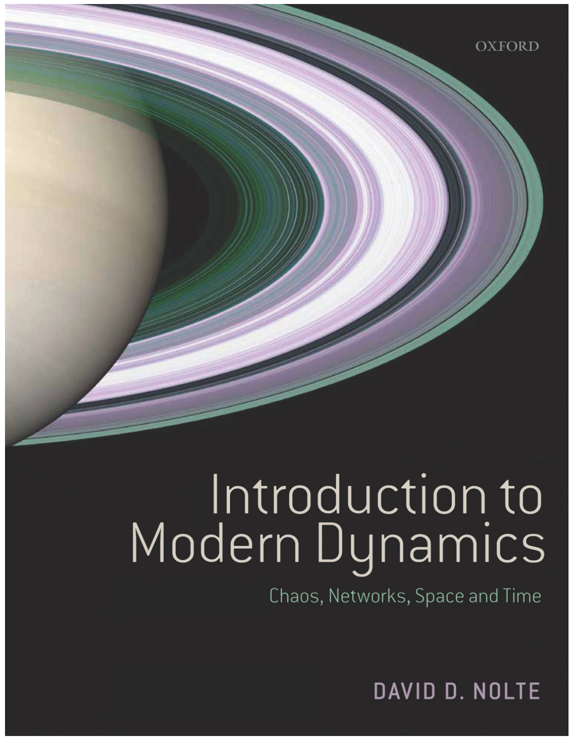
Fall 2019

Historical Notes: Modern Dynamics

David D Nolte, *Purdue University*

Historical Notes: Modern Dynamics

David D. Nolte



The stories behind the development of modern dynamics can be found in the book from Oxford University Press

Galileo Unbound: A Path Across Life, the Universe and Everything (Oxford, 2018)
https://global.oup.com/academic/product/galileo-unbound-9780198805847

and at the WordPress Blog Site

https://galileo-unbound.blog/

Part 1 Geometric Mechanics

Jean Bernard Léon **Foucault** (1819 – 1868) was a French physicist who performed one of the earliest measurements of the speed of light. In 1851 he constructed a long and heavy pendulum that was installed in the Pantheon in Paris. As it oscillated, it precessed, dramatically proving the rotation of the Earth. This made him somewhat of a celebrity, and the pendulum bears his name.

Gaspard-Gustave de **Coriolis** (1792 - 1843) was a French mathematician, mechanical engineer and scientist. He was the first to coin the term "work" for the product of force and distance, but is best known for his mathematical work on the motion of moving bodies, publishing the paper titled *Sur les équations du mouvement relatif des systèmes de corps* (On the equations of relative motion of a system of bodies) (Coriolis 1835). This work did not contain any mention of projectile trajectories on the rotating Earth, nor the cyclone motion in atmospheric science, but his mathematics were behind both of these effects, and his name was attached to these effects only in the 20th century.

2 Hamiltonian Dynamics and Phase Space
2.1 Hamilton's principle
2.2 Conservation laws
2.3 The Hamiltonian function
2.4 Central force motion
2.5 Phase space
2.6 Integrable systems and action–angle variables

Karl Gustav Jacob Jacobi

Karl Gustav Jacob Jacobi (1804 – 1851) was a Prussian mathematician best known for his development of analytical geometry of many variables (Jacobian matrix) and for his extension of Hamilton's work to dynamics. Jacobi was the first to derive the conservation of volume of phase space for a conservative dynamical system (although in his time there was no current notion of space) using a theorem of Liouville (although Liouville was not aware of its dynamical significance until much later). His lectures on dynamics from 1848 became the standard physics reference for many generations to follow.

Vladimir I. Arnold

Vladimir I. Arnold (1937—2010) was a Russian mathematician widely known for his role in the Kolomogorov-Arnold-Moser theory of nearly integrable Hamiltonian dynamics. He was a student of Kolmogorov at Moscow State University in the late 1950's. His mathematical approaches to physics were highly geometric, after the tradition of Poincaré, and he played a critical role in the application of symplectic geometry to Hamiltonian systems. His textbook *Mathematical Methods in Classical Mechanics* is the leading graduate-level textbook on geometric aspects of modern dynamics, with a strong intuitive flavor. Arnold is known for the Arnold Cat Map of the torus onto itself, which he demonstrated using the image of a cat as an example of phase space mixing in chaotic systems. He is also known for Arnold Tongues in the synchronization of oscillators on the torus.

