \( x_0 \in D(T) \) and \( LX_0 + TX_0 + CX_0 \geq 0 \). This is a contradiction to hypothesis (H2) for \( \lambda = 0 \).

For the remaining case \( s_0 \in (0,1) \), one can see that (2.13) is replaced with
\[ \limsup_{n \to \infty} \langle LX_n + T_{t_n}x_n + CX_n, x_n - x_0 \rangle \leq 0. \] (2.15)
We may assume that \( T_{t_n}x_n \to w^* (\text{some}) \in X^* \). By using the monotonicity of \( L \), \( T_{t_n} \), the continuity of \( T_t \) from Lemma 1.9 and a standard argument, we obtain \( x_n \to x_0 \in \partial G \), and hence (2.13) implies
\[ \langle LX_0 + w^* + CX_0 + \frac{1-s_0}{s_0} Jx_0, u \rangle = 0 \]
for all \( u \in Y \). By the density of \( Y \) in \( X \), we obtain
\[ LX_0 + TX_0 + CX_0 + \frac{1-s_0}{s_0} Jx_0 \geq 0, \]
which contradicts hypothesis (H2).

At this time, we replace the number \( t_0 \) chosen previously with \( t_1 \) and call it \( t_0 \) again. Let us fix \( t \in (0, t_0) \) and consider the homotopy equation
\[ \bar{H}(s,x) = s(\hat{L}x + \hat{T}_ix + \hat{C}x) + tMx + (1-s)\hat{J}x = 0, \quad s \in [0,1], \ x \in G^2_R(Y). \] (2.16)
It is already shown that (2.16) has no solution on \( \partial G^2_R(Y) \). We note that \( \bar{H} \) is an affine homotopy of bounded demicontinuous operators of type \( (S_+) \) on \( G^2_R(Y) \); namely, \( \hat{L} + \hat{T}_i + \hat{C} + tM \) and \( tM + \hat{J} \). We also note here that \( tM + \hat{J} \) is strictly monotone. Therefore \( \bar{H}(s, x) \) is an admissible homotopy for the Skrypnik’s degree, \( d_S \), which satisfies
\[ d_S(\bar{H}(1, \cdot), G^2_R(Y), 0) = d_S(\bar{H}(0, \cdot), G^2_R(Y), 0). \] (2.17)
This implies
\[ d_S(\hat{L} + \hat{T}_i + \hat{C} + tM, G^2_R(Y), 0) = d_S(tM + \hat{J}, G^2_R(Y), 0) = 1 \] (2.18)
for all \( t \in (0, t_0] \). The last equality follows from [10] Theorem 3, (iv)]. From (2.11) and (2.18), we obtain
\[ d_S(\hat{L} + \hat{T}_i + \hat{C} + tM, G^1_R(Y), 0) \neq d_S(\hat{L} + \hat{T}_i + \hat{C} + tM, G^2_R(Y), 0) \]
for all \( t \in (0, t_0] \). By the excision property of the Skrypnik’s degree, for each \( t \in (0, t_0] \), there exists a solution \( x_t \in G^1_R(Y) \setminus G^2_R(Y) \) of the equation
\[ \hat{L}x + \hat{T}_ix + \hat{C}x + tMx = 0. \]
We now pick a sequence \( \{ t_n \} \subset (0,t_0] \) such that \( t_n \downarrow 0 \), and denote the corresponding solution \( x_t \) by \( x_n \), i.e.

\[
\hat{L}x + \hat{T}_n x + \hat{C}x + t_n Mx = 0.
\]

Since \( Y \) is reflexive, we have \( x_n \to x_0 \in Y \) by passing to a subsequence. This implies \( x_n \to x_0 \in X \) and \( Lx_n \rightharpoonup Lx_0 \) in \( X^* \). By the strong quasiboundedness of \( T \), we may assume that \( T_n x_n \to w^* \in X^* \). If (2.9) holds, then we obtain a contradiction by Lemma 1.2 (i). Then (2.10) must be valid. Since \( C \) is of type \((S_+)\) w.r.t. \( D(L) \), we obtain \( x_n \to x_0 \in G_R^1(Y) \setminus G_R^2(Y) \), and by Lemma 1.1 we have \( x_0 \in D(T) \) and \( Lx_0 + w^* + Cx_0 = 0 \), and therefore \( Lx_0 + Tx_0 + Cx_0 = 0 \).

