A Companion to the Introduction to Modern Dynamics

David D Nolte, Purdue University

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A Companion to:
Introduction to Modern Dynamics
Notes and Corrections

These online Companion Notes are supplements to the textbook Introduction to Modern Dynamics: Chaos, Networks, Space and Time (Oxford University Press, 2015). This is a Junior/Senior textbook for undergraduate mechanics/dynamics taking an updated approach to teaching mechanics at the level of Thornton and Marion (Classical Dynamics) or Taylor (Classical Mechanics).
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Chapter 1 Physics and Geometry

1. Frenet-Serret for a Simple Parabola

The Frenet-Serret formulas for a simple parabola are:

\[ y = ax^2 \]
\[ \frac{dy}{dx} = 2ax = m \]
\[ ds^2 = dx^2 + dy^2 = dx^2 \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) = dx^2 \left( 1 + m^2 \right) \]
\[ T^x = \frac{dx}{ds} = \frac{1}{\sqrt{1 + m^2}} \]
\[ T^y = \frac{dy}{ds} = \frac{dy}{dx} \frac{dx}{ds} = mT^x \]
\[ \kappa N^x = \frac{dT^x}{ds} = \frac{d}{ds} \left( \frac{1}{\sqrt{1 + m^2}} \right) = \frac{-2a}{(1 + m^2)^{3/2}} mT^x = \frac{-2a}{(1 + m^2)^{3/2}} T^y \]
\[ \kappa N^y = \frac{dT^y}{ds} = \frac{2a}{(1 + m^2)^{3/2}} T^x \]
\[ \kappa = \frac{2a}{(1 + m^2)^{3/2}} \]

For a parabolic trajectory of a mass thrown in the x-direction with initial x-speed \( v_x \) and falling under gravity \( g \), the constant \( a \) is

\[ a = -\frac{g}{2v_x^2} \]

and the curvature (as a function of time) is

\[ \kappa = \frac{-g / v_x^2}{\left( 1 + \frac{g^2 t^2}{v_x^2} \right)^{3/2}} \]
2. Three-Dimensional Rotations

The combination of the three rotation matrices \( R(\phi) \), \( R(\theta) \) and \( R(\psi) \) on pg. 24 of IMD\(^1\) is

\[
U = R(\psi)R(\theta)R(\phi)
\]

\[
\begin{pmatrix}
c_\psi c_\phi - c_\phi s_\psi s_\theta & c_\psi s_\phi + c_\theta c_\phi s_\psi & s_\theta s_\psi \\
-s_\psi c_\phi - c_\phi s_\psi c_\theta & -s_\psi s_\phi + c_\theta c_\phi c_\psi & s_\theta c_\psi \\
-s_\phi s_\theta & -s_\phi c_\theta & c_\theta
\end{pmatrix}
\]

where the shorthand notation is \( c_a = \cos a \) and \( s_a = \sin a \). The angular velocity in the body frame in terms of these rotation angles is

\[
\vec{\omega}_{\text{body}} = \begin{pmatrix}
\dot{\psi} s_\theta + \dot{\phi} c_\theta \\
\dot{\phi} c_\psi s_\theta - \dot{\theta} s_\psi \\
\dot{\phi} c_\theta + \dot{\psi}
\end{pmatrix}
\]

The rotational kinetic energy of a symmetric top is then

\[
T_{\text{rot}} = \frac{I_1}{2}(\dot{\omega}_1^2 + \dot{\omega}_2^2) + \frac{I_2}{2} \dot{\omega}_3^2
\]

\[
= \frac{I_1}{2} \left[ (\dot{\phi} c_\psi + \dot{\phi} s_\psi s_\theta)^2 + (-\dot{\theta} s_\psi + \dot{\phi} c_\psi s_\phi)^2 \right] + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} c_\theta)^2
\]

\[
= \frac{I_1}{2} (\dot{\theta}^2 + \phi^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2
\]

This expression will be useful when using the Lagrangian formulation of mechanics in Chapter 2.

\(^1\) IMD = Introduction to Modern Dynamics
The rotation of basis vectors is a key analytical operation in rotational dynamics. For instance, using the general relation of Eq. (1.95), the derivatives of the basis vectors for rotating frames is

\[
\frac{d\vec{e}}{dt}_{\text{fixed}} = \frac{d\vec{e}}{dt}_{\text{rot}} + \vec{\omega} \times \vec{e}
\]

\[
= \frac{d\vec{e}}{dt}_{\text{rot}} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}
\]

where the cross product can be expressed as a matrix multiplication. The derivatives in the fixed frame vanish, yielding

\[
\frac{d\vec{e}}{dt}_{\text{rot}} = -\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{pmatrix}
\]

or

\[
\phi = \phi \sin \theta \sin \psi \\
\dot{\phi} = \phi \sin \theta \cos \psi \\
\theta = \theta \cos \psi \\
\dot{\theta} = -\theta \sin \psi \\
\psi = 0 \\
\dot{\psi} = -\psi \\
\omega_1 = \dot{\phi} + \dot{\theta} + \dot{\psi} = \phi \sin \theta \sin \psi + \theta \cos \psi \\
\omega_2 = \dot{\phi} + \dot{\theta} + \dot{\psi} = \phi \sin \theta \cos \psi - \theta \sin \psi \\
\omega_3 = \dot{\phi} + \dot{\theta} + \dot{\psi} = \phi \cos \theta + \psi
\]
\[
\frac{d\vec{e}_x}{dt} = \omega_y \vec{e}_y - \omega_z \vec{e}_z
\]
\[
\frac{d\vec{e}_y}{dt} = -\omega_z \vec{e}_z + \omega_x \vec{e}_x
\]
\[
\frac{d\vec{e}_z}{dt} = \omega_x \vec{e}_x - \omega_y \vec{e}_y
\]

which is Eq. (1.90).

3. HW Notes

In IMD:

 HW Problem 1.4. (Be aware that not all integrals will have analytical solution. It is more important that you recognize what the integral is, and how it behaves physically. You can use Mathematica, or use look-up tables, and draw a graph as the HW solution. I will accept hand-drawn graphs as long as they are clean and have important physical limits marked.)

 HW Problem 1.6 This turns out to be a very messy problem in the general case. To help keep it manageable, note that

\[
\frac{dx}{d\theta} = -y + bx
\]
\[
\frac{dy}{d\theta} = x + by
\]
\[
\frac{dz}{d\theta} = a
\]

Additional HW Problems:

1) A physical pendulum is a pendulum that has an extended body allowed to pivot around a single point or line. Consider a metal bar of mass M with dimensions w x w x L where L >> w. It is allowed to pivot around a line that is centered at a distance x from an end along the long dimension. What is the frequency of small oscillations as a function of pivot position x?

2) Find the inertia tensor for a cylinder of radius b and height \( h = 2r \) around a point at the edge (rim) of one of the flat faces. Diagonalize this tensor to find the principal moments and the principal axes. Are the principal axes directed along recognizable symmetry axes? Redo the problem for \( h = \sqrt{3}r \).
3) Use Euler’s equations to solve for the precession rate of a rapidly spinning gyroscope precessing uniformly (no wobble, also known as no nutation). Use \( I_3 \) for the moment along the \( z \) axis along the axle.

\[
\begin{align*}
\dot{\omega}_1 + \Omega \omega_2 &= N_1 \\
\dot{\omega}_2 - \Omega \omega_1 &= N_2
\end{align*}
\]

where \( \Omega = \frac{(I_3 - I_1)}{I_1} \omega_3 \).

And then solve for the precession rate of the gyroscope. (Remember the technique of adding \( i \)-times the second to the first equation, as for Focault’s pendulum.) Because Euler’s equations are in the body frame, the torque is time-varying with angular frequency \( \omega_3 \). Note that the gyroscope precession rate is very different than the force-free precession rate \( \Omega \).

4) Find the relationships among \( (\phi, \theta, \psi) \) of Eqns: 165 – 167 and the angle \( \theta’ \) of Eq. 1.68. (Hint: Note that in the book, the angle in Eq. 1.68 is not the same as the angle in Eq. 1.66. The first step is to find the axis of rotation for Eq. 1.68 by finding the eigenvectors of the rotation matrix Eq. 1.66. Once this is found, find a vector perpendicular to the rotation axis and operate on it using the matrix in Eq. 1.66. The rotation angle is the angle between the original and rotated vector.)

5) Find the frequency of small amplitude oscillations for a thin equilateral triangle suspended from one apex (oscillating in the plane).
4. Errata (Chap 1):

Page 23
The rotation matrix in Eq. (1.59) rotates vector components clockwise by an angle $\theta$. Hence, the rotation matrix in Eq. (1.60) rotates basis vectors counterclockwise by an angle $\theta$. This is not the standard definition which assumes counterclockwise rotation of vectors as positive. The matrices are altered from one to the other simply by changing the signs on the off-diagonal terms.

Page 24
The primes of Fig. 1.9 are not the usual convention. Typically the body frame is the unprimed frame, and the fixed frame is the primed frame. Therefore, the originating axes are primed (fixed frame), then in succession double-primed and triple-primed, ending finally in unprimed (body frame).

Page 30
In Fig. 1.13, the directions of the rotation arrows should be reversed. Basis vectors transform oppositely to vector components. If a rigid body is rotating counterclockwise (the convention), then the basis vectors are rotated clockwise.

Page 43
In Fig. 1.19, the direction of $\Omega$ should be flipped because $I_3 < I_1$.

Page 48
HW Problem 1.5 The parameter $\omega^*$ is related to the helical pitch, rather than the pitch itself (depending on how you define it).
Chapter 2 Hamiltonian Dynamics and Phase Space

1. Examples of Lagrangian Applications

1) Mass on a Spring

Consider a mass $m$ hanging on a spring of spring constant $k$ in a gravitational field. The kinetic and potential energies are

$$T = \frac{1}{2} \dot{z}^2$$
$$V = \frac{1}{2} k(z - z_0)^2 + mgz$$

$$L = \frac{1}{2} m\dot{z}^2 - \frac{1}{2} k(z - z_0)^2 - mgz$$

$$\frac{\partial L}{\partial \dot{z}} = m\ddot{z} \quad \frac{\partial L}{\partial z} = -k(z - z_0) - mg$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = m\ddot{z} + k(z - z_0) + mg = 0$$

The equilibrium position is

$$k(z^* - z_0) = -mg$$
$$z^* = z_0 - \frac{mg}{k}$$

The equation of motion in the new variable $z' = z - z^*$ is

$$m\ddot{z}' = -kz'$$
$$\dot{z}' = -\omega_0^2 z'$$

which is the classic linear harmonic oscillator.

2) Simple Pendulum

Consider a mass $m$ on a massless taught string of length $L$ in a gravitation field. The kinetic and potential energies are
\[ T = \frac{1}{2} m L^2 \dot{\theta}^2 \quad V = mgL(1 - \cos \theta) \]

\[ L = \frac{1}{2} m L^2 \dot{\theta}^2 - mgL(1 - \cos \theta) \]

\[
\frac{\partial L}{\partial \dot{\theta}} = mL^2 \dot{\theta} \quad \frac{\partial L}{\partial \theta} = -mgL \sin \theta
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = mL^2 \ddot{\theta} + mgL \sin \theta = 0
\]

\[
\ddot{\theta} = -\frac{g}{L} \sin \theta \quad \dot{\theta} \approx -\omega_0^2 \theta
\]

which is again the classic linear harmonic oscillator for small angles.