Notes on the Historical Development of Phase Space
(See more details in "The Tangled Tale of Phase Space", D. Nolte, Physics Today, April, pp. 33-38, (2010)

The origins of both the concept of phase space as well as its name are surprisingly obscure, especially in view of the central role it plays in virtually every aspect of modern physics. The obscurity of its origins is partially a consequence of its ubiquity: it is in such common use today that few authors reference its origins when the term first arises in physics textbooks. The obscurity is also a consequence of retro-active attribution made looking backwards with hindsight. This is especially true in the case of phase space. In virtually every textbook on dynamics, the origin of phase space is placed in the hands of Liouville, usually with a citation dated to a paper of Liouville's in 1838 [1] in which he is supposed to have derived the theorem on the conservation of volume in phase space. In fact, no mention is made in Liouville's paper of phase space let alone dynamical systems. The paper is purely mathematical on the behavior of a class of solutions to specific differential equations. Though he lived to a ripe old age (he died in 1883), he was apparently unaware of its application to mechanics [2][3]. Therefore, the paper cited routinely as the origin of phase, by even the most rigorous textbooks, is not it! How did this happen? And if not by Liouville, then by whom and when and why? And where did it get it's somewhat strange name of "phase space"?

Before answering these questions, it is interesting to first look at possible candidates who did not in fact invent it or name it. One possible candidate would be Gibbs. Gibbs' works were highly influential, and he was not afraid to coin new phrases. The phrase "statistical mechanics" was one of his inventions, as was "ensemble". But in his influential textbook on statistical mechanics from 1902 [4], though he cites Liouville (like everyone else) and rederives the conservation of phase space volume, he calls it by the rather awkward phrase "extension in phase" in place of our modern "volume of phase space". The word "phase" is certainly there, but not "space", and he turns out not to be the the originator even of the "phase" part.

Another candidate would be Planck. After all, the derivation of the mode density leading to the Planck spectrum is explicitly an integral over phase space. Furthermore the volume element in phase space is given as $dpdq = \hbar \ldots$, i.e., Planck's constant and also the phase of a quantum mechanical wave. But here, too, we do not find any mention of the phrase "phase space" in Planck's papers of 1900 [5] through 1910. He uses the concept, as did Gibbs; it is clearly defined by the turn of the last century, but it was used without a name, or at least it's modern name.

The final possible candidate that often springs to mind when discussing phase space is Poincaré. He certainly was the first to use it extensively for the solution of dynamical systems, and invented powerful analytical tools for studying complex motions in phase space. These new approaches appeared in his famous treatise on the three-body problem and celestial mechanics published between 1890 and 1896 [6]. But he neither invented phase space, nor named it. Therefore, we can determine that the concept of phase space was well established by 1900, but it's name was still missing. To answer the question on the invention of phase space, we need to look farther back in time. To answer the question on the invention of its modern name, we need to look closer in time. Let's answer the first question first by going back to Liouville and finding out what role he really did play in this story and how his contribution found its way into all our modern textbooks.

Liouville's paper of 1838 appeared only a few years after Hamiltons publication of his dynamics [7, 8], and yet it made no reference to its application to dynamics, and it is possible that Liouville was unaware of Hamilton's work. This connection was first made in 1843 by Jacobi who was the first to recognize that the conditions on the system of differential equations that Liouville had studied were in fact satisfied by Hamilton's equations as reformulated by Jacobi. Jacobi's work is the mathematical origin of the application of phase space, though in the 1840's there was no concept of "space" beyond the three dimensions of our physical space. This was the time (slightly) before Grassmann [9] and Riemann [10] and their new notions of multidimensional spaces. For Jacobi, there was no space, only products of differentials of many variables. And there certainly were no "trajectories" through the phase space, other than physical trajectories of individual particles. Therefore, Jacobi is the originator of the analytical treatment of dynamical systems of many variables, but cannot be designated the originator of phase space. The time was not right.