It remains to show that \( x_0 \in G_1 \setminus G_2 \). Since

\[
G_R^1(Y) \setminus G_R^2(Y) = (G_1 \setminus G_2) \cap Y \cap B_Y(0,R) \subset G_1 \setminus G_2,
\]

we have \( x_n \in G_1 \setminus G_2 \) for all \( n \), and so

\[
x_0 \in \overline{G_1 \setminus G_2} \subset (G_1 \setminus G_2) \cup \partial (G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.
\]

By hypotheses (H1) and (H2), \( x_0 \not\in \partial G_1 \cup \partial G_2 \). Thus, \( x_0 \in D(L) \cap D(T) \cap (G_1 \setminus G_2) \). \( \square \)

3. Nonzero solutions of \( Tx + Cx + Gx \geq 0 \)

Hu and Papageorgiou [15] generalized the degree theory of Browder [12] to the mappings of the form \( T + C + G \), where \( T \) is maximal monotone with \( 0 \in T(0) \), \( C \) bounded demicontinuous of type \((S_+)\) and \( G \) belongs to class \((P)\). In this section, with an application of Browder and Skrypnik degree theories, the existence of nonzero solutions of the inclusion \( Tx+Gx \geq 0 \) is established with an additional condition of positive homogeneity of degree \( \alpha \in \langle 0,1 \rangle \) on \( T \). The result extends and generalizes a similar result by Kartsatos and the author in [3, Theorem 6, p.1246], for \( \alpha = 1 \) and \( G = 0 \) to a multivalued \( T \) with \( \alpha \in \langle 0,1 \rangle \) and \( G \neq 0 \). This result is new for \( \alpha \in \langle 0,1 \rangle \) and applies to partial differential equations involving \( p \)-Laplacian with \( p \in \langle 1,2 \rangle \).

In what follows, the norms in \( X \) and \( X^* \) are both denoted by \( \| \cdot \| \) and will be understood from the context of their use.

**Theorem 3.1.** Assume that \( G_1, G_2 \subset X \) are open, bounded with \( 0 \in G_2 \) and \( \overline{G_2} \subset G_1 \). Let \( T : X \supset D(T) \to 2^{X^*} \) be maximal monotone, and positively homogeneous of degree \( \alpha \in \langle 0,1 \rangle \), \( C : G_1 \to X^* \) bounded, demicontinuous and of type \((S_+)\), and \( G : \overline{G_1} \to 2^{X^*} \) of class \((P)\). Moreover, assume the following:

(H3) There exists \( v_0^* \in X^* \setminus \{ 0 \} \) such that \( Tx + Cx + Gx \not\geq \lambda v_0^* \) for every \( (\lambda, x) \in \mathbb{R}_+ \times \{ (D(T) \cap \partial G_1) \}; \)

(H4) \( Tx + Cx + Gx + \lambda Jx \not\geq 0 \) for every \( (\lambda, x) \in \mathbb{R}_+ \times \{ (D(T) \cap \partial G_2) \} \).

Then the inclusion \( Tx + Cx + Gx \geq 0 \) has a nonzero solution \( x \in D(T) \cap (G_1 \setminus G_2) \).

**Proof.** We consider the inclusion

\[
Tx + Cx + Gx \geq 0
\]

and then the associated approximate equation

\[
T_\epsilon x + Cx + g_\epsilon x = 0. \tag{3.1}
\]
Here, \( \epsilon > 0 \) and \( g_\epsilon \colon \overline{G_1} \to X^* \) is an approximate continuous Cellina-selection (cf. [15], [6, Lemma 6, p. 236]) satisfying
\[
g_\epsilon x \in G(B_s(x) \cap \overline{G_1}) + B_s(0)
\]
for all \( x \in \overline{G_1} \) and \( g_\epsilon(G_1) \subset \text{conv}G(\overline{G_1}) \).

We show that equation (3.1) has a solution \( x_{t,\epsilon} \) in \( G_1 \setminus G_2 \) for all sufficiently small \( t \) and \( \epsilon \). To this end, we first show that there exist \( \tau_0 > 0 \), \( t_0 > 0 \) and \( \epsilon_0 > 0 \) such that the equation
\[
T_t x + C x + g_\epsilon x = \tau \nu_0^*
\]
has no solution in \( G_1 \) for every \( \tau \geq \tau_0 \), \( t \in (0, t_0] \) and \( \epsilon \in (0, \epsilon_0] \).

Assuming the contrary, let \( \{\tau_n\} \subset (0, \infty) \), \( \{t_n\} \subset (0, \infty) \), \( \{\epsilon_n\} \subset (0, \infty) \) and \( \{x_n\} \subset G_1 \) be such that \( \tau_n \to \infty \), \( t_n \downarrow 0 \), \( \epsilon_n \downarrow 0 \) and
\[
T_{t_n} x_n + C x_n + g_{\epsilon_n} x_n = \tau_n \nu_0^*.
\]
We may assume that \( g_{\epsilon_n} x_n \to g^* \in X^* \) in view of the properties of \( G \). Then \( \|T_{t_n} x_n\| \to \infty \) as \( \|\tau_n \nu_0^*\| \to \infty \) and \( \{C x_n\} \) is bounded.