3) Symmetric top with fixed tip

Fig. 2.1 Gyroscope at an angle \( \theta \) relative to horizontal.
The Lagrangian is

\[ L = \frac{1}{2} Md^2 \cos^2 \theta \dot{\phi}^2 + \frac{1}{2} Md^2 \dot{\phi}^2 + \frac{1}{2} I_3 \omega_3 - Mg d \sin \theta \]

and using \( I_3 = Md^2 \) is

\[ L = \frac{1}{2} I_3 \cos^2 \theta \dot{\phi}^2 + \frac{1}{2} I_3 \dot{\phi}^2 + \frac{1}{2} I_3 (\dot{\phi} \sin \theta + \dot{\psi})^2 - Mg d \sin \theta \]

and the partial derivatives are

\[ \frac{\partial L}{\partial \theta} = (I_3 - I_1) \dot{\phi}^2 \cos \theta \sin \theta + (I_3 \dot{\phi} - Mg d) \cos \theta \]
\[ \frac{\partial L}{\partial \phi} = 0 \]
\[ \frac{\partial L}{\partial \psi} = 0 \]

and

\[ p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} = I_1 \dot{\theta} \]
\[ p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \cos^2 \theta + I_3 \sin^2 \theta) \dot{\phi} + I_3 \sin \theta \dot{\psi} = \text{const} \]
\[ p_{\psi} = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\phi} \sin \theta + \dot{\psi}) = I_3 \omega_3 = \text{const} \]

where \( p_{\theta} \) and \( p_{\psi} \) are each constants of the motion. Comparing the last two equations, the angular momentum in \( \phi \) is

\[ p_{\phi} = I_1 \cos^2 \theta \dot{\phi} + p_{\psi} \sin \theta \]

which is rewritten as

\[ I_1 \cos^2 \theta \dot{\phi} = p_{\phi} - p_{\psi} \sin \theta \]

to obtain

\[ \dot{\phi} = \frac{p_{\phi} - p_{\psi} \sin \theta}{I_1 \cos^2 \theta} \]

Similarly
\[ p_\psi = I_3 \left( \frac{(p_\phi - p_\psi \sin \theta)}{I_1 \cos^2 \theta} \sin \theta + \psi \right) \]

\[ \psi = \omega_3 - \frac{(p_\phi - p_\psi \sin \theta)}{I_1 \cos^2 \theta} \sin \theta \]

The dynamics in \( \theta \) are obtained as

\[ I \dot{\theta} = (I_3 - I_1) \phi^2 \cos \theta \sin \theta + (I_3 \psi \dot{\phi} - Mgd \cos \theta) \]

To explore some of the behavior of the gyroscope, take the special case \( \sin \theta = 0 \) and find the conditions for \( \dot{\theta} = 0 \), then

\[
\begin{align*}
0 &= Mgd - I_3 \psi \dot{\phi} \\
    &= Mgd - I_3 \omega_3 \dot{\phi} \\
\dot{\phi} &= \frac{Mgd}{I_3 \omega_3} = \frac{Mgd}{p_\psi}
\end{align*}
\]

which is the case for steady precession. Now consider small oscillations (known as nutations) around \( \theta = 0 \)

\[
I_i \dot{\theta} = (I_3 - I_1) \frac{p_\phi^2}{I_1^2} \theta + \left( I_3 \left( \omega_3 - \frac{p_\phi \theta}{I_1} \right) \left( \frac{p_\phi - p_\psi \theta}{I_1} \right) - Mgd \right)
\]

which simplifies to

\[
\dot{\theta} = -\left[ \frac{p_\phi^2 + p_\psi^2}{I_1^2} \right] \theta = -\omega_3^2 \theta
\]

Small displacements in \( \theta \) oscillate with an angular frequency proportional to the sum of the squares of the constant angular momenta. If the gyroscope spins with higher speed, the frequency of oscillation is higher.
2. Examples of Lagrangian Applications with Constraints

1) Massive Pulley

Consider a mass on a massless rope attached to a massive pulley, or a massless rope wrapped around a cylinder able to spin without friction on its axis. This problem will use Lagrange’s undetermined multipliers as an example. The equation of constraint is

\[ f = y - R\theta = 0 \]

The kinetic and potential energies, using two generalized coordinates: one angle for the pulley, and one linear coordinate for the falling mass

\[ T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m\dot{y}^2 \]

\[ V = -mg y \]

\[ L = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m\dot{y}^2 + mg y \]

\[ \frac{\partial L}{\partial \theta} = I \dot{\theta} \]

\[ \frac{\partial L}{\partial \theta} = 0 \]

\[ \frac{\partial L}{\partial \dot{y}} = m \dot{y} \]

\[ \frac{\partial L}{\partial y} = mg \]
The three dynamical equations in three unknowns are

\[ I\ddot{\theta} + \lambda R = 0 \]
\[ m\ddot{y} - mg - \lambda = 0 \]
\[ \ddot{y} - R\dot{\theta} = 0 \]

\[ \lambda = m\ddot{y} - mg \]
\[ I\ddot{\theta} + (mR\dot{\theta} - mg)R = 0 \]
\[ \ddot{\theta}(I + mR^2) = mgR \]

\[ \dot{\theta} = \frac{mgR}{(I + mR^2)} \]
\[ \ddot{y} = \frac{mgR^2}{(I + mR^2)} \]
\[ \lambda = mg \left[ \frac{I}{(I + mR^2)} \right] \]

where the undetermined multiplier is equal to the tension on the rope.

2) Atwood Machine with Massive Pulley

Two masses on a massless rope around a massive pulley. This problem is the same as the last problem, but with an additional mass on the other side of the pulley. The number of degrees of freedom remain the same, but the driving mass is partially balanced across the pulley. Therefore, the equations in the preceding problem can be used directly by replacing the driving mass by the difference \( m^− = (m_1 - m_2) \) and the inertial mass by the sum \( m^+ = (m_1 + m_2) \). This yields

\[ \dot{\theta} = \frac{-m^−gR}{(I + m^+R^2)} = \frac{-(m_1 - m_2)R}{(I + (m_1 + m_2)R^2)}g \]
\[ \ddot{y} = \frac{-m^−gR^2}{(I + m^+R^2)} = \frac{-(m_1 - m_2)R^2}{(I + (m_1 + m_2)R^2)}g \]
\[ \lambda = m^−g \left( \frac{I}{I + m^+R^2} \right) = (m_1 - m_2)g \left( \frac{I}{I + (m_1 + m_2)R^2} \right) \]
**3) Cylinder Rolling down an Inclined Plane**

The cylinder rolls without slipping with the equation of constraint

\[ f = y - R\theta = 0 \]

The kinetic and potential energies are

\[ T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} My^2 \]
\[ V = Mg(L - y)\sin\alpha \]
\[ L = \frac{1}{4} MR^2 \dot{\theta}^2 + \frac{1}{2} My^2 - Mg(L - y)\sin \alpha \]

\[ \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{4} MR^2 \dot{\theta} \quad \frac{\partial L}{\partial \theta} = 0 \]

\[ \frac{\partial L}{\partial \dot{y}} = My \quad \frac{\partial L}{\partial y} = Mg \sin \alpha \]

\[ \frac{1}{2} MR^2 \ddot{\theta} + \lambda R = 0 \]

\[ M\ddot{y} - Mg \sin \alpha - \lambda = 0 \]

\[ \ddot{y} - R \ddot{\theta} = 0 \]

\[ \dot{\theta} = -\frac{2 g \sin \alpha}{3R} \quad \ddot{y} = \frac{2}{3} g \sin \alpha \quad \lambda = -\frac{1}{3} Mg \sin \alpha \]

The undetermined multiplier is now determined, and it is equal to the force of friction that is required for the cylinder to roll without slipping.

3. Dissipation in Lagrangian Systems

Lagrange’s equations admit the inclusion of velocity-dependent forces. These forces are non-conservative, leading to path dependence in the dynamics, and cannot be described by a potential function. However, their effect can be captured in a power function. For instance, in viscous damping, the drag force (at low speed)

\[ F_{\text{drag}}^a = -\gamma \dot{q}^a \]

has a dissipated power of

\[ P = \gamma \sum_a \int \dot{q}^a d\dot{q}^a \]

summed over the generalized velocities and integrated over the path. The derivative of the power with respect to velocity is a generalized force

\[ Q^a = \frac{\partial P}{\partial \dot{q}^a} = \gamma \dot{q}^a \]
The Euler-Lagrange equations then become

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_a} \right) - \frac{\partial L}{\partial q_a} + Q_a = 0
\]

As an example, consider a falling mass

\[
T = \frac{1}{2} m \dot{q}^2 \quad U = mgq \quad L = T - U = \frac{1}{2} m \dot{q}^2 - mgq
\]

\[
\frac{\partial L}{\partial \dot{q}} = m \dot{q} \quad \frac{\partial L}{\partial q} = -mg \quad Q = \gamma \dot{q}
\]

and the Euler-Lagrange equation is

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_a} \right) - \frac{\partial L}{\partial q_a} + Q_a = m \dot{q} + mg + \gamma \dot{q} = 0
\]

which describes the approach to terminal velocity. Dissipative forces that depend only on velocities are common in nonlinear dynamics as a system approaches steady state. Steady-state is a form of dynamic equilibrium, and Lagrangian approaches capture dynamic equilibrium systems. These can include abstract dynamical systems such as macroeconomic systems that are described by dynamic equilibrium theory (see below in the Companion to Chapter 8).

4. Lagrange Points in the Planar Three-Body Problem

In 1890 Poincaré proved that the three-body problem is nonintegrable\(^2\). However, there are special configurations of three bodies that can be solved. For instance, there are solutions to the planar three-body problem for three gravitationally attracting masses moving in a plane with \(m_1 < m_2 < m_3\). This problem was first considered by Leonhard Euler in 1762, and was further explored by Lagrange in 1772\(^3\). He discovered a solution to the three-body problem in which the three masses occupy the vertices of equilateral triangles as each mass executes an elliptical orbit around the center of mass of the system.

For the special case of circular orbits of constant angular frequency \(\omega\), the motion of the third mass is described by the Lagrangian


\(^3\) J.L. Lagrange Essai sur le problème des trois corps, 1772, Oeuvres tome 6
where the potential is time dependent because of the motion of the two larger masses. Lagrange approached the problem by adopting a rotating reference frame in which the two larger masses $m_1$ and $m_2$ move along the stationary line defined by their centers. The new angle variable is $\theta' = \theta + \omega t$. The Lagrangian in the rotating frame is

\[
L = \frac{1}{2} m \left( \dot{\rho}^2 + \rho^2 (\dot{\theta} - \omega t)^2 \right) - V(\rho, \theta', t)
\]

\[
= \frac{1}{2} m \left( \dot{\rho}^2 + \rho^2 \dot{\theta}'^2 \right) - m \omega \rho^2 \dot{\theta}' - \frac{1}{2} m \rho^2 \omega^2 + V(\rho, \theta')
\]

where the effective potential is now time independent. The first term in the effective potential is the Coriolis effect and the second is the centrifugal term.
Fig. 2.6 Effective potential for the planar three-body problem and the five Lagrange points where the gradient of the effective potential equals zero. (From RTB.m)

The effective potential is shown in Fig. 2.6 for $m_3 = 10m_2$ and with $\dot{\theta}' = 0$. There are five locations where the gradient of the effective potential equals zero. The point L1 is the equilibrium position between the two larger masses. The points L2 and L3 are at positions where the centrifugal force balances the gravitational attraction to the two larger masses. These are also the points that separate local orbits around a single mass from global orbits that orbit the two-body system. The last two Lagrange points at L4 and L5 are at one of the vertices of an equilateral triangle, with the other two vertices at the positions of the larger masses. The first three Lagrange points are saddle points. The last two are at maxima of the effective potential.

A key question about the five Lagrange points is their stability. The equations of motion are
\[
m\dot{\rho} - m\rho\dot{\theta}'^2 + 2m\omega\rho\dot{\theta}' - m\dot{\rho}\omega^2 + \frac{\partial V(\rho,\theta')}{\partial \rho} = 0
\]

\[
m\rho^2\dot{\theta}' + 2m\rho\dot{\theta}'\dot{\rho} - m\omega\dot{\rho} + \frac{\partial V(\rho,\theta')}{\partial \theta'} = 0
\]

The corresponding flow is
\[
\dot{\rho} = v_\rho
\]
\[
\dot{v}_\rho = \rho v'_\rho - 2\omega v_\rho + \rho\omega^2 - \frac{1}{m} \frac{\partial V(\rho,\theta')}{\partial \rho}
\]
\[
\dot{\theta}' = v'_{\theta'}
\]
\[
\dot{v}'_{\theta'} = -2\frac{v_\rho v'_\rho}{\rho} + \frac{\omega v_\rho}{\rho^2} - \frac{1}{m\rho^2} \frac{\partial V(\rho,\theta')}{\partial \theta'}
\]

The Jacobian matrix is
\[
J = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & A & B & C \\
0 & 0 & 1 & 0 \\
D & E & F & G
\end{pmatrix}
\]

with
\[
A = v'_\rho - 2\omega v_\rho + \omega^2 - \frac{1}{m} \frac{\partial^2 V(\rho,\theta')}{\partial \rho^2}
\]
\[
B = 2\rho(v'_\rho - \omega)
\]
\[
C = -\frac{1}{m} \frac{\partial^2 V(\rho,\theta')}{\partial \rho \partial \theta'}
\]
\[
D = -2\frac{v'_\rho}{\rho} + \frac{\omega}{\rho^2}
\]
\[
E = E = 2\frac{v_\rho v'_\rho}{\rho^2} - 2\frac{\omega v_\rho}{\rho^3} - \frac{1}{m\rho^2} \frac{\partial^2 V(\rho,\theta')}{\partial \theta'^2} + \frac{2}{m\rho^3} \frac{\partial V(\rho,\theta')}{\partial \theta'}
\]
\[
F = -\frac{\omega}{\rho}
\]
\[
G = -\frac{1}{m\rho^2} \frac{\partial^3 V(\rho,\theta')}{\partial \theta'^2}
\]
When the Jacobian matrix is evaluated at the Lagrange points, the saddle points L1, L2 and L3 all have one unstable solution and hence are unstable equilibria. On the other hand, even though L4 and L5 are at maxima in the effective potential (for $\theta' = 0$), the Coriolis force stabilizes motion around these points, and the third mass can orbit these points in elliptic orbits (in the moving frame). In the Jupiter-Sun system, these stable Lagrange points are occupied by the Trojan asteroids.