Jacobi's work would likely have sunk into obscurity if not for the later generosity of Ludwig Boltzmann. Boltzmann took up the reigns of the kinetic theory of heat laid down by Maxwell and he developed it into a probabilistic theory that relied on the laws of dynamics governing the atoms of the gas (at a time before the atomic theory of matter was even established). In the derivation of his dynamical probability distributions, he required the use of conservation of volume in phase space. This he rederived and published in 1871 [11, 12], unaware at the time of Jacobi's work or of his use of Liouville's theorem. By this time, Cayley's and Riemann's concepts of multidimensional spaces were becoming more widely accepted and Boltzmann specifically for the first time explicitly described systems of many particles as a *single* trajectory through a highly multi-dimensional space. Boltzmann's paper of 1871 can therefore take its rightly place as the invention of phase space and of a single trajectory describing many-body systems. He was able to give dynamical behavior a geometric interpretation (trajectory) where Jacobi was not because the time was right. Concepts of space had matured in the time between 1843 and 1871, allowing Boltzmann to visualize it.

Given Boltzmann's invention of phase space and of the system trajectory through it, why does Liouville get all the credit? Boltzmann later became aware of Jacobi's contribution [2][13], and generously placed Liouville's name on his conservation theorem in his famous Lectures on Gas Theory of 1896 [14]. Had it not been for Jacobi's reference to Liouville in his Vorlesung, Boltzmann would likely never have known of Liouville's paper, since he rederived the theorem independently. Therefore, what could very reasonably have been called Boltzmann's theorem, is called Liouville's theorem because of Boltzmann's generosity. Ironically, his naming it Liouville's theorem is one of the chief reasons for the obscurity of his own role in the invention of phase space.

Having answered the first question on the invention of the concept of phase space, we are left with the question about its name. Why phase? What phase is it refering to?

The answer to this question is also partly contributed to Boltzmann, but not the whole answer.

As Boltzmann was developing the kinetic theory of gases, specifically the system trajectory through phase space and the definition of ergodic systems, he made the analogy [15][16] [12, 17] between trajectories in 2-dimensional phase space and what are called Lissajou figures. Lissajou figures, also sometimes known as Bowditch-Lissajou figures, are two-dimensional patterns that arise when two harmonic time series are plotted against each other. These are best experienced in physics labs using oscilloscopes and function generators. When the two harmonic frequencies are rational fractions, periodic patterns occur. But when the frequency ratio is irrational, then the system trajectory visits all points on the plane bounded by the signal amplitude. In Lissajou figures, the relative phase between the harmonic signals plays an important role in determining the pattern, and the instantaneous point on the figure defines the instantaneous relative phase of the two signals. For this reason, the point on the figure is referred to as the phase point. Boltzmann borrowed this expression from Lissajou figures and applied it to the instantaneous point in phase space. Therefore, the historical origin of the term "phase" can be attributed to Boltzmann.

Ironically, Boltzmann did not take the last step and give phase space its full name. During the last decade of the 19th century, the use of the phrase "space" was still mainly restricted to refer to our physical three-dimensional space. Although the geometry of multidimensional spaces was well developed by this time, Riemann's term "manifold" was used for general spaces. It is likely that for this reason, neither Boltzmann, nor Gibbs writing at roughly the same time, called it "phase space" because of prejudice against the use of "space" for general n-dimensions.

This situation quickly changed in the first decade of the 20th century, especially with the advent of relativity and the growing common conception of four-dimensional space-time, where time takes on some of the properties of a fourth spatial dimension. Boltzmann by this time was dead (by his own hand), but one of his students, Paul Ehrenfest, was asked by Felix Klein to write a review of Boltzmann's work for the Encyclopdeia of Mathematical Sciences. Paul Ehrenfest, with his physicist wife Tatyania, published the encyclopedia article in 1911 [18]. They approached the subject highly systematically, seeking to make precise definitions. This was partly in response to the controversies that had raged during the later part of Boltzmann's life on proofs or dis-proofs of the ergodic nature of gas systems. Therefore, Paul took great pains to define a rigorous name for the multidimensional dynamical space…and invented the term "Γ-space" where the instantaneous state of the system was the "Γ-point".

There is an irony here. By this time, the stigma of using the expression "space" for n-dimensions had disappeared, and so he was very comfortable in using "space" to define Boltzmann's n-dimensions. But he dispensed with the term "phase", possibly because of its obscurity. Yet at the very beginning of the encyclopedia article, in order to set the context for his newly invented term of "Γ-space", he needed to refer back to Boltzmann's usage of "phase". He briefly refers to "Phasenraum" (phase space) in the article to set the stage, and then dispense with it. This is the first usage of the expression "phase space" in print. I suspect that this term was likely used colloqually among Boltzmann's students, in the hallways, so to speak. But it had not appeared previously in print.