Thus, from (3.3), we obtain
\[
\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} + \frac{C x_n}{\|T_{t_n} x_n\|} + \frac{g_{\epsilon_n} x_n}{\|T_{t_n} x_n\|} = \frac{\tau_n}{\|T_{t_n} x_n\|} \nu_0^*,
\]
In view of (1.7), we obtain
\[
\frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} = T_{t_n} \lambda_n \left( \frac{x_n}{\|T_{t_n} x_n\|^{1/\alpha}} \right),
\]
where
\[
\lambda_n = \|T_{t_n} x_n\|^{(\alpha - 1)/\alpha}.
\]
It clear that \( \lambda_n \to 0 \) for \( \alpha \in (0, 1) \) and \( \lambda_n = 1 \) for \( \alpha = 1 \). Then (3.4) implies
\[
1 - \frac{\|C x_n\|}{\|T_{t_n} x_n\|} - \frac{\|g_{\epsilon_n} x_n\|}{\|T_{t_n} x_n\|} \leq \frac{\tau_n}{\|T_{t_n} x_n\|} \leq 1 + \frac{\|C x_n\|}{\|T_{t_n} x_n\|} + \frac{\|g_{\epsilon_n} x_n\|}{\|T_{t_n} x_n\|}.
\]
Thus,
\[
\frac{\tau_n}{\|T_{t_n} x_n\|} \to 1 \quad \text{and} \quad \frac{T_{t_n} x_n}{\|T_{t_n} x_n\|} \to \frac{1}{\|\nu_0^*\|} \text{ as } n \to \infty.
\]
Let
\[
u_0^* = \|T_{t_n} x_n\|^{1/\alpha}.
\]
We have \( u_n \to 0 \). By (3.4), (3.5) and (3.6), we obtain \( T_{t_n} \lambda_n u_n \to h \) with
\[
h = \frac{\nu_0^*}{\|\nu_0^*\|}.
\]
Therefore
\[
\lim_{n \to \infty} \langle T_{t_n} \lambda_n u_n, u_n \rangle = \langle h, 0 \rangle = 0.
\]
Since \( t_n \lambda_n \to 0 \), by (ii) of Lemma 1.2 with \( S = 0 \) we obtain, \( 0 \in D(T) \) and \( h = T(0) \).

Since \( T(0) = 0 \), this is a contradiction to \( \|h\| = 1 \).

We now consider the homotopy mapping
\[
H_t(s, x, t, \epsilon) = T_t x + C x + g_\epsilon x - s \tau_0 \nu_0^*, \quad s \in [0, 1], \ x \in \overline{G_1},
\]
(3.7)
where \( t \in (0, t_0] \) and \( \epsilon \in (0, \epsilon_0] \) are fixed. For every \( s \in [0, 1] \) the operator \( x \mapsto Cx - s\tau_0v_0^* \) is demicontinuous and bounded on \( G_1 \). In order to see that it is of type \((S_+)\), assume that \( \{x_n\} \subset G_1 \) satisfies \( x_n \to x_0 \in X \) and
\[
\limsup_{n \to \infty} \langle Cx_n - s\tau_0v_0^*, x_n - x_0 \rangle \leq 0.
\]
Then
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0,
\]
which by the \((S_+)\)-property of \( C \), implies \( x_n \to x_0 \in \overline{G_1} \). Before we consider the Skrypnik degree of this homotopy on the set \( G_1 \), we show that the equation \( H_1(s, x, t, \epsilon) = 0 \) has no solution on the boundary of \( G_1 \) for all sufficiently small \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \) and all \( s \in [0, 1] \). To this end, assume the contrary and let \( \{x_n\} \subset \partial G_1 \), \( \{t_n\} \subset (0, t_0], \{s_n\} \subset [0, 1] \) and \( \{\epsilon_n\} \subset (0, \epsilon_0] \) such that \( t_n \downarrow 0 \), \( s_n \to s_0 \) for some \( s_0 \in [0, 1] \), \( \epsilon_n \downarrow 0 \) and
\[
T_{t_n}x_n + Cx_n + g_{\epsilon_n}x_n = s_n\tau_0v_0^*.
\]
We may assume that \( x_n \to x_0 \in X \). Since \( \{Cx_n\} \) is bounded, we may assume that \( Cx_n \to y_0^* \in X^* \) and \( g_{\epsilon_n}x_n \to g^* \). Then we have \( T_{t_n}x_n \to -y_0^* - g^* + s_0\tau_0v_0^* \). From
\[
\langle T_{t_n}x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle = \langle g_{\epsilon_n}x_n + s_n\tau_0v_0^*, x_n - x_0 \rangle,
\]
we obtain
\[
\lim_{n \to \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle + \langle Cx_n, x_n - x_0 \rangle = 0. \tag{3.8}
\]
Let us assume that
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle > 0. \tag{3.9}
\]
Then there exists a subsequence of \( \{x_n\} \), which we still denote by \( \{x_n\} \), such that
\[
\lim_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle = q. \tag{3.10}
\]
for some constant \( q > 0 \). By (3.8) and (3.10), we obtain
\[
\lim_{n \to \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle = -q < 0.
\]
Applying (i) of Lemma 1.2 with \( S = 0 \), we obtain a contradiction. Therefore (3.9) is false and we now only have
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0.
\]
Since \( C \) is of type \((S_+)\), we have \( x_n \to x_0 \in \partial G_1 \). Since \( C \) is also demicontinuous, \( Cx_n \to Cx_0 \). This implies
\[
T_{t_n}x_n \to -Cx_0 - g^* + s_0\tau_0v_0^*.
\]
Applying (ii) of Lemma 1.2 with \( S = 0 \), we obtain \( x_0 \in D(T) \cap \partial G_1 \) and
\[
Tx_0 + Cx_0 + Gx_0 \ni s_0\tau_0v_0^*,
\]
which is a contradiction to our hypothesis \((H3)\). Thus, we may now choose \( t_0 \) and \( \epsilon_0 \) further so that we also have that \( H_1(s, x, t, \epsilon) = 0 \) has no solution \( x \in \partial G_1 \) for all \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \) and all \( s \in [0, 1] \). It is clear that the mapping \( H_1(s, x, t, \epsilon) \) is an admissible homotopy for Skrypnik’s degree and the Skrypnik degree \( d_S(H_1(s, \cdot, t, \cdot), G_1, 0) \) is well-defined and is constant for all \( s \in [0, 1] \) and for
all \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \). Consequently, the Browder’s degree generalized by Hu and Papageorgiou \cite{[15]}, \( d_{HP} \), is well-defined and satisfies
\[
d_{HP}(T + C + G - \tau_0 v_0^*, G_1, 0) = d_{S}(T_i + C + g_{\epsilon}, - \tau_0 v_0^*, G_1, 0) \tag{3.11}
\]
for \( t \in (0, t_0], \epsilon \in (0, \epsilon_0] \).