5. Legendre Transforms and the Hamiltonian

Legendre transforms are used when it is desired to transform a function $f(x)$ to a new function $g(y)$. It is often desirable to do this, because $y$ might be a more “natural” variable to work with than $x$. In mechanics, momenta may be viewed as more fundamental than velocities, because momenta are often conserved quantities in dynamical motion. Therefore, one would like to transform the Lagrangian from a function of velocities $L(q^a, \dot{q}^a)$ to a new function of momenta $H(q^a, p_a)$ known as the Hamiltonian. This new function would obey new equations of motion for the $q^a$ and the $p_a$, although clearly the new equations would be consistent with the Euler-Lagrange equations because they describe the same physics. The Hamiltonian is derived from the Lagrangian by a Legendre transform. The further usefulness of the Hamiltonian in dynamics comes from the fact that if the Lagrangian is not an explicit function of time, then the Hamiltonian itself is a constant of the motion. In addition, Hamilton’s equations of motion are first-order in time derivatives, compared to the second-order time derivatives that arise from the Euler-Lagrange equations.

Going deeper into the geometric aspects of Legendre transforms, it is important to note a momentum is not in general related linearly to the conjugate generalized velocity. In Cartesian coordinates, one may have $p = m\dot{q}$, but this is not generally true for generalized coordinates. Note, too that momenta $p_a$ are strictly covariant vectors, while generalized coordinates $q^a$ are contravariant vectors. This is seen directly in the definition of the conjugate momenta

$$p_a = \frac{\partial L}{\partial \dot{q}^a}$$

which is a covariant derivative. For non-Cartesian or non-Euclidean geometries, this distinction must be maintained. The Legendre transform guarantees that the transformed function of momenta $H(q^a, p_a)$ has the correct form in these non-Cartesian and non-Euclidean geometries.

**Example of the Legendre Transform**: A mass constrained to move on the surface of a sphere
The line element is

\[ ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

and the mass has a speed

\[ \dot{s}^2 = r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \]

The kinetic energy and potential energy are

\[ T = \frac{1}{2} m \dot{s}^2 \quad V = mgr \cos \theta \]

The Lagrangian is

\[ L = \frac{1}{2} m \left( r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - mgr \cos \theta \]

Applying Hamilton’s equations

\[ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi} \]

\[ \dot{\theta} = \frac{p_\theta}{mr^2} \quad \dot{\phi} = \frac{p_\phi}{mr^2 \sin^2 \theta} \]

Therefore, the Legendre transform is

\[ H = \dot{\theta} p_\theta + \dot{\phi} p_\phi - L \]

\[ = \frac{p_\theta^2}{mr^2} + \frac{p_\phi^2}{mr^2 \sin^2 \theta} p_\phi - \frac{1}{2} m \left( r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + mgr \cos \theta \]

which simplifies to

\[ H = \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2 \theta} + mgr \cos \theta \]

which is recognized as the total energy of the system.
6. Examples of the use of Poisson Brackets

Applying Poisson brackets to position and momentum lead directly back to Hamilton’s equations

\[
\dot{q} = \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} - \frac{\partial q}{\partial p} \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p}
\]

\[
\dot{p} = \frac{\partial p}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial p}{\partial q} \frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q}
\]

The Poisson bracket between two conjugate variables leads to a non-zero commutator relation

\[
\{p, q\} = \frac{\partial p}{\partial q} \frac{\partial q}{\partial p} - \frac{\partial p}{\partial q} \frac{\partial q}{\partial p} = -1
\]

This non-zero commutator has a direct analog in the commutation of operators in quantum mechanics, where, for instance

\[
[\hat{p}, \hat{q}] = -i\hbar
\]

This kind of commutator is a key element in Heisenberg’s uncertainty principle, where

\[
\sigma_A \sigma_B \geq \frac{1}{2} \left|\left\{\hat{A}, \hat{B}\right\}\right|
\]

for two quantum operators \(\hat{A}\) and \(\hat{B}\). If they commute, then a single state (wave function) can be an eigenstate of each. However, if they do not commute, then no state can be found that is simultaneously an eigenstate of each operator.

For general problems, the Poisson bracket can be used to monitor the time change of quantities of interest. For example, in the case of the simple harmonic oscillator with a slowly-varying spring constant, the potential energy varies as

\[
\frac{d}{dt} V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial V}{\partial p} \frac{\partial H}{\partial q}
\]

\[
= \frac{1}{2} q^2 \frac{\partial k}{\partial t} + kq \frac{p}{m} - 0 = \frac{1}{2} q^2 \dot{k} + kq \dot{q}
\]

The first term on the right is the change in internal energy of the spring. The last term is recognized as the work per time (power is force times speed) performed on the system by the changing spring constant.
7. Action-Angle Variables

*Action-Angle Canonical Transformation Example: Straight-line Motion*

Action-angle variables can be defined even for non-periodic motion. Consider a mass moving in 1-dimension without forces. The linear momentum is a constant of the motion, but the action-angle transformation seeks an action variable with units of angular momentum (angular momentum and action have the same units). The angular momentum is also a constant of this motion, and is

\[ p_\phi = mxr \cos \phi \]
\[ = mxb \]

where \( r \cos \phi = b \) the impact parameter. Expressing the Hamiltonian in this new variable gives

\[ H = \frac{1}{2} m \dot{x}^2 = \frac{p_\phi^2}{2mb^2} \]

The action integral is

\[ J = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} m\dot{xb}d\phi = \dot{m}b = p_\phi \]

and the energy is

\[ H = \frac{J^2}{2mb^2} \]

The equations of motion are

\[ \theta = \frac{\partial H}{\partial J} = \frac{J}{mb^2} = \omega \]
\[ j = -\frac{\partial H}{\partial \theta} = 0 \]

The action angle is

\[ \theta = \frac{x}{b} = \tan \phi \]

Note that the action angle \( \theta \) is not the physical angle \( \phi \) in configuration space. The action angle \( \theta \) changes uniformly in time, while the rate of change of \( \phi \) does not.

A generating function of Type I can be constructed for this simple problem. Note that
\[ m\dot{x} = \frac{\partial F_i}{\partial x} \quad \text{and} \quad J = -\frac{\partial F_i}{\partial \theta} \]

which imply

\[ F_i \sim m\dot{x} \quad \text{and} \quad F_i \sim -J\theta \]

which, using \( \dot{x} = b\omega \), yields the generating function

\[ F_i(x,\theta) = bm\omega x - J\theta \]

for this example.

8. Errata (Chap 2):

pg. 54
The sentence should read ... “The action integral of Euler can be written ...”

pg. 55
The sentence at the top should read ... “Applying the Euler equations to ...”

pg. 60
The expression in Eq. (2.46) defines a gauge transformation of the Lagrangian using a gauge function M. The gauge functions M are defined by four types of canonical transformations, described in Section 2.3.2, all of which are Legendre transforms (enabling transformation from one set of independent variables to another set). However, the function in Eq. (2.47) is not a gauge function, rather it is a Legendre transform of the Lagrangian, namely the Hamiltonian.

pg. 61
Eq. (2.49) shows the Legendre transform from the Lagrangian \( L(q^a,\dot{q}^a) \) to the Hamiltonian \( H(q^a,p_a) \). Note that the Hamiltonian must be rewritten in terms of the conjugate momenta.

pg. 63
Equation 2.58 that defines the conjugate momentum should be written in terms of the Lagrangian, rather than the Hamiltonian, as
\[ p_a = \frac{\partial L}{\partial q_a^\prime} \]

pg. 65

In Table 2.1 the expression in the second row for \( Q^a \) should be \( Q^a = \frac{\partial F}{\partial P_a} \).

pg. 66

Should read “... into an action-angle form, which we shall see in Section 2.6.”

pg. 67

Eq. 2.88 should be

\[
\frac{d}{dt} \left( \mu r^2 \dot{\phi} \right) = 0
\]

\[
\mu \ddot{r} - \mu r^2 \dot{\phi}^2 + \frac{dV}{dr} = 0
\]

pg. 85

The sentence after Eq. (2.160) should read: “…meaning that the \( \Theta^a \) are ignorable coordinates. The constants \( J_a \) are called the action, and the variables ...”

The sentence before Eq. (2.161) should read: “The Hamiltonian for the harmonic oscillator takes on the very simple form ...“

pg. 86

Equation (2.167) should be divided by \( 2\pi \).

pg. 87

Equation (2.170) should have a square on the momentum

\[
J = \frac{H}{\omega} = \frac{p^2}{2m\omega} + \frac{k q^2}{2\omega}
\]

pg. 88

Eq. (2.177) needs to be expressed in terms of action-angle variables for the nonlinear pendulum. For libration, the action is

\[
J(\epsilon) = \epsilon \int_{-\pi/2}^{\pi/2} \frac{\cos^2 \phi d\phi}{\sqrt{1 - \epsilon \sin^2 \phi}}
\]

\[
= \frac{2}{\pi} \left[ E\{\epsilon\} - (1 - \epsilon) K(\epsilon) \right]
\]
(with normalized variables) for the action variable, where $K(\varepsilon)$ and $E(\varepsilon)$ are the first and second complete elliptic integrals, respectively. The corresponding frequency is

$$\omega(\varepsilon) = \left( \frac{\partial J(\varepsilon)}{\partial \varepsilon} \right)^{-1} = \frac{\pi}{2K(\varepsilon)}$$

At low energies, when the pendulum approximates an harmonic oscillator

$$\varepsilon = \omega J$$
$$\omega = 1$$

as expected for a simple harmonic oscillator. However, for libration energies approaching rotation ($\varepsilon \approx 1$), the action variable approaches the constant $J = \frac{2}{\pi}$, while $\omega$ approaches zero.

pg. 92 HW Exercises

Problem 1. The function should be $f = \left( \frac{dY}{dx} \right)^2$.

Problem 2. The function should be $f = \sqrt{1 + \left( \frac{dY}{dx} \right)^2}$.
Chapter 3 Nonlinear Dynamics and Chaos

1. Nullclines and Separatrixes

Nullclines and separatrixes do not in general overlap. Nullclines are the nonlinear functions that define the flow. Separatrixes are the stable and unstable manifolds with directions defined by the eigenvectors at the fixed point. An example of the general case is shown in Fig. 3.1.

\[
\begin{align*}
\dot{x} &= x + \exp(-y) \\
\dot{y} &= -y + \exp(-x)
\end{align*}
\]

Fig. 3.1 A 2D flow showing nullclines and separatrixes attached to a saddle fixed point. (S147ext.m)

2. Bistability and Hysteresis: The Biased Double Well Potential

Bistability and hysteresis are phenomena that arise from nonlinear dynamics that have importance for technology applications. For instance, the hysteresis associated with the flipping of magnetic domains under magnetic fields is the central mechanism for magnetic memory, and bistability is a key feature of switching technology.
One of the simplest models for bistability and hysteresis is the one-dimensional double-well potential biased by a linear potential. A double-well potential can be expressed as

\[ V_{DW}(x) = x^4 - x^2 \]

This potential can be biased by a linear potential, so that the net potential is

\[ V(x) = x^4 - x^2 - cx \]

where the parameter \( c \) is a control parameter. The net potentials for several values of the parameter \( c \) are shown in Fig. 3.2. With no bias, there are two degenerate energy minima. As \( c \) is made negative, the left well has the lowest energy, and as \( c \) is made positive the right well has the lowest energy.

![Potential Energy](dwpoten.qpc)

**Fig. 3.2** A biased double-well potential in one dimension. The thresholds to destroy the local metastable minima are \( c = \{-0.55, 0.55\} \). For values beyond threshold, only a single minimum exists and no barrier. Hysteresis is caused by the mass being stuck in the metastable minimum because it has insufficient energy to overcome the potential barrier, until the barrier disappears at threshold. (dblwellhysteresis.m)

This potential energy profile can be probed by a mass that responds to the local forces exerted on the mass by the potential. For large negative values of \( c \) the mass will have its minimum energy in the left well. As \( c \) is increased, the energy of the left well increases, and rises above the energy of the right well. If the mass began in the left well, even when the left well has a higher energy than the right, there is a potential barrier that the mass cannot overcome and it remains on the left. This local minimum is a stable equilibrium, but it is called “metastable” because it is not a global minimum of the system. Metastability is the origin of hysteresis.

Once sufficient bias is applied that the local minimum disappears, the mass will roll downhill to the new minimum on the right. The bias can then be slowly removed, reversing this process, eventually bringing the mass back to the left well. However, because of the potential
barrier, the bias must change sign and be strong enough to remove the stability of the metastable fixed point. This “overshoot” defines the extent of the hysteresis.

For illustration, assume the mass obeys the flow equation

\[
\dot{x} = v \\
\dot{v} = -\gamma v - 4v^3 + 2x + c
\]

including a damping term, where the force is the negative gradient of the potential energy. The bias parameter \(c\) can be time dependent, beginning beyond the negative threshold and slowly increasing until it exceeds the positive threshold, and then reversing and increasing again. The position of the mass is locally a damped oscillator until a threshold is passed, and then the mass falls into the global minimum, as shown in Fig. 3.3. As the bias is reversed, it remains in the metastable minimum until the control parameter passes threshold, and then the mass drops into the other minimum that is now a global minimum.