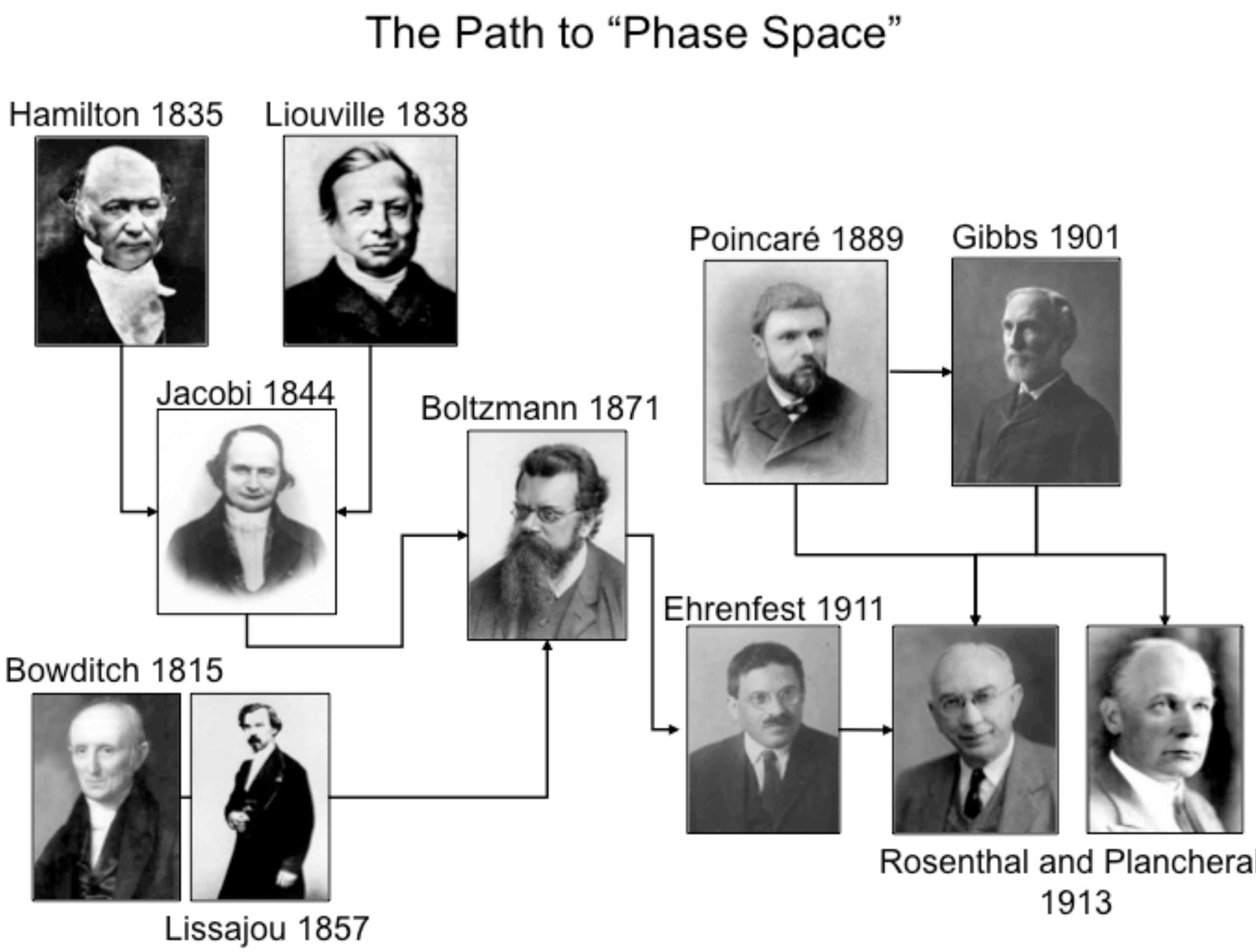
Encyclopedia articles in this day were widely read, somewhat like Reviews of Modern Physics today, and the Ehrenfest's article was no exception. And here is the irony: what stuck in reader's minds was his toss-away phrase "phase space". Virtually everyone ignored his "Γ-space".