Assume that
\[
d_S(H_1(1, t_1, \epsilon_1), G_1, 0) \neq 0,
\]
for some sufficiently small \( t_1 \in (0, t_0] \) and \( \epsilon_1 \in (0, \epsilon_0] \). Then, the equation
\[
T_i x + C x + g_{\epsilon}, x = \tau_0 v_0^*
\]
has a solution in the set \( G_1 \). However, this contradicts our choice of the number \( \tau_0 \) in \( (3.2) \). Consequently,
\[
d_S(T_i + C + g_{\epsilon}, G_1, 0) = d_S(H_1(0, t_1, \epsilon_1), G_1, 0) = 0, \quad t \in (0, t_0], \epsilon \in (0, \epsilon_0].
\]

We next consider the homotopy mapping
\[
H_2(s, x, t, \epsilon) = s(T_i x + C x + g_{\epsilon}, x) + (1 - s)J x, \quad (s, x) \in [0, 1] \times \overline{G}.
\tag{3.12}
\]
We first show that there exist \( t_1 \in (0, t_0], \epsilon_1 \in (0, \epsilon_0] \) such that the equation \( H_2(s, x, t, \epsilon) = 0 \) has no solution on \( \partial G_2 \) for any \( s \in [0, 1] \), any \( t \in (0, t_1] \) and any \( \epsilon \in (0, \epsilon_1] \).

Let us assume the contrary. Then there exist sequences \( t_n \in (0, t_0], \epsilon_n \in (0, \epsilon_1], \)
\( s_n \in [0, 1], \) and \( x_n \in \partial G_2 \) such that \( t_n \downarrow 0, \epsilon_n \downarrow 0, s_n \to s_0 \in [0, 1], x_n \to x_0 \in X, \)
\( C x_n \to y_0^* \in X^*, g_{\epsilon_n} x_n \to g^* \in X^* \), \( J x_n \to z_0^* \in X^* \), and
\[
s_n (T_i x_n + C x_n + g_{\epsilon_n} x_n) + (1 - s_n)J x_n = 0. \tag{3.13}
\]
\( s_n = 0 \) is impossible because \( J(0) = 0 \) and \( J \) is injective, we may assume that \( s_n > 0 \), for all \( n \). If \( s_n \to 0 \),
\[
\langle T_i x_n + C x_n, x_n \rangle = - \left( \frac{1}{s_n} - 1 \right) \langle J x_n, x_n \rangle - \langle g_{\epsilon_n} x_n, x_n \rangle \to -\infty \tag{3.14}
\]
because \( \{\|x_n\|\} \) is bounded below away from zero. Since \( \langle T_i x_n, x_n \rangle \geq 0 \) and \( \{\langle C x_n, x_n \rangle\} \) is bounded, we see that \( (3.14) \) is impossible. Thus \( s_0 \in (0, 1] \) and \( (3.13) \) implies that
\[
T_i x_n \to -y_0^* - g^* - \left( \frac{1}{s_0} - 1 \right) z_0^*.
\]
Also, from \( (3.13) \),
\[
\langle T_i x_n + C x_n, x_n - x_0 \rangle \]
\[
= - \left( \frac{1}{s_n} - 1 \right) \langle g_{\epsilon_n} x_n + J x_n, x_n - x_0 \rangle \]
\[
= - \left( \frac{1}{s_n} - 1 \right) \left[ \langle J x_n - J x_0, x_n - x_0 \rangle + \langle g_{\epsilon_n} x_n + J x_0, x_n - x_0 \rangle \right] \tag{3.15}
\]
\[
\leq - \left( \frac{1}{s_n} - 1 \right) \langle g_{\epsilon_n} x_n + J x_0, x_n - x_0 \rangle,
\]
by the monotonicity of the duality mapping \( J \). Since \( s_0 \in (0, 1] \) and \( x_n \to x_0 \), we see from \( (3.15) \) that
\[
\limsup_{n \to \infty} \{q_n := \langle T_i x_n + C x_n, x_n - x_0 \rangle \} \leq 0.
\]
Let
\[
\limsup_{n \to \infty} \{C x_n, x_n - x_0 \} > 0. \tag{3.16}
\]
Then, for some subsequence of \( \{n\} \) denoted by \( \{n\} \) again, we have
\[
\lim_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle = q > 0.
\] (3.17)