Fig. 3.3 Hysteresis diagram. The mass begins in the left well. As the parameter \(c\) increases, the mass remains in the well, even though it is no longer the global minimum when \(c\) becomes positive. When \(c\) passes the positive threshold, the mass falls into the right well, with damped oscillation. Then the control parameter \(c\) is decreased until the negative threshold is passed, and the mass returns to the left well with damped oscillations. (DoubleWellrk.m)

3. Secular Perturbation Theory for the van der Pol Oscillator

Secular perturbation theory seeks to find distinct time scales into which a problem can be separated. For the autonomous van der Pol oscillator the fast time scale is the natural oscillation
A slow time scale is the approach to the limit cycle. Let’s assign $T_0 = t$ and $T_1 = \varepsilon t$, where $\varepsilon$ is a small parameter. The solution in terms of these time scales is

$$x(t) = X(T_0, T_1) = X_0(T_0, T_1) + \varepsilon X_1(T_0, T_1)$$

To write the van der Pol equation, we need the first and second derivatives with respect to time. The first derivative is

$$\frac{dx}{dt} = \frac{\partial x}{\partial T_0} + \varepsilon \frac{\partial x}{\partial T_1}$$

and the second derivative is

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial^2 T_0} + \varepsilon \frac{\partial^2 x}{\partial T_0 \partial T_1} + \varepsilon \frac{\partial^2 x}{\partial T_1 \partial T_0} + O(\varepsilon^2)$$

The vDP equation is

$$\ddot{x} + \varepsilon (x^2 - 1) \dot{x} + x = 0$$

where the gain $\varepsilon$ is taken as small. Therefore, the vDP equation in terms of $X_0$ and $X_1$ is

$$\frac{\partial^2 X_0}{\partial^2 T_0} + 2\varepsilon \frac{\partial^2 X_0}{\partial T_0 \partial T_1} + \varepsilon \frac{\partial^2 X_1}{\partial^2 T_0} + \varepsilon \left(X_0^2 - 1\right) \frac{\partial X_0}{\partial T_0} + X_0 + \varepsilon X_1 = 0$$

to lowest order in $\varepsilon$. Now separate the orders to zeroth and first orders

$$\frac{\partial^2 X_0}{\partial^2 T_0} + X_0 = 0$$

$$\frac{\partial^2 X_1}{\partial^2 T_0} + X_1 = -2 \frac{\partial^2 X_0}{\partial T_0 \partial T_1} - \left(X_0^2 - 1\right) \frac{\partial X_0}{\partial T_0}$$

Solve the first equation (a simple harmonic oscillator)

$$X_0 = A \cos T_0 + B \sin T_0$$

and plug it into the right-hand side of the second equation to give
The key to secular perturbation theory is to confine dynamics to their own time scales. In other words, the slow dynamics provide the envelope that modulates the fast carrier frequency. The envelope dynamics are contained in the time dependence of the coefficients A and B. Furthermore, the dynamics of $X_1$ should be an homogeneous function of time, which requires each term in the last equation to be zero. Therefore, the dynamical equations for the envelope functions are

$$\frac{\partial A}{\partial T_1} = A + \frac{1}{4} A \left( A^2 + B^2 \right)$$

$$\frac{\partial B}{\partial T_1} = -\frac{1}{4} B \left( A^2 + B^2 \right)$$

These can be transformed into polar coordinates using

$$A = R \cos \phi$$

$$B = R \sin \phi$$

Because the envelope functions do not depend on the fast time scale, the time derivatives are

$$\dot{A} = \varepsilon \frac{\partial A}{\partial T_1}$$

$$\dot{B} = \varepsilon \frac{\partial B}{\partial T_1}$$

With these expressions, the slow dynamics become

$$\dot{R} = \varepsilon \frac{R \left( 1 - \frac{R^2}{4} \right)}{2 R}$$

$$\dot{\phi} = 1$$

or making the final substitution $\rho = R / 2$ gives

$$\dot{\rho} = \varepsilon \rho \left( 1 - \rho^2 \right)$$

$$\dot{\phi} = 1$$

where the “dot” is with respect to $T_1$. This last equation is Eq. (3.45) on pg. 113 of *Introduction to Modern Dynamics* that has a Floquet multiplier.
\[ M = e^{-4\pi x} \]

which is an attractor—a stable limit cycle.

In this analysis, using secular perturbation theory, an equation for the approach to the limit cycle is extracted. The limit cycle is not a fixed point of the original dynamics, and hence cannot be located through the method of intersecting nullclines. However, using first-order perturbation theory, the quality of the limit cycle (stable or unstable), as well as the rate of approach, was derived.

4. Cantor Sets and Cantor-Like Sets

A famous example of an infinite set is Cantor’s ternary set that was published in 1883\(^4\). This is an infinite set of points composed of all the numbers that are generated by the formula

\[
z = \frac{c_1}{3} + \frac{c_2}{3^2} + \ldots + \frac{c_m}{3^m} \quad \lim m \to \infty
\]

where the \(c_m\) take on all permutations of the two integers \([0, 2]\). The set generates a function that has derivative zero almost everywhere, yet whose area is equal to unity. This was an example of a function that was not equal to the integral of its derivative. Cantor demonstrated that the size of his set is \(\aleph_1\), which is the cardinality of the real numbers, but whereas the real numbers are uniformly distributed, Cantor’s set is “clumped”. This clumpiness was an essential feature that distinguished it from the one-dimensional number line, and which, thirty five years later, would raise important questions of its dimensionality that would be answered Felix Hausdorff. Cantor’s ternary set is a self-similar structure with fractal dimension

\[ D = \frac{\ln N}{\ln b} = \frac{\ln 2}{\ln 3} = 0.6309 \]

The Cantor set is a dust with zero measure, meaning zero total length. An example of the dust is shown in Fig. 3.4 on the left. At each stage in the construction, as the length is scaled by a factor of 3, the middle third is removed. However, there are other related infinite sets that are not self similar. For instance, Fig. 3.4 on the right shows a Cantor-like set in which the middle 3\(^{rd}\) is removed at the first level, then the middle 9\(^{th}\) at the next level, then the middle 27\(^{th}\), etc. The final set has a nonzero total length of 4/7.

\(^4\) G. Cantor, "Grundlagen einer allgemeinen Mannigfaltigkeitslehre" ("Foundations of a General Theory of Aggregates") Leipzig B. G. Teubner (1883)
Fig. 3.4 The ternary Cantor set that is a fractal with zero measure. On the right is a Cantor-like set that has non-zero measure.

Cantor sets, and Cantor-like sets with finite measure, are both important structures that appear in Hamiltonian chaos. Cantor sets appear in homoclinic tangles, and Cantor-like sets appear in KAM theory.

pg. 146

HW Problem 3.7

Try \( \dot{r} = r(1-r)^2 \) This leads to a saddle spiral!

Errata:

pg. 146

HW Problem 3.11.

The Rössler oscillator does not have a fixed point exactly at (0,0,0). For (a,b,c) = (0.3, 0.4, 8) there is a fixed pt near (0.0150, -0.0501, 0.0501), but it is a function of the parameter \( a \). The Rössler also has a fixed point near (8, -26, 26).
Chapter 4 Coupled Oscillators and Synchronization

1. Coupled Rational Resonances

In Hamiltonian perturbation theory, and in the sine-circle map, the locking of phase oscillators whose frequency ratio is at or near a rational fraction (a resonance) can be described by the coupled equations

\[
\dot{\theta}_1 = \omega_1 - \frac{g}{p^2 + q^2} \sin(p\theta_1 - q\theta_2)
\]

\[
\dot{\theta}_2 = \omega_2 - \frac{g}{p^2 + q^2} \sin(q\theta_2 - p\theta_1)
\]

for \( p \) and \( q \) integers. The argument of the sine function defines the slow phase as

\[ \phi = p\theta_1 - q\theta_2 \]

Multiplying the top equation by \( p \), the bottom equation by \( q \) and subtracting gives

\[
p\dot{\theta}_1 - q\dot{\theta}_2 = p\omega_1 - q\omega_2 - \frac{(p + q)g}{p^2 + q^2} \sin(p\theta_1 - q\theta_2)
\]

\[
\phi = (p\omega_1 - q\omega_2) - \frac{(p + q)g}{p^2 + q^2} \sin(\phi)
\]

This has the same form as Eq. (4.41) in Introduction to Modern Dynamics pg. 161. The critical threshold for locking is therefore

\[ g_c = \frac{p^2 + q^2}{p + q} |p\omega_1 - q\omega_2| \]

As an example, for \( p = 3 \) and \( q = 2 \) and assuming that \( \omega_2 = 1.65\omega_1 \) (a little off the 3:2 resonance) requires a coupling \( g_c = 0.78\omega_1 \). However, if \( \omega_2 = 1.5\omega_1 \) then locking occurs at minimal coupling. The regions of synchronization for rational resonances are displayed as an Arnold tongue map in Fig. 4.1. However, rational numbers are dense on the real number line, so the density of Arnold tongues would cover the real number line and all frequencies would be sufficiently close to a rational resonance for locking. Compare this with Fig. 4.12 on pg. 165 of Introduction to Modern Dynamics that was obtained for the sine-circle map. Note that in Fig. 4.12, the tongues are not perfectly vertical, but higher-order resonances are “bent” away from
lower-order resonances that prevents overlap. This is related to the KAM effect in Hamiltonian theory that protects certain “irrational” resonances from chaos. Therefore, synchronization of rational resonances and the invariant tori of KAM theory share a mathematical similarity within Diophantine analysis.

Fig. 4.1 An Arnold tongue map of rational resonances p/q for q up to and equal to q = 23. The width of the tongues is made to scale as g/q^{2.5}. All tongues are vertical and overlap (there is no interaction here—just numbers). On the other hand, in the sine-circle map, nearby resonances are “repelled” by interactions with the broader tongues, ensuring that not all resonances overlap. (From listfrac.m fractongue.tif)

2. Some HW Help

**Huygens’s pendulum clocks.** Construct a model with two pendulum clocks whose pendula are connected with a linear spring. What conditions are necessary to cause them to synchronize? When don’t they synch?

```matlab
% Huygens's pendulum clocks
function couplepend
    a = 1;  % damping
```
\begin{verbatim}
b1 = 1;  \% freq1
fac = 1.14159;
b2 = fac*b1;  \% freq2
g = -0.5;  \% coupling
F = 1.05;  \% velocity-dependent force

y0 = [.1 0 0 0.1];
tspan = [0 1200];

[t,y] = ode45(@f5,tspan,y0);

figure(1)
plot(t,y(:,1),t,y(:,3))

figure(2)
plot(y(:,1),y(:,3))

function yd = f5(t,y)
yp(1) = y(2);
yp(2) = -a*y(2) - b1*sin(y(1)) + F*(sin(y(2))) + g*(sin(y(3)) -
    sin(y(1)));
yp(3) = y(4);
yp(4) = -a*y(4) - b2*sin(y(3)) + F*(sin(y(4))) + g*(sin(y(1)) -
    sin(y(3)));
yd = [yp(1);yp(2);yp(3);yp(4)];
end  \% end f5
\end{verbatim}

3. Errata (Chap 4):

pgs. 155-156
The spring constant in Example 4.2 should be represented by a “kappa” \( \kappa \) to distinguish it from the \( k \)-vector.

pg. 167
The limits of integration on Eq. 4.58 should be \(-\pi\) to \( \pi \). Similarly for Eqs. 4.59 and 4.60.

pg. 177
HW Problem 4.13. The equations in this problem should read

\[
\begin{align*}
\dot{x}_i &= -0.97y_i - z_i + g(x_2 - x_i) \\
\dot{y}_i &= 0.97x_i + ay_i + g(y_2 - y_i) \\
\dot{z}_i &= z_i(x_i - 8.5) + 0.4 + g(z_2 - z_i)
\end{align*}
\]
\[ \dot{x}_2 = -0.97y_2 - z_2 + g(x_1 - x_2) \]
\[ \dot{y}_2 = 0.97x_2 + ay_2 + g(y_1 - y_2) \]
\[ \dot{z}_2 = z_2(x_2 - 8.5) + 0.4 + g(z_1 - z_2) \]
Chapter 5 Network Dynamics

pg. 187
Another example of a scale-free network is generated using node = makeSF(200,1) and is shown here

Fig. 5.1 This figures shows the tree structure of scale-free graphs.

1. Discrete Evolution for Diffusion on Networks

As an alternative to the continuous-time evolution of the diffusing concentration, a discrete-time evolution is given by

\[
c_i^{n+1} = c_i^n (1 - \beta' k_i) + \beta' \sum_j A_{ij} c_j^n
\]

This recursion equation converges to the differential equation in the limit of small diffusion coefficient \( \beta' = \beta \Delta t \) as \( \Delta t \) goes to zero. The vector equation is
\[ c^{n+1} = (I - \beta D + \beta A)c^n \]
\[ = (I - \beta L)c^n \]
\[ = (I - \beta L)^n c^0 \]

which is defined in terms of the graph Laplacian, and \( M = I - \beta L \) is the Floquet multiplier. The solution is obtained by recursive application of the Floquet multiplier on an initial concentration vector.

At early times, the rate of diffusion is proportional to the product of the mean degree with the largest eigenvalue of the graph Laplacian. Therefore, different network structures that have the same mean degree can have different characteristic diffusion times, depending on the graph Laplacian.

