Appendix

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A.1 The Complex Plane

Linear, or linearized, ordinary differential equations (ODEs) have exponential solutions, and the arguments of the exponential functions are often complex. Euler’s formula for the imaginary exponential is

\[ \exp(i \omega t) = \cos \omega t + i \sin \omega t \]  

(0.1)

with the inverse compositions into real and imaginary parts

\[ \cos \omega t = \frac{1}{2} \left( e^{i \omega t} + e^{-i \omega t} \right) \]  

(0.2)

\[ \sin \omega t = \frac{1}{2i} \left( e^{i \omega t} - e^{-i \omega t} \right) \]

The real and imaginary parts of the exponential are sines and cosines. On the complex plane, these are the x and y components of a phasor, shown in Fig. A.1. The rotation direction of the phasor is counterclockwise for \( \exp(i \omega t) \) and clockwise for \( \exp(-i \omega t) \).
**A.2 Solution of Linear and Linearized ODEs**

Many of the analytical solutions of dynamical systems are achieved for systems of linear ordinary differential equations, or for linearized solutions of nonlinear systems around fixed points. In either case, the coupled differential equations can be represented through the equation

$$\dot{x}^a = \sum_{b=1}^{N} A^a_b \lambda^b$$  \hspace{1cm} (0.3)

for $a = 1:N$, where $N$ is the number of equations. In vector form this is

$$\dot{\mathbf{x}} = \tilde{A} \mathbf{x}$$  \hspace{1cm} (0.4)

In the case of linearized equations, the matrix $A^a_b$ is the Jacobian matrix $J^a_b$. The eigenvalues of the linear matrix are $\lambda_a$, and the eigenvectors are $\mathbf{v}^a$. The general solution of the set of coupled linear differential equations is
\[ x^a = \sum_{b=1}^{N} C^a_b \nu^b \exp(\lambda_b t) \]  \hspace{1cm} (0.5)

where the coefficients in \( C^a_b \) are uniquely determined by initial conditions

\[ x^a_0 = \sum_{b=1}^{N} C^a_b \nu^b \]  \hspace{1cm} (0.6)

and are solved using linear algebra (Cramer’s rule).

When \( N = 2 \), and if the coupled equations have simple symmetry, it is often convenient to adopt a complex-plane solution rather than a matrix approach. For instance, consider the set of two coupled differential equations

\[
\begin{align*}
\dot{x} &= x + y \\
\dot{y} &= -x + y
\end{align*}
\]  \hspace{1cm} (0.7)

Add the first equation to \( i \) times the second equation (known as adding the equations in quadrature) to get

\[
\begin{align*}
(\dot{x} + i\dot{y}) &= (x + iy) + (-ix + y) \\
&= (x + iy) - i(x + iy) \\
&= (1 - i)(x + iy)
\end{align*}
\]  \hspace{1cm} (0.8)

Make the substitution

\[ q = x + iy \]  \hspace{1cm} (0.9)

to convert the two equations into the one-dimensional complex ODE

\[ \dot{q} = (1 - i)q \]  \hspace{1cm} (0.10)

The assumed solution is

\[ q(t) = q_0 e^{it\omega} \]  \hspace{1cm} (0.11)

which is inserted into the equation to give

\[ \omega = -1 + i \]  \hspace{1cm} (0.12)

and the general solution is

\[ q(t) = q_0 e^{-t} e^{-i\omega} \]  \hspace{1cm} (0.13)
Writing this out explicitly

\[ x(t) + iy(t) = q_0 e^{-t} (\cos t - i \sin t) \]  

(0.14)

Separating out the real and imaginary terms gives

\[
\begin{align*}
  x(t) &= q_0 e^{-t} \cos t \\
  y(t) &= q_0 e^{-t} \sin t
\end{align*}
\]  

(0.15)

which are decaying sinusoidal solutions.

The complex-plane approach can also be applied to second-order equations that have sufficient symmetry, such as

\[
\begin{align*}
  \ddot{x} + 2\omega_1 \dot{y} + \omega_0^2 x &= 0 \\
  \ddot{y} - 2\omega_1 \dot{x} + \omega_0^2 y &= 0
\end{align*}
\]  

(0.16)

Add the equations in quadrature to get

\[
(\ddot{x} + i \ddot{y}) - 2i\omega_1 (\dot{x} + i \dot{y}) + \omega_0^2 (x + iy) = 0
\]  

(0.17)

and make the substitution \( q = x + iy \) to convert the two equations into the one-dimensional complex ODE

\[ \ddot{q} - 2i\omega_1 \dot{q} + \omega_0^2 q = 0 \]  

(0.18)

The assumed solution is

\[ q(t) = q_0 e^{i\omega t} \]  

(0.19)

which is inserted into the equation to give the secular equation

\[ -\omega^2 + 2\omega_1 \omega + \omega_0^2 = 0 \]  

(0.20)

The solution to this quadratic equation in \( \omega \) is

\[ \omega = -\omega_1 \pm \sqrt{\omega_1^2 + \omega_0^2} \]  

