Integrated Mathematics Proof of Pythagorean Theorem

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Advanced Mathematics Made Easy:
Interrelationships of: Algebra, Geometry, Trigonometry, Calculus
Probable Independent solvers: Archimedes, Erasthenes, Pythagoras, Leibnitz, Newton
Proposed as: The original derivation of the proof of the Pythagorean Theorem.

angle $\alpha$

Figure 1: Circle C1

radius $r = 1$

Figure 2: Isosceles Triangle A

Figure 3: Isosceles Triangle T

radius $r = 1$

Figure 4: Circle C2; with center point O, (origin $= O$).

radius $r = 1$

Figure 5: Isosceles Triangle T
Advanced Mathematics Made Easy:
Algebra, Geometry, Trigonometry and Calculus are really all the same thing, since slope (algebra) and area (geometry) are simply the derivative and integral functions of calculus. The area of a triangle within a unit circle is trigonometry. Givens and Intuitive relationships and Intuitive proofs. Intuitive proofs of triangles, areas, and geometry.

It is given that the center of the circle is the origin (O). The radius of any circle is equal to any other radius of the same circle. See Figure 1: Circle C1.

It is given that one radius shall lie on the x axis. Any triangle may be created by drawing 2 radii from the center (origin) of a circle, and connecting the two radii to form the base of the triangle. Such a triangle has three sides, with two sides of the same length and hence, it is an isosceles triangle. See Figure 2: Isoceles Triangle A.

A perpendicular may be dropped from the origin to the center of the base of the triangle so as to create two right triangles that are each one half the area of the original isosceles triangle. See Figure 5: Isoceles Triangle T. For labeling purposes, the two right triangles that lie top and bottom shall be labeled “c” and “d” respectively. That is triangle “d” includes the x axis radius.

A perpendicular may be dropped from the corner of the triangle that joins one radius and the base to the opposite radius (this radius shall fall on the x axis for the convenience of computations.) See Figure 3: Isoceles Triangle T. For labeling purposes, the two right triangles shall be labeled “a” and “b” from left to right. That is triangle “a” includes the origin and “b” includes the base connecting the radii (isosceles legs.)

It is intuitive that the area of the entire original triangle which has a perimeter of both radii and the base connecting them, is equal to the area of the sum of the areas of triangle “a” plus triangle “b”; and is also equal to the sum of the areas of triangle “c” plus triangle “d”.

The area of a rectangle is the base times the height (b x h); or the length times the width (l x w). This may be intuitively proven with a tile floor in the bathroom or the kitchen. The area of a triangle created by bisecting a rectangle with a diagonal line segment will create two right triangles of equal area. Each newly created triangle is ½ the area of the rectangle. A(triangle) = ½(b x h) = A = ½bh.

Any triangle may have a perpendicular dropped from at least one angle to an opposite base so as to create two right triangles. The sum of the area of those two newly created right triangles is equal to the area of the original triangle. This may be intuitively proven with a tile kitchen floor or with drawings. This may be drawn with any triangle, even if the first triangle is also a right triangle.

It is given: Triangle T is Triangle O,R1,R2. Area Triangle O,R1,R2 = Area “a” + Area “b” = Area “c” + Area “d”
And, since triangle “c” is the mirror image of triangle “d”; we know that: Area “c” = Area “d”. The label “A” shall be Area. \[A OR1R2 = (a + b) = (c + d) = 2c.\]

Distance along the x axis is called the “run.”
Distance along the y axis is called the “rise.”
The ratio of the rise divided by the run is called the slope. [\(\text{rise}/\text{run} = \text{slope}\).]
The variable label we shall assign for rise/run is “m”; (slope = m); \[\text{rise}/\text{run} = \text{slope} = m = \sin/cos.\]
The variable label we shall assign to the “run” is “cos.”
The variable label we shall assign to the “rise” is “sin.”

The Angle created at the Origin (O) of Triangle T is alpha (\(\alpha\)), the angle at the origin of triangles “c” and “d” are equal and are \(\frac{\alpha}{2}\).

The radius of the circle is one, it is a unit circle.
The height of the perpendicular that divides Triangle T into triangles “a” and “b” is the rise of angle alpha = rise \(\alpha = \sin \alpha\).
The base of triangle “a” is the “run” \(\alpha = \cos \alpha\).
The hypotenuse of triangle “a” is the radius, which is = 1.
The height of triangle “a” is \(\sin \alpha\), and the base of triangle “a” is \(\cos \alpha\).
The radius is = 1.
The height of triangle “b” is = \(\sin \alpha\). Height of triangle “b” = height of triangle “a”.
The base of triangle “b” is the radius less the base of triangle “a” = \(1 - \cos \alpha\).

Triangles “c” and “d” are right triangles, with equal heights, equal bases, and equal areas.
Triangles “c” and “d” have hypotenuses of equal length, and that length being the radius = 1.
The base of triangle “c” = the base of triangle “d”; since the perpendicular bisects the base line connecting the radii. The base to triangles “c” and “d” is created by the line connecting the radii of the original Triangle T; and each base is one half the length of that connecting line segment.