Fig. 5.2 Eigenvalues of the graph Laplacian for ER, SF and SW\((p = 0.1)\) networks with mean degree equal to four. The SF network has the largest eigenvalue and hence will support the fastest diffusion rates. The SF maximum eigenvalue is approximately three time larger than for the SW network. (From avgeig.m)
2. SIRS (Susceptible-Infected-Removed-Susceptible) Model

A further infection model can allow the removed population to become susceptible again. This occurs for malaria and tuberculosis infections. The dynamics are

\[
\begin{align*}
\frac{di}{dt} &= -\mu i + \langle k \rangle \beta is \\
\frac{dr}{dt} &= \mu i - \nu r \\
\frac{ds}{dt} &= \nu r - \langle k \rangle \beta is
\end{align*}
\]

for all \( \mu, \nu, \beta \) positive. The total population can be assumed to be constant so that \( s + i + r = 1 \). This constraint reduces the dynamics to two-dimensional

\[
\begin{align*}
\frac{di}{dt} &= -\mu i + \langle k \rangle \beta is \\
\frac{ds}{dt} &= \nu (1 - s - i) - \langle k \rangle \beta is
\end{align*}
\]

Despite the simple-looking dynamical equations, this disease model exhibits thresholds and bifurcations for disease spread as well as steady-state solutions.
Home-Work Problem: Identify the fixed points and their stability of the SIRS model. What is the threshold for the establishment of steady-state disease in the population? For a well-established disease population, draw the phase portrait and nullclines.

Home-Work Problem: Verify the mean-field equations using simulations of discrete-time evolution of infection on SF, SW and ER graphs. For which type of graph is the mean-field approximation most accurate and least accurate?

3. Discrete Epidemic Model on Networks

Epidemic spread shares much in common with diffusion, but with an important difference. In the diffusion process, the net amount of diffusing material is conserved. However, there is no similar conservation for infection (other than a constant total population). In addition, in diffusion, each node has a concentration between zero and unity, while in the infection model a node is either infected or not (it cannot be “half” sick). This changes the dynamics from being dominated by the graph Laplacian (for diffusion) to the adjacency matrix (for infection spread). Finally, a stochastic element must enter into disease spread. The spreading parameter $\beta$ must now be interpreted as a probability of infection. The infection spread (for an SI model without recovery) is described as

$$c_i^{n+1} = H\left(c_i^n + \tilde{S}(\beta') \sum_j A_{ij} c_j^n\right)$$

where $H(...)$ is the Heaviside function (equal to unity for positive argument, and equal to zero for negative or zero argument), and $\tilde{S}(\beta')$ is a stochastic function that is equal to 1 with a probability $\beta' = \beta \Delta t$ and equal to zero otherwise. The solution is obtained by recursive application on an initial concentration vector.

For a discrete SIS model, the iterative evolution must now include a recovery step

$$c_i^{n+1} = H\left(c_i^n + \tilde{S}(\beta') \sum_j A_{ij} c_j^n\right) - H\left(\tilde{S}(\mu') c_i^n\right)$$

where there is a recovery probability given by $\mu' = \mu \Delta t$. Once a node has recovered, it is susceptible again for reinfection.
4. Graph Laplacian Eigenvalue Spectra

The eigenvalues of the graph Laplacian are important properties of networks. For instance, the rapidity of transport processes depends on the eigenvalue spectrum, with the fastest transport rates being proportional to the maximum eigenvalue. As another example, the robustness of network phase-locking of identical oscillators depends on the ratio of the maximum to the minimum eigenvalues $\lambda_{\text{max}}/\lambda_{\text{min}}$ of the graph Laplacian. The average eigenvalue spectra are shown in Fig. 5.5 for SF, SW and ER graphs for an average degree $<k> = 6$ and $N = 100$ nodes. The SF graph has the broadest spectrum, which implies that phase-locking would not be robust, but transport rates (and rates of infection) would be high. Note that global coupling has all equal eigenvalues $\lambda = N$ providing both greatest ease of phase-locking oscillators across the net and the fastest transport.

Fig. 5.4 Infected SIS population as a function of time for an SF network with $N = 50$, $m = 2$, $\beta = 0.2$ and $\mu = 0.6$. Note that the epidemic dies out at long time in this example. See SIRS.m.
Fig. 5.5  Average eigenvalue histograms of the graph Laplacian for SF, SW and ER graphs with average degree $<k> = 6$ for graphs with $N = 100$ nodes.

5. Errata (Chap 5):

pg. 194
At the bottom of Table 5.2, the variable $s(t)$ should read $s(t) = 1 - i(t) - r(t)$.

pg. 207
In problem 5.7, track the result as a function of the edge probability $p$.

In problem 5.8, use the same networks as in problem 5.7.
Chapter 6 Neurodynamics and Neural Networks

1. Errata (Chapter 6)

pg. 227

The XOR figure (for a single-sided sigmoid response function) should be

![XOR Boolean Function Diagram]

\[
\begin{align*}
\text{Inputs} & : \quad w, w, -2w, b = 3w/2, \quad b = w/2 \\
\text{Output} & : \quad \text{truth table}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Inputs</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fig. 6.15 Note that the bias sets a threshold and is subtracted from the net input to determine if the neuron fires.
Chapter 7 Evolutionary Dynamics

1. Kauffman’s NK Model of Epistatic Fitness Landscapes

The study of the evolutionary dynamics of genomes is facilitated by adopting simple models such as binary (also known as Boolean) genomes in which each location (known as a locus) along a binary string takes only values of 0 or 1. For instance, a binary genome with \( N = 9 \) loci (locations) could be \((101100100)\). This genome differs from \((100100100)\) by the flip of the third locus from 1 to 0. Therefore, these two genomes have a Hamming distance of 1. In fact, for \( N = 9 \) binary loci, there are \( 2^9 = 512 \) distinct genomes. By counting up from 0 to 511, and considering links between genomes with Hamming distance of 1, one constructs a network with the topology of an \( N \)-dimensional hypercube. Each node in the network has \( N \) nearest neighbors that are genomes that differ by on bit flip. Because networks do not have an intrinsic embedding dimension, hypercubes such as for \( N = 7 \), are easily visualized on the 2-dimensional page. The network in the figure shows such a 7-dimensional hypercube. Each link connects two genomes that differ at a single loci (a single bit flip).

![Fig. 7.6 A 7-dimensional hypercube representing 128 different binary genomes. Each link connects two genomes that are related by a single bit flip. The genome with the deepest red is the genome with the maximum fitness. See NK.m with \( N = 7 \) and \( K = 2 \).](image)

One of the central questions in the evolutionary dynamics of genomes, (in other words, the evolution of species) is how new genomes arise from old one through mutation. The concept
of a fitness landscape introduced by Sewall Wright can help visualize the principle that some genomes are more fit than others. The landscape is a high-dimensional landscape in which each dimension corresponds to each locus. This may be considered the dimension of the space because each locus is a degree of freedom. This landscape is populated with a population with genetic diversity. Groups with more fit genomes reproduce faster. As mutations sample fitter genomes, the average population shifts towards these genomes. In this process, a population shifts to successively more fit genomes, represented on the landscape as climbing the fitness peaks.

One way to construct model fitness landscapes on binary hypercubes is to assign fitness values for each locus. For instance, consider: \( f_1(0) = 0.3, f_1(1) = 0.7, f_2(0) = 0.2, f_2(1) = 0.8 \), etc. This gives a fitness weight of 0.3 if the first locus has 0, and a fitness of 0.7 if the first locus has 1, etc. The total fitness of the genome is

\[
f = \frac{1}{N} \sum_{n=1}^{N} f_n(b_n)
\]

where \( b_n \in \{0,1\} \) is the binary value on the n-th locus. If a bit is flipped, then the total fitness changes by a small amount. Stuart Kauffman has shown that a fitness landscape in this case has a single optimum, and any starting point on the network (hypercube) can lead steadily uphill, node by node, to the maximum. This type of walk is a directed walk on a network, in which each successive node visited has a higher fitness. Such a fitness landscape is strongly correlated, so each node has a similar fitness to its neighbors. However, actual genomes are much more complicated than this.

A recent development in genomics was the recognition that a locus on a gene does not work in isolation. The outcome of having a certain value at a locus depends on values of other loci on the genome. Each of the loci can influence the effects of the others. These effects are called epistatic interactions. Kauffman extended the fitness model of the binary genome to include the effects of K other loci influencing the fitness of a given locus. An example is shown in Fig. 7.1 for \( N = 9 \) loci and \( K = 2 \) epistatic interactions. Each locus has two arrows pointing to it originating from two other loci. For 2 epistatic interactions, there are \( 2^3 = 8 \) combinations that describe each locus. As an example, let’s consider a genome that has \( N = 9 \) loci, such as

\[
(101100100)
\]

Then consider the epistatic interactions shown in Fig. 7.2, so that the first locus (1) is affected by loci 8 (0) and 5 (0), the second locus is affected by loci 3 and 6, and so on. Then the combination for the first locus is 100, and the combination for the second locus is 010, and so on. Each of the eight possible combinations of locus plus two epistatic loci has a fitness associated with it when the locus is in the first location, another fitness when the locus is in the second location, etc. The fitness is once again the average of \( N \) fitness contributions, but each fitness contribution depends on the values of 3 loci (\( K + 1 \)) across the genome, not just one. This is expressed as

\[
f = \frac{1}{N} \sum_{n=1}^{N} f_n(b_n b_m b_r)
\]
This introduces a roughness to the landscape, so that changing a single bit value influences many of the fitness contributions at the other loci. In the limit that $K = N - 1$, so that every locus affects every other, the fitness landscape is uncorrelated and consists of numerous local maxima. A dynamic walk (a diffusing population) that only moves from a node of lower fitness to higher becomes trapped at these local maxima and the genome becomes frozen, unable to reach the global maximum in fitness. The color values in Fig. 7.6 represent the fitness of each genome from blue (lowest) to red (highest). At $K = 2$, there are still correlations, and a global maximum can be reached. However, for larger $K$, more local maxima occur, and the genome cannot reach its highest fitness through single bit flips.

Kauffman’s model is called the NK model and has been used to study the effects of epistatic interactions in genomic evolution, as well as interactions among agents (in agent-based modeling of game theory) and in economic models.

Fig. 7.7 Example of epistatic interactions for $N = 9$ and $K = 2$. The genome has 9 binary units (loci). The fitness contribution of the first locus to the total fitness of the genome is given by the combination of locus 0 with loci 8 and 5, etc..
2. Nearly Neutral Networks in Extremely High Dimensions

The NK model is a toy model that is helpful in studying effects of epigenetic interactions. However, it does not directly capture a key process in genomic evolution known as neutral drift. The theory of neutral drift occurs in landscapes with extremely high dimension, which is appropriate for biological genomes. After all, there are over a million loci in the human genome, constituting an evolutionary space of over a million dimensions. When the dimensionality is this large, there are many possible mutations that do not affect the overall fitness of the genome significantly. Furthermore, because each genome is connected to a million others by a single point mutation, the new genome can mutate again and take it a step farther in the landscape. With successive mutations, the average genome (remember that it is a diffusing population of closely related genomes spreading out in the genome space) can diffuse far from the starting genome without significant change in the fitness.

This is called neutral drift. It provides one of the central methods for timing the rate of evolution. Mutations in a genome tend to accumulate at a relatively fixed rate. By measuring the number of neutral mutations between two related genomes, the time of the divergence of the two species can be calculated using the average rate of neutral mutation. For neutral drift to be tenable, there must be far more ways to mutate neutrally than to mutate with increased fitness (or decreased fitness). With a million dimensions, this condition is satisfied. The importance of neutral drift is the tendency not to get hung up in local maxima of the fitness landscape.

Another important feature of neutral drift is the ability to cross wide valleys of low fitness to arrive at a higher fitness peak. There is far more room in sequence space to explore at altitudes around the mean fitness of the landscape than near its peak. This is illustrated in Fig. 7.8. The current population may be near a local fitness peak, but a higher fitness peak exists in the landscape. However, between the local and the global maximum is a wide valley of low fitness. If the population of species descends the local peak into the valley, the population may die out without getting to the maximum fitness peak. The theory of neutral drift, with the implicit participation of a large number of dimensions, can explain how evolution crosses such valleys. In high dimensions, there are many extra dimensions in which to move—there is literally a lot of room to move about in the extra dimensions. This feature of higher dimensions was illustrated nicely by Abbott in his famous book *Flatland* about a civilization that lives in two dimensions and observes strange phenomena, such as circles that can disappear and reappear on the far side of impenetrable barriers. From our perspective in 3D, we know that the circle was actually the cross section of a sphere, and the sphere avoids the barrier by simply moving above it, as illustrated in Fig. 7.9. Crossing the valley of low fitness, which seems impenetrable, is made possible by moving through neutral mutations in the high dimensions of sequence space.

Although the NK model is typically run with low dimensions (in the hundreds), it can be extended to include some of the effects of neutral drift. This is accomplished by making the fitness of the genomes take on discrete values. Therefore, there will be many genomes with the same fitness, representing the neutral network of neutral mutations. This simple procedure increases the distance that the genome can drift before getting frozen into a local maximum. As the discreteness is made coarser, so that a larger fraction of genomes share equal fitnesses, the average distance travelled through sequence space increases, until it approaches the network diameter. Therefore, neutral networks, even within the modified NK model at moderate dimensionality (this modification is called the NKP model), increase the ability of a population to sample sequence space and propagate longer distances before ultimately climbing higher peaks.
Fig. 7.8 The paradox of the valley of death (of low fitness). A population will die out if it descends into the valley and attempts to cross it to escape the local peak to get to the global fitness peak.