From
\[
\langle T_{t_n}x_n, x_n - x_0 \rangle = q_n - \langle Cx_n, x_n - x_0 \rangle,
\]
we see that
\[
\limsup_{n \to \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle \leq \limsup_{n \to \infty} q_n + \lim_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq -q < 0.
\]
This implies
\[
\limsup_{n \to \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle < 0.
\]
Using (i) of Lemma 1.2, we conclude that (3.16) is impossible, and therefore (3.16) holds with \( \leq \) in place of \( > \). Since \( C \) is of type \( (S_+) \), we have \( x_n \to x_0 \in \partial G_2 \).

This implies \( Cx_n \to Cx_0 \) and \( Jx_n \to Jx_0 \) and
\[
T_{t_n}x_n \to -Cx_0 - g^* - \left( \frac{1}{s_0} - 1 \right) Jx_0.
\]

Since \( x_n \to x_0 \), we have
\[
\lim_{n \to \infty} \langle T_{t_n}x_n, x_n - x_0 \rangle = 0.
\]
Using ii of Lemma 1.2, we have \( x_0 \in D(T) \) and
\[
-Cx_0 - g^* - \left( \frac{1}{s_0} - 1 \right) Jx_0 \in Tx_0.
\]

By a property of the selection \( g_{n, x_n} \) (cf. [15, p. 238]), we have \( g^* \in G(x_0) \). This implies
\[
Tx_0 + Cx_0 + Gx_0 + \left( \frac{1}{s_0} - 1 \right) Jx_0 \ni 0.
\]

We arrived at a contradiction to our hypothesis (H4) because \( x_0 \in D(T) \cap \partial G_2 \).

For the sake of convenience, we assume that \( t_0 \) and \( \epsilon_0 \) are sufficiently small so that we may take \( t_1 = t_0 \) and \( \epsilon_1 = \epsilon_0 \).

It is now clear that the mapping \( H_2(s, x, t, \epsilon) \) is an admissible homotopy for Skrypnik’s degree and so the Skrypnik degree \( d_S(H_2(s, x, t, \epsilon), G_2, 0) \) is well-defined and constant for all \( s \in [0, 1] \), all \( t \in (0, t_0] \) and all \( \epsilon \in (0, \epsilon_0] \). By the invariance of the Skrypnik degree, for all \( t \in (0, t_0] \), \( \epsilon \in (0, \epsilon_0] \), we have
\[
d_S(H_2(1, \cdot, t, \epsilon), G_2, 0) = d_S(T_t + C + g_{\epsilon}, G_2, 0)
\]
\[
= d_S(H_2(0, \cdot, t, \epsilon), G_2, 0)
\]
\[
= d_S(J, G_2, 0) = 1.
\]

Thus, for all \( t \in (0, t_0] \), \( \epsilon \in (0, \epsilon_0] \), we have
\[
d_S(T_t + C + g_{\epsilon}, G_1, 0) \neq d_S(T_t + C + g_{\epsilon}, G_2, 0).
\]