Within two years of the Ehrenfests article of 1911, two paper appeared in the same issue of Annalen der Physik that used the expression "phase space" for the first time in journal publications. These were papers in 1913 on ergodic theory by Rosenthal [19] and Plancheral [20]. The usage of "phase space" stuck, first appearing in a journal paper title in 1918, and becoming increasingly common after that.

Interestingly, Rosenthal later became a professor of mathematics at Purdue University, passing away in 1959. Professor Ramdas remembers him. So the somewhat obscure history of phase space, after a convoluted path that starts with Liouville, finishes at Purdue.

[1] J. Liouville, "Note sur la théorie de la variation des constantes arbitraires," *Liouville Journal*, vol. 3, pp. 342-349, 1838.
[2] J. Lutzen, *Joseph Liouville 1809-1882: Master of Pure and Applied Mathematics*: Springer-Verlag, 1990.
[3] Lutzen, pg. 51
[4] J. W. Gibbs, *Elementary Principles in Statistical Mechanics*. New York: C. Scribners & Sons, 1902.
[5] M. Planck, "Distribution of energy in the normal spectrum," *Verhandlungen der Deutschen Physikalischen Gesellschaft*, vol. 2, pp. 237, 1900.
[6] H. Poincaré and D. L. Goroff, *New methods of celestial mechanics. Edited and introduced by Daniel L. Goroff*. New York: American Institute of Physics, 1993.
[7] W. R. Hamilton, "On a general method in dynamics I," *Phil. Trans. Roy. Soc.*, pp. 247-308, 1834.
[8] W. R. Hamilton, "On a general method in dynamics II," *Phil. Trans. Roy. Soc.*, pp. 95-144, 1835.
[9] H. Grassmann and L. C. Kannenberg, *A new branch of mathematics: The "Ausdehnungslehre" of 1844 and other works. Translated by Lloyd C. Kannenberg*. Chicago: Open Court, 1995.
[10] B. Riemann and R. Narasimhan, *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachtrage. Collected papers. Nach der Ausgabe von Heinrich Weber und Richard Dedekind neu hrsg. von Raghavan Narasimhan*. Berlin: Teubner, 1990.
[11] L. Boltzmann, "Über das Wärmegleichgewicht zwischen mehratomigen Gasmolekülen," *Wien. Ber.*, vol. 63, pp. 397-416, 1871.
[12] L. Boltzmann, "Einige allgemeine Sätze über Wärmegleichgewicht," *Wien. Ber.*, vol. 63, pp. 676, 1871.
[13] Lutzen, pg. 665
[14] L. Boltzmann, *Vorlesungen über Gastheorie*. Leipzig, 1898.
[15] S. Brush, *The Kinetic Theory of Gases*. London: Imperial College Press, 2003.
[16] Brush, pg. 507
[17] L. Boltzmann, *Sitzunsber. k. Akad. Wiss. Wien*, vol. Kl. II, 90, pp. 231, 1884.

[18] P. Ehrenfest and T. Ehrenfest, "Begriffliche Grundlagen der statischen Auffassung in der Mechanik," in *Encyklopädie der mathematischen Wissenschaften*, vol. IV, Part 32. Leipzig: B. G. Teubner, 1911.
[19] A. Rosenthal, "Proof of the Impossibility of Ergodic Systems," *Annalen der Physik*, vol. 42, pp. 796-806, 1913.
[20] M. Plancheral, "Proof of the Impossibility of Ergodic Mechanical Systems," *Annalen der Physik*, vol. 42, pp. 1061-1063, 1913.



The path to phase space: Jacobi synthesized the unrelated work of Hamilton and Liouville into the first derivation of conservation of phase-space volume, but without concepts of space. Boltzmann synthesized the unrelated work of Jacobi and Lissajou-Bowditch into a probabilistic theory of phase space, while Poincaré applied phase space concepts to systems of small number. Notions of a trajectory in a 2n-dimensional space became common after Gibbs, and phase space (Phasenraum) was finally named by Ehrenfest. The first explicit uses of the term "phase-space" in a paper were separately by Rosenthal and Plancheral, writing on the ergodic theory motivated by the work of Boltzmann and Poincaré.