Fig. 7.9 A sphere in Flatland avoids an impenetrable barrier simply by moving above it. But to the citizens of Flatland, they see a circle disappear and reappear mysteriously on the far side of the barrier.
Chapter 8 Economic Dynamics

1. Dynamic General Equilibrium Theory

Dynamic general equilibrium (DGE) theory attempts to model an entire macro economy from the bottom up, assuming that economies are always in a balance of economic forces. Constraints may change in time, but the economy tracks these changes in a condition of general equilibrium. Equilibrium theories are a different “breed” of economic theory compared with Keynesian IS-LM models and at times make different predictions about the economy.

Because DGE seeks to optimize a value function, it uses the approach of undetermined Lagrange multipliers. Therefore, the economic dynamics originate from a Lagrangian function, making a connection between economic dynamics and physics. A simple DGE model assumes that a value function takes on its optimal value under conditions of dynamic general equilibrium. This value function is based on a utility function of consumption with a discount on value in future years. The value function is

$$V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$$

where the sum is over successive years and $0 < \beta < 1$ is the annual discount rate. This value function downweights the contribution to the current value, by future consumption and utility in future years because of economic uncertainty. The time horizon is given by $s$, which could be taken to infinity. The utility function $U(c_{t+s})$ is assumed to have diminishing returns, meaning that it saturates at large values of consumption. The goal is to maximize the value function $V_t$, subject to the constraints of the economy.

A simple economic constraint is that the national output $y_t$ in a given year $t$ is equal to consumption plus investment (savings is set equal to investment in this simple model)

$$y_t = c_t + i_t$$

This is a discrete time model in which the time index $t$ is taken as the year index. In addition, capital stock $k_{t+1}$ next year is equal to the capital stock this year plus investment and minus depreciation

$$k_{t+1} = k_t + i_t - \varepsilon k_t$$

$$= k_t (1 - \varepsilon) + i_t$$

---

5 General equilibrium theory had its origin with Léon Walras in 1874, but emerged in recent years in reaction to Keynesian macroeconomics.

where the factor $0 < \varepsilon < 1$ captures depreciation of the capital stock, representing obsolescence.

Finally, the national output is a nonlinear function of capital stock

$$y_t = F(k_t)$$

The output function $F(k_t)$ has diminishing returns, which means that it saturates at large values of its argument $k_t$. These equations combine to give the constraint equation

$$k_{t+1} = F(k_t) - c_t + (1 - \varepsilon)k_t$$

To optimize the value function $V_t$ under the economic constraints, we use the method of undetermined Lagrange multipliers. The Lagrangian function is

$$L_t = \sum_{s=0}^{s=\infty} \left[ \beta^s U(c_{t+s}) + \lambda_{t+s} \left( F(k_{t+s}) - c_{t+s} - k_{t+s+1} + (1 - \varepsilon)k_{t+s} \right) \right]$$

Note that this Lagrangian has no explicit velocity dependence. Therefore, the Euler-Lagrange equations are simply

$$\frac{\partial L_t}{\partial c_{t+s}} = \beta^s \frac{\partial U}{\partial c_{t+s}} - \lambda_{t+s} = 0$$

$$\frac{\partial L_t}{\partial k_{t+s}} = \lambda_{t+s} \left[ \frac{\partial F}{\partial k_{t+s}} + (1 - \varepsilon) \right] - \lambda_{t+s-1} = 0$$

$$\frac{\partial L_t}{\partial \lambda_{t+s}} = F(k_{t+s}) - c_{t+s} - k_{t+s+1} + (1 - \varepsilon)k_{t+s} = 0$$

The undetermined multiplier is eliminated to yield the DGE Euler equation for this optimization problem as

$$\beta \frac{U'(c_{t+1})}{U'(c_t)} \left[ F'(k_{t+1}) + (1 - \varepsilon) \right] = 1$$

expressed in terms of the annual discount rate $\beta$, the annual depreciation rate $\varepsilon$, the derivative of the diminishing returns utility function $U'$ with respect to consumption and the derivative of the diminishing returns output function $F'$ with respect to the capital stock. Collecting the Euler equation with the resource constraint defines the two-variable discrete map for the DGE model

$$k_{t+1} = F(k_t) - c_t + (1 - \varepsilon)k_t \quad (8.2)$$

$$\beta \frac{U'(c_{t+1})}{U'(c_t)} \left[ F'(k_{t+1}) + (1 - \varepsilon) \right] = 1 \quad (8.3)$$
which is not expressed in the usual form. Note that the bottom equation is a nonlinear transcendental equation. The top equation is used to obtain \(k_{t+1}\), and then the bottom equation is solved for \(c_{t+1}\).

The fixed point of this discrete map is

\[
F'(k^*) = \epsilon + \frac{1}{\beta} - 1
\]

\[
c^* = F(k^*) - \epsilon k^*
\]

Linearizing around the fixed point gives

\[
\frac{U''}{U'}(c_{t+1} - c^*) + \beta (k_{t+1} - k^*) F'' = \frac{U''}{U'}(c_t - c^*)
\]

\[
(k_{t+1} - k^*) = -(c_t - c^*) + \frac{1}{\beta}(k_t - k^*)
\]

which is expressed in matrix form as

\[
\begin{pmatrix}
  c_{t+1} - c^* \\
  k_{t+1} - k^*
\end{pmatrix} = \begin{pmatrix}
  1 + \beta \frac{U'F''}{U''} & -\frac{U'F''}{U''} \\
  -1 & 1/\beta
\end{pmatrix} \begin{pmatrix}
  c_t - c^* \\
  k_t - k^*
\end{pmatrix}
\]

where the transformation matrix is the Floquet (Jacobian) matrix of the discrete map. The trace and determinant are

\[
\tau = 1 + \beta \frac{U'F''}{U''} + 1/\beta
\]

\[
\Delta = 1/\beta
\]

with eigenvalues

\[
\lambda_{1,2} = \frac{1}{2} \left(1 + \beta \frac{U'F''}{U''} + 1/\beta\right) \pm \frac{1}{2} \sqrt{\left(1 + \beta \frac{U'F''}{U''} + 1/\beta\right)^2 - 4/\beta}
\]  \hspace{1cm} (8.4)

Here we use the fact that \(U\) and \(F\) are diminishing return functions (saturate for large values of their arguments). These functions have \(U' > 0\) and \(F'' < 0\) and \(U'' < 0\). This guarantees that \(U'F''/U'' > 0\). Therefore, one eigenvalue has absolute value less than unity, and the other has absolute value greater than unity. This represents a saddle-point in the discrete map. To be explicit, give \(U\) and \(F\) saturating forms as
These functions are shown in the Fig.

![Saturation Function and Derivatives](image)

Fig. 8.1 Saturation function and its first and second derivatives. The first derivative is positive and decreasing. The second derivative is negative and decreasing in magnitude.

A discrete map that captures both the linearized as well as nonlinear behavior of the DGE model is

\[
\begin{align*}
  k_{n+1} &= k_n + \varepsilon k_n \left( a - c_n^3 \right) \\
  c_{n+1} &= c_n + \varepsilon c_n \left( b - k_n \right)
\end{align*}
\]

This has simple nullclines (a vertical line at \( k = b \) and a horizontal line at \( c = a^{3/2} \)). The Floquet multiplier is

\[
J = \begin{pmatrix}
  1 & -3a^{3/2}b^{1/2} \\
  -a^{3/2} & 1
\end{pmatrix}
\]

with

\[
\begin{align*}
  \tau &= 2 \\
  \Delta &= 1 - 3a^{3/2}b^{1/2}\varepsilon^2 \\
  \lambda &= 1 \pm \varepsilon \sqrt{3a^{3/2}b^{1/2}}
\end{align*}
\]

whose parameters can be matched to Eq. 8.4. A related, continuous-valued flow that captures the general behavior of the dynamics is
\[ k = k(a - c^3) \]
\[ \dot{c} = c(b - k) \]

These dynamics describe a saddle with stable and unstable manifolds as shown in Fig. 8.2

Fig. 8.2 Flow field for the discrete map (modeled as a continuous flow in capital stock (k) and consumption (c)). The equilibrium point is a saddle point. (From dsge.m dsge.ai)

A central assumption of dynamic general equilibrium theory is that the system state always resides on a stable manifold and hence approaches the equilibrium point as a stable equilibrium point. This is equivalent to having 2 degrees of freedom (capital stock and consumption) with a constraint that reduces the dynamics to 1-dimensional dynamics. The constraint is the stable manifold. This 2D approach with constraint describes the macroeconomic properties in terms of two variables (k, c), but the constraint guarantees that the system is stable (cannot fall off the stable manifold or follow the unstable manifold). This principle operates even in the presence of a shock, a sudden change in the system description, as shown in Fig. 8.3. The system moves from the original equilibrium point to the new stable manifold, which it follows to the new equilibrium point.

The constraint used by DGE theory that forces the economy always to be on a stable equilibrium is a premise rather than a physical principle. It guarantees stability to the economy, while retaining a 2-dimensional description, but it is difficult to guess the exact mechanism that allows the system to find the new stable manifold. The DGE model is interesting, because it has
more structure than IS-LM models, and it is based on optimization principles (through the Lagrange multipliers). But it still must be viewed as highly idealized and hence not necessarily an accurate model of economic reality.

Fig. 8.3 The guiding assumption of dynamic general equilibrium theory that the system state always resides on a stable manifold of the saddle equilibrium. When the system experiences a sudden change in parameters, the system moves to the new stable manifold and then relaxes along this manifold to the new saddle equilibrium.

2. Discrete Random Walks

Consider a 1-dimensional random walk in which equal steps to the right or left are possible. The probability of taking a step right or left is equal to \( p \) or \( q \), respectively. These probabilities are independent, and \( q = 1 - p \). If a total of \( N \) steps are taken, then \( N = n_1 + n_2 \), where \( n_1 \) is the number of steps to the right, and \( n_2 \) is the number of steps to the left. The probability after \( N \) steps to be \( m \) steps away from the starting point, for a probability of \( p \) for steps to the right and \( q \) for steps to the left is
where \( m = 2n_1 - N \). Note that \( m \) takes on integer values that are separated by an amount \( \Delta m = 2 \), where all the \( m \) are either even or odd, depending on whether \( N \) is even or odd.

The mean value of \( m \) is

\[
\bar{m} = N(p - q) \tag{8.6}
\]

which is zero for \( p = q \). The variance of \( m \) is

\[
\Delta m^2 = 4Npq \tag{8.7}
\]

which, for \( p = q = 0.5 \), is

\[
\Delta m^2 = N \tag{8.8}
\]

This states the important result that the mean squared displacement of an unbiased random walk is equal to the number of steps. Equivalently, the root-mean-squared displacement grows as the square root of the number of steps.

For large \( N \), the binomial distribution is approximated by a continuous Gaussian function

\[
P(m) = \frac{1}{\sqrt{2\pi Npq}} \exp \left\{ -\frac{(m - N(p - q))^2}{8Npq} \right\} \tag{8.9}
\]

If the step size is uniform and equal to \( \ell \), then the distance from the origin after \( N \) steps is \( x = m\ell \). The distance between possible values of \( x \) is equal to \( 2\ell \). Therefore, the probability density for \( x \) is

\[
P(x) dx = P(m) \frac{dx}{2\ell} = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \tag{8.10}
\]

where the mean and standard deviation are

\[
\mu = (p - q)N\ell \tag{8.11}
\]

\[
\sigma = 2\sqrt{Npq}\ell
\]

Note that for \( p = q = \frac{1}{2} \), then
\( \mu = 0 \)
\( \sigma = \sqrt{N \ell} \)
\( \langle \Delta x^2 \rangle = N \ell^2 \) (8.12)

Diffusion is often described in terms of random walks. The mean squared displacement for 1D diffusion is

\[ \langle \Delta x^2 \rangle = N \ell^2 = 2Dt \] (8.13)

and the mean-squared displacement grows linearly with time. This is an essential result of random walks and diffusion, contributing to stochastic calculus and Ito’s lemma.

When the random walk is in 3D, many of these 1D results are easily extended as

\[ \langle \Delta r^2 \rangle = N_x \ell_x^2 + N_y \ell_y^2 + N_z \ell_z^2 \]
\[ = \frac{N}{3} \ell^2 = 6Dt \] (8.14)

The mean-squared displacement again grows linearly with time, but there are three independent contributions to the random walk.
3. Eratta (Chapter 8)

pg. 286  The sentence should read: “The other parameters, such as \( i \) the response of investment to the income, are called endogenous because they cannot directly be controlled and are properties of the economy.”

pg. 291  In Example 8.6, the variable \( y \) and \( y_n \) are the GDP \( g \) and \( g_n \).

pg. 295  In the figure legend, the values should be \( \alpha = -1 \) and \( \beta = 1 \). In Eq. (8.73), the V should be a U.
Chapter 9 Metric Spaces

1. Light Orbit

Consider a radial refractive index profile

\[ n(r) = 1 + n_0 r \exp \left( - \frac{r^2}{2\sigma^2} \right) \]

where the gradients are

\[ \partial_a n = n_0 \exp \left( - \frac{r^2}{2\sigma^2} \right) \left( \frac{2x^a}{r} - r \frac{2x^a}{2\sigma^2} \right) \]

The refractive index for this case is shown in Fig. 9.1a. A ray path is shown in Fig. 9.1b for an initial propagation direction in the xy-plane. The ray is confined within the region of high index and orbits the low-index core.