The hypotenuse of triangle “c” shall be “hyp c.”
The hypotenuse of triangle “d” shall be “hyp d.” And, hyp c = hyp d = radius = 1.
The base of triangle “c” is equal to the rise of angle \(\frac{\alpha}{2} = \text{rise} \frac{\alpha}{2} = \sin \alpha/2\).
The base of triangle “d” is equal to the rise of angle \(\frac{\alpha}{2} = \text{rise} \frac{\alpha}{2} = \sin \alpha/2\).
The height of triangle “c” is the perpendicular that is dropped to create triangles “c” and “d”.
The height of triangle “c” = height triangle “d” = run of angle \(\frac{\alpha}{2} = \cos \alpha/2\).
The height of triangle “d” = run of angle \(\frac{\alpha}{2} = \cos \alpha/2\).

Area = A.
\[A(Triangle \ T) = A(a) + A(b) = A(c) + A(d) = 2A(c)\].
\[A(a) = \frac{1}{2}\{(\sin \alpha)(\cos \alpha)\} = \frac{1}{2} \cos \alpha \sin \alpha\]
\[A(b) = \frac{1}{2}\{(1-\cos \alpha)\sin \alpha\}\]
\[A(c) = \frac{1}{2}\{(\sin \alpha/2)(\cos \alpha/2)\}\]
\[A(d) = \frac{1}{2}\{(\cos \alpha/2)(\sin \alpha/2)\}\]
\[A(c) = A(d) = \frac{1}{2}(\cos \alpha/2)(\sin \alpha/2)\]
\[A(c) + A(d) = 2[\frac{1}{2}(\cos \alpha/2)(\sin \alpha/2)] = A(c) + A(d) = \cos \alpha/2 \sin \alpha/2.\]
\( A(a) + A(b) = \frac{1}{2}(\sin \alpha)(\cos \alpha) + \frac{1}{2}(1-\cos \alpha)(\sin \alpha) \)

\( A(a) + A(b) = \frac{1}{2} \sin \alpha \cos \alpha + \left( \frac{1}{2} - \frac{1}{2} \cos \alpha \right)(\sin \alpha) \)

\( A(a) + A(b) = \frac{1}{2} \sin \alpha \cos \alpha + \frac{1}{2} \sin \alpha - \frac{1}{2} \cos \alpha \sin \alpha \)

\( A(a) + A(b) = \frac{1}{2} \sin \alpha + \left( \frac{1}{2} \sin \alpha \cos \alpha \right) - \left( \frac{1}{2} \sin \alpha \cos \alpha \right) \)

\( A(a) + A(b) = \frac{1}{2} \sin \alpha + 0 \)

\( A(a) + A(b) = \frac{1}{2} \sin \alpha \)

\( A(c) + A(d) = 2A(c) = A(a) + A(b) = \)

\( A(c) + A(d) = \frac{1}{2}(\cos \alpha/2 \sin \alpha/2) + \frac{1}{2}(\cos \alpha/2 \sin \alpha/2) = \cos \alpha/2 \sin \alpha/2 \)

\( A(c) + A(d) = A(a) + A(b) = \cos \alpha/2 \sin \alpha/2 = \frac{1}{2} \sin \alpha \)

\( \frac{1}{2} \sin \alpha = \cos \alpha/2 \sin \alpha/2 \)

Place alpha in terms of alpha/2; that is: \( \sin \alpha = \left( \frac{1}{2} \right)(\cos \alpha/2 \sin \alpha/2) \)

\( A(a) = \frac{1}{2}(\sin \alpha)(\cos \alpha) \)

\( A(b) = \frac{1}{2}(\sin \alpha)(1- \cos \alpha) \)

\( A(c) = \frac{1}{2}(\cos \alpha/2)(\sin \alpha/2) \)

\( A(d) = \frac{1}{2}(\cos \alpha/2)(\sin \alpha/2) \)

\( A(a) + A(b) = A(c) + A(d) = 2A(c) \)

\( A(a) + A(b) = \frac{1}{2}(\sin \alpha)(\cos \alpha) + \frac{1}{2}(\sin \alpha) - \frac{1}{2}(\sin \alpha \cos \alpha) = \frac{1}{2} \sin \alpha = A(c) + A(d) \)

\( A(c) + A(d) = \cos \alpha/2 \sin \alpha/2 \)

\( A(a) + A(b) = \frac{1}{2} \sin \alpha = \cos \alpha/2 \sin \alpha/2 \)

\( A(c) + A(d) = \frac{1}{2} \cos \alpha/2 \sin \alpha/2 + \frac{1}{2} \cos \alpha/2 \sin \alpha/2 = \cos \alpha/2 \sin \alpha/2 \)

Equivalent areas for triangles “a” plus “b” and triangles “c” plus “d”:
Final equality: \( \cos \alpha/2 \sin \alpha/2 = \cos \alpha/2 \sin \alpha/2 \)

Or the alpha angle equals the alpha angle; and alpha = alpha. \([\alpha = \alpha]\)

Critical result: Area of Triangle T is \( \frac{1}{2} \sin \alpha \); which makes sense, since the area of the isosceles triangle formed within a unit circle (circle of radius = 1) should be a function of the angle formed by the radii.

This also demonstrates nature’s preference for radial coordinates.

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