Part 2 Nonlinear Dynamics

Henri **Poincaré** (1854-1912) was a highly influential French mathematician and theoretical physicist in his day. He introduced the necessary requirement that physics be independent of coordinate system, which became the First Postulate of Special Relativity. He was the first to discover chaos, which he stumbled upon in solutions to the three-body problem that he had prepared for a mathematics prize in honor of the King of Sweden in 1887. It was also with this work on celestial mechanics that he developed his integral invariants, one of which gave the first methematically rigorous proof of what has become known as Liouville's Theorem on the conservation of phase space volume.

Edward **Lorenz** (1917-2008) was an American mathematician who is best known for the Lorenz attractor also called the Lorenz Butterfly. His professional interests were in meterology and the introduction of nonlinear methods. He published a paper in 1963 *Deterministic Nonperiodic Flow* in the Journal of Atmospheric Science that established several of the tools used today in nonlinear dynamics. In 1969 he coined the phrase "Butterfly Effect" as a metaphor for sensitivity to initial conditions (SIC).

4 Coupled Oscillators and Synchronization
4.1 Coupled linear oscillators
4.2 Simple models of synchronization
4.3 External synchronization of an autonomous phase oscillator
4.4 External synchronization of a van der Pol oscillator
4.5 Mutual synchronization of two autonomous oscillators

Christian Huygens

Christian Huygens (1629 – 1695), a Dutch physicist, was the first to explore the phenomena of oscillations and waves in great detail. He was a prolific inventor and tinkerer of clocks and watches. In 1665 he observed that two pendulum clocks mounted near each other on a beam in his workshop could start out of phase, but would eventually come into phase and remain there in an "odd sympathy". Huygens also proposed that light is made of waves, which went counter to Newton's theory of light as particles. This theory had to wait until the time of Thomas Young (1803) before being generally accepted. Huygens was the first to propose that Saturn had rings; one of the major gaps bears his name today. Interestingly, the gap is caused by a dynamical resonance not unlike the "odd sympathy" of his pendulum clocks, although he could not have known this at the time.

Arthur Winfree

Arthur Winfree (1942 – 2002) was a theoretical biologist at the University of Arizona. He was one of the leaders in the mathematical description of biological rythms and of coupled oscillators, drawing inspiration from brainwaves, fireflies and pacemaker cells in the heart. His key insight was to ignore the amplitudes of oscillations and concentrate on the phases. This provided the foundation for studies of phase singularities in complex and dynamical systems. His work showed that synchronization could be either life-sustaining or life-arresting, depending on the system and circumstances, and that only mild perturbations are sometimes sufficient to stop spontaneous oscillations.

5 Network Dynamics
5.1 Network structures
5.2 Random network topologies
5.3 Diffusion and epidemics on networks
5.4 Linear synchronization of identical oscillators
5.5 Nonlinear synchronization of coupled phase oscillators on regular graphs

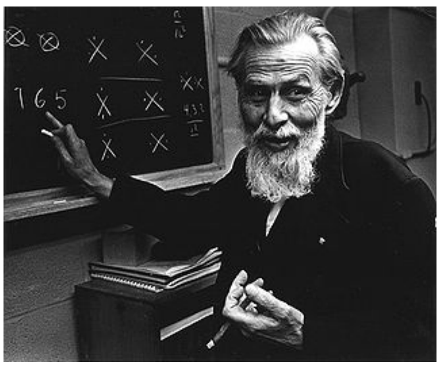
Leonard **Euler** (1701-1783) was a Swiss mathematician and physicist who was amazingly prolific and broad in his interests. His reworking of Newton's mechanics into the language of calculus and functions, published in 1736, was the first to give physics its modern "look" and set the stage for Lagrange. Euler was the father of many new fields. His solution in 1735 of the *Seven Bridges of Königsberg* problem introduced new mathematical methods and initiated the fields of graph theory and topology. This problem concerns a town in Prussia (now in Russia) that had two islands and two river banks that were connected by seven bridges. By reducing the land masses to nodes and the bridges to edges, while ignoring the spatial distances involved, he was able to prove that no one could cross each bridge only once during a single walk around town.