(0.21)

and the general solution to the coupled equation is

\[ q(t) = q_1 e^{-i\omega_1 t} \exp \left( i \sqrt{\omega_1^2 + \omega_0^2} t \right) + q_2 e^{-i\omega_0 t} \exp \left( -i \sqrt{\omega_1^2 + \omega_0^2} t \right) \]  

(0.22)
where the 2 coefficients are determined by the 2 initial conditions. To return to the x and y representation, rewrite the equation as

\[ q(t) = e^{-i \omega_0 t} q'(t) \]  

(0.23)

and write out explicitly as

\[ x(t) + iy(t) = \left( \cos \omega_1 t - i \sin \omega_1 t \right) \left( x'(t) + iy'(t) \right) \] \nonumber

(0.24)

Separating out the real and imaginary parts gives, individually

\[ x(t) = x'(t) \cos \omega_1 t + y'(t) \sin \omega_1 t \]

\[ y(t) = -x'(t) \sin \omega_1 t + y'(t) \cos \omega_1 t \] \nonumber

(0.25)

This is recognized as the matrix equation

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} =
\begin{pmatrix}
  \cos \omega_1 t & \sin \omega_1 t \\
  -\sin \omega_1 t & \cos \omega_1 t
\end{pmatrix}
\begin{pmatrix}
  x'(t) \\
  y'(t)
\end{pmatrix}
\] \nonumber

(0.26)

which is a rotation matrix applied to the primed solution. The frequency \( \sqrt{\omega_1^2 + \omega_0^2} \) of the primed representation is always larger than \( \omega_1 \). Therefore, this is interpreted as a slow rotation of the “fast” coordinates. These equations and solutions are applied directly to Focault’s Pendulum in Chapter 1. The fast solution is the swinging of the pendulum, while the slow rotation is the precession of the pendulum as the Earth spins on its axis.

Complex-plane approaches are particularly useful when there are two coordinates, coupled linearly to arbitrary order, but with sufficient symmetry that the substitution \( q = x + iy \) can be made. The solutions are oscillatory or decaying, or both, and have easy physical interpretations. There are many examples in general physics that use this complex-plane approach, such as mutual induction of circuits, polarization rotation of electromagnetic fields propagating through optically active media, as well as Focault’s Pendulum. The symmetry of Maxwell’s equations between the E-field and the B-field make the complex-plane approach common. However, general coupled equations often lack the symmetry to make this approach convenient, and the more general matrix approach is then always applicable.
A.3 Index Notation: Rows, Columns and Matrices

This textbook uses extensive index notation to denote the elements of vectors, matrices and metric tensors. In terms of linear algebra, the correspondence between indexes on the one hand and rows and columns on the other uses a simple convention: superscripts are column vectors where the rows are indexed, and subscripts are row vectors where the columns are indexed. Similarly, a matrix has a superscript that indexes the rows and a subscript that indexes the columns. In terms of metric spaces, superscripts on tensors are contravariant indexes and subscripts on tensors are covariant indexes.

An explicit example of column vector notation is

\[ x^a = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \]  \hspace{1cm} (0.27)

and a row vector is

\[ x_a = \begin{pmatrix} a & b & c \end{pmatrix} \]  \hspace{1cm} (0.28)

Matrix multiplication uses the implicit Einstein summation notation in which a repeated index, one a superscript and the other a subscript, imply summation over that index. For instance, an inner product is

\[ x_a x^a = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 + b^2 + c^2 \]  \hspace{1cm} (0.29)

Matrix multiplication of a column vector on the right is

\[ w_b^a x^b = \begin{pmatrix} d & e & f \\ g & h & i \\ j & k & l \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} da + eb + fc \\ ga + hb + ic \\ ja + kb + lc \end{pmatrix} \]  \hspace{1cm} (0.30)

Matrix multiplication with a row vector on the left is

\[ x_a w^a_b = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} d & e & f \\ g & h & i \\ j & k & l \end{pmatrix} = \begin{pmatrix} (ad + bg + cj) & (ae + bh + ck) & (af + bi + cl) \end{pmatrix} \]  \hspace{1cm} (0.31)
which have different resulting components than when multiplied on the right. Matrix multiplication is not commutative. The right eigenvectors of a matrix are solutions of

$$w^b_a x^b = \lambda x^b$$

while the left eigenvectors of a matrix are solutions of

$$x^a w^a_b = \lambda x^a$$

The left and right eigenvectors are related to each other: the left eigenvectors are the transpose of the right eigenvectors of the transposed matrix.

In terms of linear algebra, one can choose to work with either column vectors or row vectors, which are simply transposes of each other. Many chapters in this text use column vectors, but Chapter 6 on neurodynamics and Chapter 7 on evolutionary dynamics use row vectors. In metric spaces, column vectors (also known as contravariant vectors) are usually preferred for configuration-space representation over row vectors (also known as covectors), and Chapters 9-11 mainly use contravariant vectors. In Cartesian coordinates the components of contravariant vectors are identical to covariant vectors, but this is not generally true for coordinates that are encountered in relativity theory.