From the excision property of the Skrypnik degree, which is an easy consequence of its finite-dimensional approximations, we obtain a solution \( x_{t, \epsilon} \in G_1 \setminus G_2 \) of \( T_t x + Cx + g_{\epsilon}x = 0 \) for every \( t \in (0, t_0] \) and every \( \epsilon \in (0, \epsilon_0] \). We let \( t_n \in (0, t_0] \) and \( \epsilon_n \in (0, \epsilon_0] \) be such that \( t_n \downarrow 0 \), \( \epsilon_n \downarrow 0 \) and let \( x_n \in G_1 \setminus G_2 \) be the corresponding solutions of \( T_t x + Cx + g_{\epsilon}x = 0 \). We have
\[
T_{t_n} x_n + Cx_n + g_{\epsilon_n} x_n = 0.
\]
We may assume that \( x_n \rightarrow x_0 \) and \( g_{e_n}x_n \rightarrow g^* \in X^* \). We have
\[
\langle T_{e_n}x_n, x_n - x_0 \rangle = -\langle Cx_n + g_{e_n}x_n, x_n - x_0 \rangle.
\]
If
\[
\limsup_{n \to \infty} \langle Cx_n + g_{e_n}x_n, x_n - x_0 \rangle > 0,
\]
then we obtain a contradiction from (i) of Lemma 1.2. Consequently,
\[
\limsup_{n \to \infty} \langle Cx_n + g_{e_n}x_n, x_n - x_0 \rangle \leq 0,
\]
and hence
\[
\limsup_{n \to \infty} \langle Cx_n, x_n - x_0 \rangle \leq 0.
\]
By the \((S_+)\)-property of \( C \), we obtain \( x_n \rightarrow x_0 \in \overline{G_1 \setminus G_2} \). Then \( Cx_n \rightarrow Cx_0 \) and \( T_{e_n}x_n \rightarrow -Cx_0 - g^* \). Using this in (ii) of Lemma 1.1, we obtain \( x_0 \in D(T) \) and \( -Cx_0 - g^* \in T_0 \). By a property of the selection \( g_{e_n}x_n \) (cf. [15, p. 238]), we have \( g^* \in G(x_0) \) and therefore \( T_0x_0 + Cx_0 + Gx_0 \ni 0 \) by Lemma 1.1. We also have
\[
x_0 \in \overline{G_1 \setminus G_2} = (G_1 \setminus G_2) \cup \partial(G_1 \setminus G_2) \subset (G_1 \setminus G_2) \cup \partial G_1 \cup \partial G_2.
\]
By conditions (H3) and (H4), we have \( x_0 \notin \partial G_1 \cup \partial G_2 \). Thus, \( x_0 \in D(T) \cap (G_1 \setminus G_2) \) and the proof is complete. \( \square \)

4. Applications

Application 1. We consider the space \( X = W_0^{m,p}(\Omega) \) with the integer \( m \geq 1 \), the number \( p \in (1, \infty) \), and the domain \( \Omega \subset \mathbb{R}^N \) with smooth boundary. We let \( N_0 \) denote the number of all multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_N) \) such that \( |\alpha| = \alpha_1 + \cdots + \alpha_N \leq m \). For \( \xi = (\xi_\alpha)|_{|\alpha| \leq m} \in \mathbb{R}^{N_0} \), we have a representation \( \xi = (\eta, \zeta) \), where \( \eta = (\eta_\alpha)|_{|\alpha| \leq m-1} \in \mathbb{R}^{N_1} \), \( \zeta = (\zeta_\alpha)|_{|\alpha| = m} \in \mathbb{R}^{N_2} \) and \( N_0 = N_1 + N_2 \). We let
\[
\xi(u) = (D^\alpha u)|_{|\alpha| \leq m}, \quad \eta(u) = (D^\alpha u)|_{|\alpha| \leq m-1}, \quad \zeta(u) = (D^\alpha u)|_{|\alpha| = m},
\]
where
\[
D^\alpha u = \prod_{i=1}^N \left( \frac{\partial}{\partial x_i} \right)^{\alpha_i}.
\]
Also, let \( q = p/(p-1) \).

We now consider the partial differential operator in divergence form
\[
(Au)(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u(x), \ldots, D^mu(x)), \quad x \in \Omega.
\]
The coefficients \( A_\alpha : \Omega \times \mathbb{R}^{N_0} \rightarrow \mathbb{R} \) are assumed to be Carathéodory functions, i.e., each \( A_\alpha(x, \xi) \) is measurable in \( x \) for fixed \( \xi \in \mathbb{R}^{N_0} \) and continuous in \( \xi \) for almost all \( x \in \Omega \). We consider the following conditions:

(H5) There exist \( p \in (1, \infty), c_1 > 0 \) and \( \kappa_1 \in L^1(\Omega) \) such that
\[
|A_\alpha(x, \xi)| \leq c_1|\xi|^{p-1} + \kappa_1(x), \quad x \in \Omega, \quad \xi \in \mathbb{R}^{N_0}, \quad |\alpha| \leq m.
\]

(H6) The Leray-Lions Condition
\[
\sum_{|\alpha| = m} [A_\alpha(x, \eta, \zeta_1) - A_\alpha(x, \eta, \zeta_2)](\zeta_1 - \zeta_2) > 0
\]
is satisfied for every \( x \in \Omega, \eta \in \mathbb{R}^{N_1}, \zeta_1, \zeta_2 \in \mathbb{R}^{N_2} \) with \( \zeta_1 \neq \zeta_2 \).
If an operator $T$ is continuous, it is maximal monotone. Similarly, condition (H5), with $A$ replaced by $B$, implies that the operator $T$ is of class ($S_+$) (cf. Kittila [22, p. 27]). Since $T$ is continuous, it is bounded, continuous and monotone (cf. Kittila [22, pp. 25-26], Pascali and Sburlan [23, pp. 274-275]). Since (H5), (H7) imply that it is bounded, continuous and monotone, we know that conditions (H5), (H6) and (H8), with $B$ in place of $A$ everywhere, imply that the operator $C$ is of type $(S_+)$ (cf. Kittila [22, p. 27]). We also consider a multifunction $H : \Omega \times \mathbb{R}^N \to 2^\mathbb{R}$ such that