Paul **Erdös** (1913-1996) was a Hungarian mathematician who was possibly more prolific than Euler. He was an itinerant scholar, moving frequently among numerous international universities and institutions. Of the many fields and topics he explored, it was combinatorics that lead him to study the statistical properties of random graphs, which he published in 1959 with Alfréd Rényi. Erdös was also famous for a series of prizes that he offered personally to anyone who would solve important mathematical problems of the day. Many of these problems have yet to be solved.

Part 3 Complex Systems

6 Neurodynamics and Neural Networks
6.1 Neuron structure and function
6.2 Neuron dynamics
6.3 Network nodes: artificial neurons
6.4 Neural network architectures
6.5 Hopfield neural network
6.6 Content-addressable (associative) memory

Warren McCulloch (1898-1969) and Walter Pitts (1923-1969)



Warren McCulloch was a neurophysiologist who worked at MIT, Yale and Chicago. He was a founding member and early president of the American Society for Cybernetics. Walter Pitts was a student of logician Rudolph Carnap ("The Logical Syntax of Language" 1934) and met McCulloch at Chicago. In 1943 they proposed a simple artificial neural network model published in "A Logical Calculus of Ideas Immanent in Nervous Activity". Later, Pitts worked with Norbert Wiener at MIT.

John Hopfield (1933 -)

John Hopfield (b. 1933) is a solid state theorist who received his PhD at Cornell University as the student of Albert Overhauser. He has held faculty positions at UC Berkeley, Princeton and Cal Tech. He is best known for an associative memory model he developed in 1982, now generally known as the Hopfield Network Model.

7 Evolutionary Dynamics
7.1 Population dynamics
7.2 Virus infection and immune deficiency
7.3 The replicator equation
7.4 The quasi-species equation
7.5 The replicator–mutator equation
7.6 Dynamics of finite numbers (optional)

Sewall **Wright** (1889-1988) was an American theoretical biologist who coined the idea of a *fitness landscape* in the context of theoretical population genetics. He was one of the founders of theoretical population genetics, together with J. B. S. Haldane and R. A. Fisher, who combined ideas of evolutionary theory with genetics in the 1930s.

Manfred **Eigen** (1927 -) is a German biophysical chemist who won the 1967 Nobel Prize in Chemistry for his work on the quasi-species model that he developed with Peter Schuster. The idea of the hypercycle was originally developed to explain the emergence of order in prebiotic systems that may have led to the origin of life. The quasispecies equation lead to more general explorations within the field of evolutionary dynamics.

8 Economic Dynamics

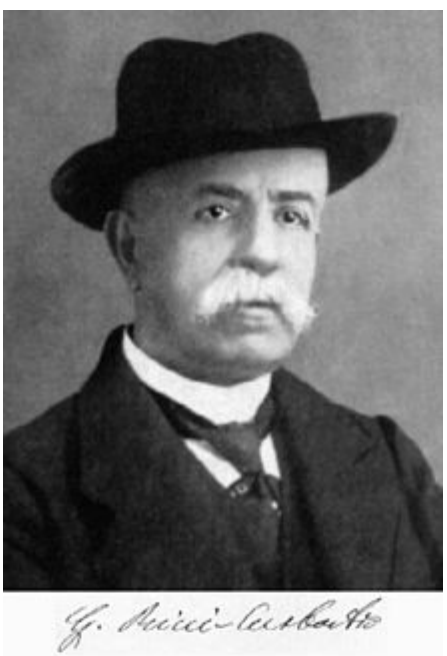
Fischer **Black** (1938 – 1995) was an American economist who is most famous for the paper published with Myron Scholes on “The Pricing of Options and Corporate Liability” in the Journal of Political Economy in 1973. He entered Harvard as a physics graduate student, but eventually received his PhD based on work he did under Marvin Minsky at MIT on the topic of artificial intelligence. He was first exposed to economic theory at a consultancy firm. Black died in 1995 and was not able to share in the Nobel Prize in Economics in 1997 although his contribution was mentioned by the Nobel committee.

Myron **Scholes** (1941 -) received the Nobel Prize in Economics in 1997 for his work with Fischer Black on the pricing of options and the development of the Black-Scholes equation.

Robert C. **Merton** (1944 -) is an American economist who received the Nobel Prize in Economics in 1997 together with Myron Scholes. Merton expanded the work of Black and Scholes.