Projection operators play a role in several chapters, notably on neurodynamics (Chapter 6). Projection operators are outer products that are constructed as the Kronecker products of two vectors

$$P^a_b = x^a y_b = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} d & e & f \end{pmatrix} = \begin{pmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{pmatrix}$$

A projection operator projects from one vector onto another

$$P^a_b y^b = \begin{pmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} ad^2 + ae^2 + af^2 \\ bd^2 + be^2 + bf^2 \\ cd^2 + ce^2 + cf^2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} d^2 + e^2 + f^2 \end{pmatrix}$$

$$= x^a (y^b y_b)$$
If \( y_b \) is a unit vector, then the term in parentheses on the last line is unity, and the resultant of the projection is \( x^d \). With appropriate normalization the projection operator provides a quantitative measure of similarity or correlation between two vectors.

### A.4 Runge Kutta Numerical Solvers for ODEs

The solution of time-dependent ODEs is a broad topic consisting of many methods with trade-offs among accuracy and stability and time. The simplest (and oldest) is Euler’s Method. Given a one-dimensional ODE \( y = \dot{x} = f(x) \) with an initial condition \((y_n, x_n)\), the \( n+1 \)st solution is

\[
y_{n+1} = y_n + hf'(x_n, y_n)
\]

for a step size \( h \). The error is of order \( O(h) \) and hence can be large.

Fig. A.2 Euler’s method linearizes the function by taking the slope. It is only accurate to first order.

A much more accurate approximate solution is the Runge-Kutta approach. The iterated solution in this case is
Runge-Kutta Method

\[
y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)
\]

\[
k_1 = hf'(x_n, y_n)
\]

\[
k_2 = hf'\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)
\]

\[
k_3 = hf'\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)
\]

\[
k_4 = hf'(x_n + h, y_n + k_3)
\]

Fig. A.3 Runge-Kutta method for the solution of ODEs with four intermediate approximates.

The Runge-Kutta Method is accurate and computationally efficient, and is sufficient for many applications in dynamical systems that have well-behaved flows. However, in numerical analysis, caution is always a virtue, and some systems may have unstable solutions, or nearly so, for which other techniques are necessary. It is also important to understand the role of iterative error in a system. For dissipative dynamical systems with attractors (limit cycles or strange attractors), even with successive iteration errors, the solution stays on the stable manifold, even though it may be a fractal (strange) attractor. However, for some non-dissipative systems, or systems that are unstable, the errors can accumulate and drive the system off the stable manifold, for which the solutions then become inapplicable.
A.5 Matlab Programs for Numerical Homework

Many of the chapters have computational homework problems at the end of the chapter. The Matlab programs listed here are meant to help on these homework problems. They are meant to be used as starting points, but most often will need to be modified to complete the homework assignment.

These Matlab programs can be found at

www.physics.purdue.edu/nlo/ModernDynamics

Chapter 1  Physics and Geometry

Chapter 2  Hamiltonian Dynamics

Problem 18. body3.m

Chapter 3  Nonlinear Dynamics and Chaos

Problem 17: AHBif.m

Problems 18 and 19: logistic.m

Problem 20 and 23: vdp.m

Problem 21: Pendulum3.m

Problem 24: lorenz.m

Problems 25 and 26: rossler.m

Problem 27: Henon.m

Problem 28: Lozi2.m

Problem 29: Heiles.m
Chapter 4 Synchronization

Problem 9: coupleNdriver.m
Problem 10: sinecircle1.m
Problem 12: coupledvdp.m

Chapter 5 Network Dynamics

Problems 5 and 6: makeSW.m
Problems 7 and 8: coupledrossler.m
Problem 9: makeSW.m, makeER.m, make SF.m

Chapter 6 Neural Networks

Problems 8, 9 and 10: NaK.m
Problems 11, 12 and 13: threelayer.m

Chapter 7 Evolutionary Dynamics

Problem 10: PredPrey.m
Problem 11: repeq.m
Problems 12-17: quasiSpec.m, quasiHam.m

Chapter 8 Economic Dynamics

Problem 6: Shone182.m
Problem 7: SDlogistic.m
Problems 8-9: mutrep.m

Chapter 9 Geodesic Motion

Chapter 10 Special Relativity

Chapter 11 General Relativity
Equation Check

Approx. 1000 equations. $0.35/equation = $350 at 40 equations/hr = $14/hr
Double if equation has a demonstrated error.

Prof. Nolte seeks a physics theory grad student for freelance equation checking for a textbook at the Phys 410 and Phys 411 level. The fee is $0.35 per equation, and an additional $0.35 per equation for a demonstrated error. The textbook has around 1000 equations and will be published by Oxford University Press. The checker’s name will be included in the book acknowledgements. A minimum GPA in graduate level physics courses of 3.5 is required. Interested grad students should contact Prof. Nolte by email at nolte@purdue.edu with the subject heading “Oxford”.