(H9) $H(x, r) = \{\varphi(x, r), \psi(x, r)\}$ is measurable in $x$ and u.s.c. in $r$, where $\varphi, \psi : \Omega \times \mathbb{R}^N \to \mathbb{R}$ are measurable functions;

(H10) $|H(x, r)| = \max(|\varphi(x, r)|, |\psi(x, r)|) \leq a(x) + c_2 r$ a.e. on $\Omega \times \mathbb{R}^N$ and $a(\cdot) \in L^q(\Omega)$, $c_2 > 0$.

Define $G : W_0^{m,p}(\Omega) \to 2W^{-m,q}(\Omega)$ by

$$G\!u = \left\{ h \in W^{-m,q}(\Omega) : \exists w \in L^q(\Omega) \text{ such that } w(x) \in H(x, u(x)) \right\},$$

$$\langle h, v \rangle = \int_{\Omega} w(x)v(x) \text{ for all } v \in W_0^{m,p}(\Omega).$$

It is well-known that $G$ is u.s.c and compact with closed and convex values (cf. [13, p. 254]), and therefore is of class $(P)$.

We now state the following theorem as an application of Theorem 3.1.

**Theorem 4.1.** Assume that the operators $T$, $C$ and $G$ defined as above with $T(0) = 0$, $C(0) = 0$. Assume, furthermore, that the rest of the conditions of Theorem 3.1 are satisfied for two balls $G_1 = B_r(0)$ and $G_2 = B_q(0)$, where $0 < q < r$. Then the Dirichlet boundary value problem

$$\begin{align*}
(Au)(x) + (Bu)(x) + (Hu)(x) &\ni 0, \quad x \in \Omega, \\
(D^\alpha u)(x) &\ni 0, \quad x \in \partial \Omega, \quad |\alpha| \leq m - 1,
\end{align*}$$

has a “weak” nonzero solution $u \in B_r(0) \setminus B_q(0) \subset W_0^{m,p}(\Omega)$, which satisfies the equation $Tu + Cu + Gu \ni 0$. 

In light of recent degree theories for more general combinations of operators, such as the ones in [4], the results of this paper may be generalized. For the triplet $T + C + G$ in Theorem 2.2, the existence of nonzero solutions for the homogeneity condition for degree $\alpha > 1$ ($p > 2$ for $p$-Laplacian operator $A$ in Theorem 4.1) needs further work.

**Application 2.** Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ with smooth boundary, $m \geq 1$ an integer, and $T > 0$. Set $Q = \Omega \times [0, a]$. We consider the differential operator

$$
\frac{\partial u(x, t)}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, t, u(x, t), Du(x, t), \ldots, D^m u(x, t))
+ \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha B_\alpha(x, t, u(x, t), Du(x, t), \ldots, D^m u(x, t))
$$

(4.3)

in $Q$. The coefficients $A_\alpha = A_\alpha(x, t, \xi)$, are defined for $(x, t) \in Q$, $\xi = \{\xi_\gamma, |\gamma| \leq m\} = (\eta, \zeta) \in \mathbb{R}^N_0$ with $\eta = \{\eta_\gamma, |\gamma| \leq m - 1\} \in \mathbb{R}^{N_1}$, $\zeta = \{\zeta_\gamma, |\gamma| = m\} \in \mathbb{R}^{N_2}$, and $N_1 + N_2 = N_0$. We assume that each coefficient $A_\alpha(x, t, \xi)$ satisfies the usual Carathéodory conditions. We consider the following conditions.

(H11) (Continuity) For some $p \geq 2$, $c_1 > 0$, $g \in L^q(Q)$ with $q = p/(p - 1)$, we have

$$|A_\alpha(x, t, \eta, \zeta)| \leq c_1 (|\xi|^{p-1} + |\eta|^{p-1} + g(x, t)),$$

for $(x, t) \in Q$, $\xi = (\eta, \zeta) \in \mathbb{R}^{N_0}$, $|\alpha| \leq m$.

(H12) (Monotonicity)

$$\sum_{|\alpha| \leq m} (A_\alpha(x, t, \xi_1) - A_\alpha(x, t, \xi_2))(\xi_1 - \xi_2) \geq 0, \quad (x, t) \in Q, \; \xi_1, \xi_2 \in \mathbb{R}^{N_0}.$$

(H13) (Leray-Lions)

$$\sum_{|\alpha| = m} (A_\alpha(x, t, \eta, \zeta) - A_\alpha(x, t, \eta, \zeta^*)) \langle \zeta - \zeta^* \rangle > 0,$$

for $(x, t) \in Q$, $\eta \in \mathbb{R}^{N_1}$, $\zeta, \zeta^* \in \mathbb{R}^{N_2}$.