Fig. 9.1 A graded refractive index and the light orbits for \( n_2 = 0.3 \) and \( \sigma = 10 \). (From raysimple.m)
Chapter 10 Special Relativity

1. Angular Doppler

The Doppler effect varies between blue shifts in the forward direction to red shifts in the backward direction, with a smooth variation in Doppler shift as a function of the emission angle.

![Diagram of Doppler effect](TransDoppler2.ai)

Fig. 10.1 Configuration for detection of Doppler shifts for emission angle $\theta_0$. The light source travels a distance $vT$ during the time of a single cycle, while the wavefront travels a distance $cT$ towards the detector.

The observed wavelength is given by

$$\lambda = cT - vT \cos \theta$$  \hspace{1cm} (10.1)

where $T$ is the emission period of the moving source. The emission period is time dilated relative to the proper emission time of the source

$$T = \gamma T_0$$  \hspace{1cm} (10.2)

This gives

$$\lambda = \gamma T_0 (c - v \cos \theta)$$

$$= \gamma \lambda_0 (1 - \beta \cos \theta)$$  \hspace{1cm} (10.3)
which leads to the Angular Doppler Effect

\[
\text{Angular Doppler} \quad \lambda = \lambda_0 \frac{1 - \beta \cos \theta}{\sqrt{1 - \beta^2}} \quad (10.4)
\]

This expression has the expected limits:

\( \theta = 0 \)

\[
\lambda = \lambda_0 \frac{(1 - \beta \cos \theta)}{\sqrt{1 - \beta^2}}
= \lambda_0 \frac{(1 - \beta)}{\sqrt{1 - \beta^2}} = \lambda_0 \sqrt{1 + \beta} \quad (10.5)
\]

\( \theta = \pi \)

\[
\lambda = \lambda_0 \frac{(1 - \beta \cos \theta)}{\sqrt{1 - \beta^2}}
= \lambda_0 \frac{(1 + \beta)}{\sqrt{1 - \beta^2}} = \lambda_0 \sqrt{1 - \beta} \quad (10.6)
\]

\( \theta = \pi/2 \)

\[
\lambda = \lambda_0 \frac{(1 - \beta \cos \theta)}{\sqrt{1 - \beta^2}}
= \lambda_0 \frac{1}{\sqrt{1 - \beta^2}} = \gamma \lambda_0 \quad (10.7)
\]

Note that this last Doppler effect for emission at right angles is a red shift, caused only by the time dilation of the moving light source. This result is not corrected for the changing angle to the detection point as the light source moves. For the corrected “transverse Doppler effect” see problem 10.6 of the Introduction to Modern Dynamics.

The emission angle for which there is no Doppler effect can be obtained by setting

\[
\frac{(1 - \beta \cos \theta)}{\sqrt{1 - \beta^2}} = 1
\]

\[
\lambda = \lambda_0 \frac{(1 - \beta \cos \theta)}{\sqrt{1 - \beta^2}} = \lambda_0 \quad (10.8)
\]
The solution for the emission angle for which there is no observed Doppler effect is

\[ \cos \theta = \frac{\gamma - 1}{\beta \gamma} \quad \text{(10.9)} \]

2. Invariant Mass Reconstruction

A powerful analysis method in high-energy particle physics that makes it possible to discover the mass of a new particle (like the top quark, the Z-boson or the Higgs) is invariant mass reconstruction. Consider a particle of rest mass \( m_0 \) that decays into multiple particles, some of which may be photons without mass. The energy and momentum of each particle is given by \((E_a, \vec{p}_a)\).

![Multi-Particle Decay](image)

**Fig. 10.2** A particle of rest mass \( m_0 \) decays into multiple particles carrying energy and momentum.

The rest mass of the original particle is an invariant property of the system. Therefore, the invariant mass of the original particle is calculated by summing the decay product energies and momenta to yield

\[ m_0 = \left( \sum_{a=1}^{N} E_a \right)^2 - \left( \sum_{a=1}^{N} \vec{p}_a \right) \cdot \left( \sum_{a=1}^{N} \vec{p}_a \right) \]
This approach yields the same invariant mass independently of the frame in which the particles’ energies and momenta are measured regardless whether it is the lab frame, or the rest-mass frame, or any other frame. This makes this invariant a powerful tool in the search for new particles.

3. Relativistic (An)harmonic Oscillator

Harmonic oscillators are one of the fundamental elements of physical theory. They arise so often in so many different contexts, that they can be viewed as a central paradigm that spans all aspects of physics. Linear harmonic (non-relativistic) oscillators are purely harmonic, with a frequency that is independent of amplitude. However, relativistic effects would be expected to modify the linearity, especially because of time dilation effects, rendering the harmonic oscillator anharmonic. The theory of the relativistic one-dimensional linear-spring-constant oscillator is derived from the relativistic Lagrangian. What is the relativistic Lagrangian?

The relativistic Lagrangian of a free particle (no potential) should yield the generalized relativistic momentum

$$\frac{\partial L}{\partial \dot{x}_a} = \gamma m v^a$$

(10.10)

The Lagrangian that accomplishes this is

$$L' = -m_0 c^2 \sqrt{1-\beta^2}$$

(10.11)

$$= -m_0 c \sqrt{-\dot{x}_a \dot{x}^a} \sqrt{1-\beta^2}$$

where the invariant 4-velocity is

$$-\dot{x}_a \dot{x}^a = -\eta_{ab} \dot{x}^a \dot{x}^b = c^2$$

(10.12)

When the particle is in a potential, the Lagrangian becomes

$$L' = -m_0 c \sqrt{-\dot{x}_a \dot{x}^a} \sqrt{1-\beta^2} - U$$

(10.13)

The action integral that is minimized is

$$S = \int_{t_i}^{t_f} L' dt = \int_{\tau_i}^{\tau_f} \gamma L' d\tau$$

(10.14)

and the Lagrangian for integration of the action integral over proper time is
\[ L = \gamma L' = -m_0c\sqrt{-\dot{x}_a \dot{x}^a} - \gamma U \] (10.15)

The relativistic modification in the potential energy term of the Lagrangian is not in the spring constant, but rather is purely a time dilation effect. This is captured by the relativistic Lagrangian

\[ L = -m_0c\sqrt{-\dot{x}_a \dot{x}^a} - \frac{1}{2c} k(x^1)^2 \dot{x}^0 \] (10.16)

where the dot is with respect to proper time \( \tau \). The classical potential energy term in the Lagrangian is multiplied by the relativistic factor \( \gamma \), which is position dependent because of the non-constant speed of the oscillator mass. The Euler-Lagrange equations are

\[ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}_a} - \frac{\partial L}{\partial x_a} = 0 \] (10.17)

where the subscripts in the variables are \( a = 0, 1 \). The derivative of the time component of the 4-vector is

\[ \dot{x}^0 = \frac{dx^0}{d\tau} = \frac{dt}{d\tau} \frac{dx^0}{dt} = \frac{c}{\sqrt{1 - (dx^1/dt)^2 / c^2}} \] (10.18)

From the derivative of the Lagrangian with respect to \( \dot{x}_0 \), the following result is derived

\[ \frac{\partial L}{\partial \dot{x}_0} = m_0x^0 + \frac{1}{2c} k(x^1)^2 = \frac{E}{c} \] (10.19)

where \( E \) is the constant total relativistic energy. Therefore,

\[ \frac{dt}{d\tau} + \frac{\omega_0^2 x^2}{2c^2} = \gamma_0 \] (10.20)

which provides an expression for the derivative of the coordinate time with respect to the proper time where \( \gamma_0 = E / m_0c^2 \). The position-dependent \( \gamma(x) \) factor is then

\[ \gamma(x) = \gamma_0 - \frac{\omega_0^2 x^2}{2c^2} \] (10.21)

The Euler-Lagrange equation with \( a = 1 \) is

7 The derivatives with respect to covectors provide the appropriate contravariant vectors from the Euler-Lagrange equations.
\[ m_0 \ddot{x}^i + \frac{k}{c} x^i x^0 = 0 \]  

(10.22)

which gives

\[ \frac{d^2 x}{d\tau^2} + \omega_0^2 x \frac{dt}{d\tau} = 0 \]  

(10.23)

Inserting Eq.(10.21) into Eq.(10.23) provides the flow equations for the (an)harmonic oscillator with respect to proper time

\[ \frac{dx}{d\tau} = v \]  

\[ \frac{dv}{d\tau} + \omega_0^2 x \left( \frac{\gamma_0 - \omega_0^2 x^2}{2c^2} \right) = 0 \]

(10.24)

This flow represents a harmonic oscillator modified by the \( \gamma(x) \) factor, due to time dilation, multiplying the spring force term. Therefore, at relativistic speeds, the oscillator is no longer harmonic even though the spring constant remains truly a constant. The term in parentheses effectively softens the spring for larger displacement, and hence the frequency of oscillation becomes smaller.

The state-space diagram of the anharmonic oscillator is shown in Fig. 10.3 with respect to proper time (the time read on a clock co-moving with the oscillator mass). At low energy, the oscillator is harmonic with a natural period \( T = 2\pi \). As the maximum speed exceeds \( \beta = 0.8 \), the period becomes longer and the trajectory less sinusoidal. The position and speed for \( \beta = 0.9999 \) is shown in Fig. 10.4. The mass travels near the speed of light as it passes the origin, producing significant time dilation at that instant. The average time dilation through a single cycle is only about a factor of three, despite the large instantaneous \( \gamma = 70 \) when the mass passes the origin.
Fig. 10.3 State-space diagram in relativistic units relative to proper time of a relativistic (an)harmonic oscillator with a constant spring constant for several relative speeds $\beta$. The anharmonicity becomes pronounced above $\beta = 0.8$.

Fig. 10.4 Position and speed in relativistic units relative to proper time of a relativistic (an)harmonic oscillator with a constant spring constant for $\beta = 0.9999$. The period of oscillation is nearly three times longer than the natural frequency at small amplitudes.
4. Constant Acceleration

For the elevator accelerating with constant acceleration $g$, it would be useful to derive how time and space transform. Consider the 4-velocity of the accelerating mass $u^2 = -c^2$ which is a constant, and its 4-acceleration has a magnitude

$$a^a a_a = g^2$$ (10.25)

Because the inner product of the 4-acceleration with the 4-velocity is an invariant, and the value equals zero at the instant as elevator begins moving, the 4-acceleration must be orthogonal to the 4-velocity, such as

$$u^a a_a = -c a^0 + u^1 a^1 = 0$$ (10.26)

Therefore

$$a^0 = \frac{du^0}{d\tau} = \frac{g}{c} u^1$$
$$a^1 = \frac{du^1}{d\tau} = \frac{g}{c} u^0$$

These can be integrated once to give

$$u^0 + u^1 = c \exp\left(\frac{g}{c} \tau\right) \quad u^0 - u^1 = c \exp\left(-\frac{g}{c} \tau\right)$$ (10.28)

and integrated a second time to give

$$ct + x^1 = \frac{c^2}{g} \exp\left(\frac{g}{c} \tau\right) \quad ct - x^1 = -\frac{c^2}{g} \exp\left(-\frac{g}{c} \tau\right)$$ (10.29)

with the solution

$$ct \propto \frac{c^2}{g} \sinh\left(\frac{g}{c} \tau\right) \quad x \propto \frac{c^2}{g} \cosh\left(\frac{g}{c} \tau\right)$$ (10.30)
For a general case the relationship of x and t to x’ and t’ is found by considering the integration constants, when the origin is taken as the event P, and gives

\[ x(x', \tau') = -\frac{c^2}{g} + \left( x' + \frac{c^2}{g} \right) \cosh \frac{g}{c} \tau' \]

\[ ct(x', \tau') = \left( x' + \frac{c^2}{g} \right) \sinh \frac{g}{c} \tau' \]

The instantaneous transformation between the inertial system O and the accelerating frame O’ is

\[ \Lambda^a_b = \frac{\partial x^a}{\partial x'^b} = \begin{pmatrix} \cosh \frac{g}{c} \tau' \left( 1 + \frac{g}{c^2} x' \right) \sinh \frac{g}{c} \tau' \\ \sinh \frac{g}{c} \tau' \left( 1 + \frac{g}{c^2} x' \right) \cosh \frac{g}{c} \tau' \end{pmatrix} \]

and the metric tensor becomes (for x-t)

\[ g_{\alpha\beta} = \Lambda^\alpha_a \Lambda^\beta_b \eta_{cd} = \begin{pmatrix} - \left( 1 + \frac{g}{c^2} x' \right)^2 & 0 \\ 0 & 1 \end{pmatrix} \]

where \( \eta_{cd} \) is Minkowski, and the invariant interval becomes

\[ ds^2 = -\left( 1 + \frac{g}{c^2} x' \right)^2 c^2 dt^2 + \left( dx'^2 + dy'^2 + dz'^2 \right) \]

Note that only the time-component of the metric is modified by the constant acceleration. It is also position dependent, which means that clocks at different locations along the direction of acceleration run at different speeds.