Part 4 Relativity and Space–Time

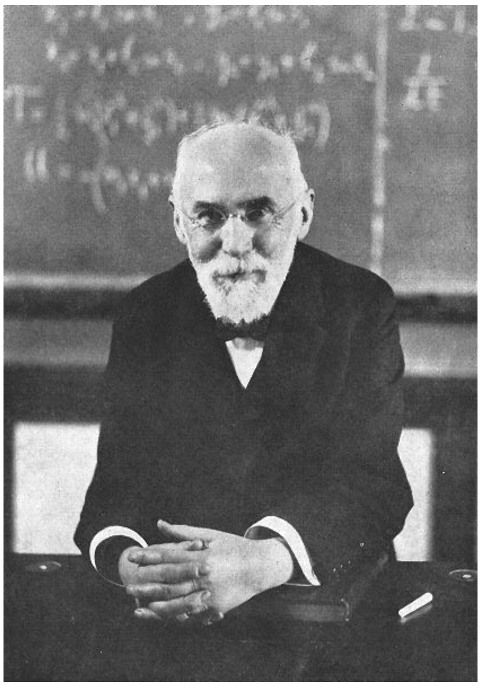
Johann Carl Friedrich **Gauss** (1777-1855) was a German mathematician and physicist who is sometimes called the Prince of Mathematicians. He was an extremely influential figure in the development of mathematics during his lifetime and originated several new fields of study, including differential geometry. His famous *Theorema Egregium* (remarkable theorem) was published in 1828. In it, he established the concept of Gaussian curvature and demonstrated that curvature can be measured within a manifold (German Mannigfältigkeit, a term coined later by Riemann) by measuring angles and distances without any need to consider the embedding dimension.



Gregorio **Ricci**-Curbastro (1853-1925) was the inventor of tensor calculus. He published, with his student Levi-Civita, the theory of tensor calculus in 1900. This book made tensor analysis generally accessible and was a major reference for Albert Einstein as he applied tensor calculus to the general theory of relativity. In turn, Einstein's formidable application of tensor calculus to General Relativity gained broad acceptance for the work of Ricci and Levi-Civita.

10 Relativistic Dynamics

Hendrik Lorentz (1853-1928)



The transformation properties of space and time were developed by Hendrik Lorentz between the years 1892 and 1905, starting long before Einstein. He introduced these relations to explain the transformation properties of electromagnetic waves that generate the famous null result of the Michelson-Morley experiment. Yet he failed to grasp the simple underlying principle that Einstein put forward in 1905. Lorentz was a Dutch physicist, a professor at the University of Leiden during the years when he was working on the electromagnetic transformation properties. He received the Nobel Prize in Physics in 1902 for his work on the Zeeman effect.

Hermann Minkowski (1864-1909)

The concept of space-time was introduced by Hermann Minkowski in an article published in 1908 titled *Raum und Zeit*. He showed that space-time constituted a pseudo-Riemann manifold. Prior to this paper, although space and time were known to be "mixed" by the Lorentz transformations, there had been no sense of ordinary space and time being components of a single 4-space. Minkowski's work on relativity was performed while a professor in Göttingen as a close colleague of David Hilbert. The introduction to his address on space and time is famous:

"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Hermann Minkowski (1907)

11 The General Theory of Relativity and Gravitation

Bernard Riemann

Bernard Riemann (1826–1866) was a German mathematician who made fundamental contributions to differential geometry with the introduction of metric spaces in his Habilitationsschrift delivered at Göttingen in 1854 with the title *On the Hypotheses which lie at the Foundation of Geometry*. The work treated multidimensional manifolds of arbitrary curvature (he coined the term *manifold* from German Manigfältigkeit). The work was not widely known until it was translated into English in 1868 by W. K. Clifford. Riemannian geometry is in general non-Euclidean, and provided the mathematical background for the later development of General Relativity.

Karl Schwarzschild

Karl Schwarzschild (1873-1916) was a German physicist and astronomer whose name is used to describe the event horizon of a black hole—the Schwarzschild radius. He received his PhD in 1896 on a topic posed by Poincaré. He joined the faculty at Göttingen where he was colleagues with David Hilbert and Hermann Minkowski. At the outbreak of World War I in 1914 he volunteered as an officer in the German army and saw action on both the western and eastern fronts. It was while he was on the eastern front in 1915 and early 1916 that he wrote his three famous papers on exact solutions to Einstein's field equations for spherical symmetry.