(H14) (Coercivity) There exist $c_0 > 0$ and $h \in L^1(Q)$ such that

$$\sum_{|\alpha| \leq m} A_\alpha(x, t, \xi) \geq c_0 |\xi|^p - h(x, t), \quad (x, t) \in Q, \; \xi \in \mathbb{R}^{N_0}.$$

Under the condition (H11), the second term of (4.3) generates a continuous bounded operator $T : X \to X^*$, where $X = L^p(0, a; V)$, $X^* = L^{q}(0, a; V^*)$, and $V = W^{m,p}_0(\Omega)$. It is defined by

$$\langle Tu, v \rangle = \sum_{|\alpha| \leq m} \int_Q A_\alpha(x, t, u, Du, \ldots, D^m u) D^\alpha v, \quad u, v \in X.$$

This operator is also maximal monotone under the condition (H12). Under (H11), (H13) and (H14) (with “$A$” replaced by “$B$” and the other necessary changes) the third term of (4.3) generates a continuous, bounded operator $C$ which satisfies the condition $(S_*)$ w.r.t. $D(L)$, where the operator $L$ is defined below. The operator $C$ is defined by

$$\langle Cu, v \rangle = \sum_{|\alpha| \leq m} \int_Q B_\alpha(x, t, u, Du, \ldots, D^m u) D^\alpha v, \quad u, v \in X.$$
The operator $\partial/\partial t$ generates an operator $L : X \supset D(L) \to X^*$, where

$$D(L) = \{v \in X : v' \in X^*, \ v(0) = 0\},$$

via the relation

$$\langle Lu, v \rangle = \int_0^a \langle u'(t), v(t) \rangle \, dt, \quad u \in D(L), \ v \in X.$$

The symbol $u'(t)$ above is the generalized derivative of $u(t)$, i.e.

$$\int_0^a \langle u'(t), \varphi(t) \rangle \, dt = -\int_0^a \langle \varphi'(t), u(t) \rangle \, dt, \quad \varphi \in C_c^\infty(0, a; X).$$

One can verify, as in Zeidler [28], that $L$ is a linear densely defined maximal monotone operator.

Let $K$ be an unbounded closed convex proper subset of $X$ with $0 \in \overset{\circ}{K}$. Let $\varphi_K : X \to \mathbb{R}_+ \cup \{\infty\}$ be defined by

$$\varphi_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{otherwise.} \end{cases}$$

The function $\varphi_K$ is proper convex and lower semicontinuous on $X$, and $x^* \in \partial \varphi_K(x)$, for $x \in K$, if and only if

$$\langle x^*, y - x \rangle \leq 0, \quad \text{for all } y \in K.$$

Also,

$$D(\partial \varphi_K) = K \quad \text{and} \quad 0 \in \partial \varphi_K(x), \quad x \in K,$$

$$\partial \varphi_K(x) = \{0\}, \quad x \in \overset{\circ}{K}.$$ 

The operator $\partial \varphi_K : X \supset K \to 2^{X^*}$ is maximal monotone with $0 \in \overset{\circ}{D(\partial \varphi_K)}$ and $0 \in \partial \varphi_K(0)$. It is thus strongly quasibounded. For these facts see, e.g., Kenmochi [21]. In addition, the sum $\partial \varphi_K + T$ is a multivalued strongly quasibounded maximal monotone operator from $K$ to $2^{X^*}$.

As an application of Theorem 2.2, we state the following theorem.

**Theorem 4.2.** Assume that the operators $L, T, C$ are as above with $A_\alpha$ satisfying (H11), (H12) and $T(0) = 0$, $C(0) = 0$, and $B_\alpha$ satisfying (H13) and (H14) with the necessary notational changes. Assume, further, that the rest of the conditions of Theorem 2.2 are satisfied for two balls $G_1 = B_r(0)$ and $G_2 = B_q(0)$, in $X = L^p(0, a, V)$, where $0 < q < r$ and $V = W_0^{m}(\Omega)$. Then the inclusion

$$Lu + \partial \varphi_K(u) + Tu + Cu \ni 0$$

has a nonzero solution $u \in B_r(0) \setminus B_q(0)$.

The mapping $\partial \varphi_K$ above is essential because the operator $T + C$ is demicontinuous, bounded and of type $(S_\infty)$ w.r.t. $D(L)$, and therefore it reduces to another operator exactly like $C$ (cf. [5, p.41]).

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References


**Dhruba R. Adhikari**

**Department of Mathematics, Kennesaw State University, Georgia 30060, USA**

*E-mail address: dadhikar@kennesaw.edu*