5. **Errata (Chapter 10)**

**pg. 372**
The Eq. (10.60) should be
\[ E_{\infty} = m(\frac{1}{2}H) c^2 + m_n c^2 \]

And Eq. (10.62) should start

\[ \Delta E_B = E_{\infty} - E_D \]

pg. 373
In the paragraph after Eq. (10.64) it should say that the deuterium is lighter than the constituent parts, and the missing mass becomes the energy given off during fusion.
Chapter 11 General Relativity and Gravitation

1. Newtonian Dynamics

Trajectories in general relativity derive from the geodesic equation subject to the metric tensor. Because 4-momentum is related (by an affine transformation) to the tangent vector of a curve through the particle location, it also satisfies the geodesic equation for parallel transport

\[ \nabla_{\dot{\rho}} \dot{\rho} = 0 \tag{11.1} \]

Examining the zero-th term

\[ p^a \partial_a p^0 + \Gamma^0_{ab} p^b = 0 \tag{11.2} \]

and using the identity

\[ p^a \partial_a = mU^a \partial_a = m \frac{d}{d\tau} \tag{11.3} \]

gives

\[ m \frac{dp^0}{dt} + \Gamma^0_{00} (p^0)^2 = 0 \tag{11.4} \]

because \( p^0 \gg p^a \) for a slow (non-relativistic \( \gamma \approx 1 \)) particle. The connection to the metric is

\[ \Gamma^0_{00} = \frac{1}{2} g^{0b} \left( \partial_b g_{00} + \partial_0 g_{0b} - \partial_b g_{00} \right) \tag{11.5} \]

and only the \( b = 0 \) components are nonzero, where

\[ \Gamma^0_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = \frac{1}{2} \left[ \frac{1}{-(1+2\phi)} \right] \partial_0 (-2\phi) \approx \frac{\partial}{\partial x^0} \phi \tag{11.6} \]

In the slow-particle limit, the approximate expression \( (p^0)^2 \approx (mc)^2 \) holds, which gives
\[ \frac{d}{dt} p^0 = -mc \frac{\partial \phi}{\partial t} \quad (11.7) \]

and recognizing \( p^0 \) as the energy component of the 4-momentum, this is finally

\[ \frac{dE}{dt} = -m \frac{\partial \Phi}{\partial t} \quad (11.8) \]

This is the statement of energy conservation if the gravitational potential has no explicit time dependence.

The space components obey a similar condition

\[ p^a \partial_a p^c + \Gamma_{ab}^c p^a p^b = 0 \quad (11.9) \]

with explicit equations

\[ m \frac{dp^a}{dt} + \Gamma_{0b}^a (p^0)^2 = 0 \quad (11.10) \]

\[ \frac{dp^a}{dt} = -mc^2 \Gamma_{00}^a \]

The metric component

\[ g^{ca} = \frac{1}{1 - 2\phi} \delta^{ca} \quad (11.11) \]

gives

\[ \Gamma_{00}^a = \frac{1}{2} \frac{1}{1 - 2\phi} \delta^{ab} \left( 2 \frac{\partial}{\partial x^c} g_{00} - \frac{\partial}{\partial x^b} g_{00} \right) \]

\[ \approx \frac{1}{2} \frac{\partial}{\partial x^b} g_{00} \delta^{ab} \quad (11.12) \]

\[ \approx \frac{1}{2} \frac{\partial}{\partial x^b} (-2\phi) \delta^{ab} \]

which converts Eq.(11.10) into Newton’s Second Law as

\[ \frac{dp^a}{dt} = -m \frac{\partial \Phi}{\partial x^b} \delta^{ab} \quad (11.13) \]

which is just the expression from Newtonian gravity (expressed in tensor notation). Therefore in the slow-particle weak-field limit, Newton’s equations and gravity emerge from the Einstein Field Equations.
2. Black Holes

Stars during a supernova can undergo gravitational collapse when the gravitation pressure exceeds the light pressure. The collapse can stop when electrons combine with protons to create a neutron star for which the nuclear pressure exceeds the gravitational pressure. However, if the original star mass was sufficiently large and the residual mass of the neutron star is 1.5 to 3 times a solar mass, then the nuclear pressure is insufficient to support the gravitational pressure and the neutron star will continue gravitational collapse until its physical size decreases below the Schwarzschild radius. A black hole remains, and all further information from the star is removed from our sensible universe. Black holes can have masses ranging from 1.5 times a solar mass to super massive black holes that can contain the mass of over a billion suns. The Schwarzschild radius is

\[ R_s = \frac{2GM}{c^2} = 2.75 \text{km} \frac{M}{M_\odot} \]  

The Schwarzschild solution at constant theta and phi

\[ ds^2 = \frac{dr^2}{(1 - \frac{2GM}{c^2r})} - (1 - \frac{2GM}{c^2r})d\tau^2 \]  

has an apparent singularity in the space term at the Schwarzschild radius. This can be either a coordinate singularity (such as the origin in polar coordinates), or a fundamental (physical) singularity. To state this another way, what would an observer experience as he travels through the Schwarzschild radius? What happens to the proper properties in his frame?

To answer this, let’s first get a feel for the light curves by finding the null geodesics. The null geodesic has \( ds^2 = 0 \) and gives

\[ dr = \pm \left(1 - \frac{2GM}{c^2r}\right)d\tau \]  

This integrates to

\[ \tau = \pm \left(r - \frac{2GM}{c^2} \ln \left| r - \frac{2GM}{c^2} \right| \right) + \text{const.} \]  

This result can be used to make a coordinate transformation to a new time parameter.
\[ \tau^* = \tau + \frac{2GM}{c^2} \ln \left| r - \frac{2GM}{c^2} \right| \]  

(11.18)

with the differential

\[ d\tau = d\tau^* - \frac{2GM}{c^2} \frac{dr}{r - \frac{2GM}{c^2}} \]  

(11.19)

When this is put into the Schwarzschild metric, it becomes

\[ ds^2 = \left( 1 + \frac{2GM}{c^2 r} \right) dr^2 + \frac{4GM}{c^2 r} drd\tau^* - \left( 1 - \frac{2GM}{c^2 r} \right) (d\tau^*)^2 \]  

(11.20)

This is called the Eddington-Finkelstein metric, and there is no longer a divergent term at the Schwarzschild radius.

The null geodesics in the Eddington-Finkelstein metric are described by \( ds^2 = 0 \), which gives the differential equation

\[ \left( 1 + \frac{2GM}{c^2 r} \right) \left( \frac{dr}{d\tau^*} \right)^2 + \frac{4GM}{c^2 r} \frac{dr}{d\tau^*} - \left( 1 - \frac{2GM}{c^2 r} \right) = 0 \]  

(11.21)

This simplifies to

\[ \frac{dr}{d\tau^*} = \begin{cases} 
-1 & , \quad \frac{1 - \frac{2GM}{c^2 r}}{1 + \frac{2GM}{c^2 r}} 
\end{cases} \]  

(11.22)

for the two sets of null geodesics (analogous to the slopes of +1 and -1 in Minkowski space). Note that the -1 slope solution remains as for Schwarzschild space, but the + null-geodesic solution does not. The + null geodesics are

\[ \tau^* = \int \frac{r + \frac{2GM}{c^2}}{r - \frac{2GM}{c^2}} dr \]  

\[ = r + \frac{4GM}{c^2} \ln \left| 1 - \frac{2GM}{c^2 r} \right| \]  

(11.23)

which still show something important happening at the Schwarzschild radius.
The null geodesics are shown in the Eddington-Finkelstein metric in Fig. 11.1 for a spherically symmetric black hole. These coordinates are what an inertial observer sees who is far from the black hole. For large radii, the light cone has its usual $45^\circ$ angles. However, as the event horizon nears, the light cone tilts towards the origin. At the event horizon, the light cone is tilted sufficiently that a photon emitted in the radial direction remains stationary (as observed by the distant observer). Locally, an astronaut emitting the photon sees the photon recede from her at the speed of light. If the astronaut is even a little inside the event horizon, the radially emitted photon is dragged inward and asymptotically approaches the true singularity at $r = 0$. So too for the astronaut. There is no amount of rocket thrust that will keep her from the singularity.

Fig. 11.1 Null geodesics in the Eddington-Finkelstein metric of a black hole. The minus geodesics remain the same as for Minkowski space, even through the event horizon. The plus geodesics have infinite slope at the event horizon, but the invariant interval remains finite. Note that the light cones at large radius have the normal $45^\circ$ angles, but tip left for smaller radii. A photon emitted in the radial direction at the even horizon remains stationary (as observed by an inertial observer far from the black hole).
3. Binding Energy of an Object at the ISCO

A massive object orbiting a black hole has a binding energy that is an appreciable fraction of the rest mass energy of the object. The energy conservation equation including angular momentum for a massive object orbiting a black hole is

$$\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \left( 1 - \frac{R_s}{r} \right) - \frac{G M m}{r} = \frac{1}{2} m c^2 \left( E^2 - m^2 c^4 \right)$$ \hspace{1cm} (11.24)

where the term on the right is the kinetic energy of the object at infinity, and the second and third terms on the left are the effective potential

$$m \Phi_{\text{eff}} = \frac{1}{2} m r^2 \left( 1 - \frac{R_s}{r} \right) - \frac{G M m}{r}$$ \hspace{1cm} (11.25)

When the object is orbiting at the ISCO with $r = 3R_s$, the angular momentum is $\ell = \sqrt{3} m c R_s$ and the effective potential energy is

$$m \Phi_{\text{eff}} = - \frac{m c^2}{18}$$ \hspace{1cm} (11.26)

Solving for the binding energy gives

$$m \Phi_{\text{eff}} = \frac{1}{2} m c^2 \left( E_{\text{ISCO}}^2 - m^2 c^4 \right)$$

$$E_{\text{ISCO}} = \sqrt{\frac{2 m \Phi_{\text{eff}}}{m c^2}} + 1 = \sqrt{\frac{8}{9}} = 0.94$$ \hspace{1cm} (11.27)

Therefore, 6% of the rest energy of the object is given up when it is orbiting at the ISCO. The accretion disk occurs near the ISCO, which explains why matter falling into an accretion disk emits such large amounts of energy.
Chapter 12 Appendix

1. Elliptic Integrals

Elliptical integrals are encountered routinely in the study of periodic systems such as gravitational orbits and pendula.

The incomplete elliptic integral of the second kind is expressed as

$$E(\alpha, k) = \int_{0}^{\alpha} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

The complete elliptic integral of the second kind is expressed as

$$E(k) = \int_{0}^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

The circumference of an ellipse with semimajor axis $a$ is expressed in terms of the complete integral as

$$C = 4a E(e)$$

where the eccentricity $e$ of the ellipse is given by

$$e = \sqrt{1 - b^2 / a^2}$$

$E(k)$ is a weakly varying function of its argument, varying from $\pi/2$ at $k = 0$ to 1 at $k = 1$.

The incomplete elliptic integral of the first kind is expressed as

$$K(\alpha, k) = \int_{0}^{\alpha} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

and the incomplete integral has as its limit the complete integral when $\alpha = \pi/2$

$$K(k) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$
The period of a pendulum is expressed in terms of the incomplete elliptic integral of the first kind. Beginning with the Hamiltonian

\[ H = \frac{p_\phi^2}{2I} + mgd(1 - \cos \phi) = E \]

the momentum is

\[ md^2 \frac{d\phi}{dt} = \sqrt{2md^2E \left[ 1 - \frac{2mgd}{E} \sin^2 \frac{\phi}{2} \right]} \]

which can be reexpressed as

\[ dt = \frac{md^2 d\phi}{\sqrt{2md^2E \left[ 1 - \frac{2mgd}{E} \sin^2 \frac{\phi}{2} \right]}} \]

This is integrated to give the quarter period of the pendulum.
\[ \frac{T}{4} = \frac{md^2}{\sqrt{2md^2E}} \int_0^{\phi_{max}} \frac{d\phi}{\sqrt{1 - \frac{2mgd}{E} \sin^2 \frac{\phi}{2}}}, \]

where

\[ \sin \left( \frac{\phi_{max}}{2} \right) = \frac{E}{\sqrt{2mgd}}. \]

Hence the period is given by

\[ T = \frac{4md^2}{\sqrt{2md^2E}} \int_0^{\phi_{max}} \frac{d\phi}{\sqrt{1 - \frac{2mgd}{E} \sin^2 \frac{\phi}{2}}}, \]

\[ = \frac{8d}{\sqrt{\frac{2E}{m} \frac{\phi_{max}}{2} \sqrt{\frac{2mgd}{E}}}} \]


A helpful integral:

\[ \int_0^T t \sin(\omega t) \cos(\omega t) dt = -4\pi^2 \]