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Abstract

We compare oscillations of a fundamental string ending on a D3-brane in two different settings: (1) a test-string radially threading the horizon of an extremal black D3-brane and (2) the spike soliton of the DBI effective action for a D3-brane. Previous work has shown that overall transverse modes of the test-string appear as $l = 0$ modes of the transverse scalar fields of the DBI system. We identify DBI world-volume degrees of freedom that have dynamics matching those of the test-string relative transverse modes. We show that there is a map, resembling T-duality, between relative and overall transverse modes for the test-string that interchanges Neumann and Dirichlet boundary conditions and implies equality of the absorption coefficients for both modes. We give general solutions to the overall and relative transverse parts of the DBI coupled gauge and scalar system and calculate absorption coefficients for the higher angular momentum modes in the low frequency limit. We find that there is a nonzero amplitude for $l > 0$ modes to travel out to infinity along the spike, demonstrating that the spike remains effectively $3 + 1$-dimensional.

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1. Introduction

The DBI world-volume field theory of a Dp-brane has BPS soliton solutions, first studied in [1][2][3], that carry both electric and scalar charge. Because the world-volume scalar fields record the spacetime position of the brane, the diverging scalar at the soliton core means that the surface of the brane has a spike reaching off to infinity, as illustrated in the figure below. Both the soliton's shape and its electric charge suggest that it represents a fundamental string ending on the Dp-brane. This interpretation is supported [1] by a matching of the divergent self-energy of the soliton with the mass of a semi-infinite string. In addition it has been shown [4] that coupling of the spike soliton to bulk supergravity fields yields the Neveu-Schwarz $B_{\mu\nu}$ field appropriate to a fundamental string at linear order.

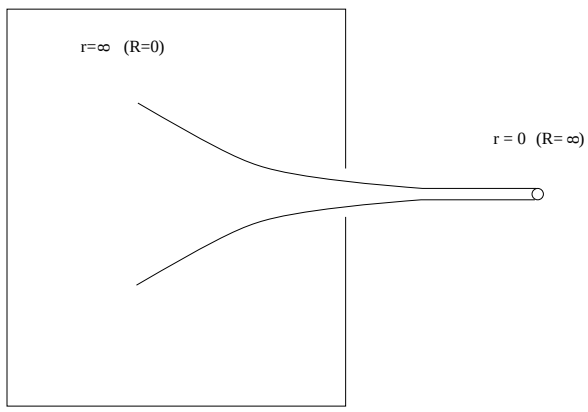

Fig. 1: The basic spike soliton. The world-volume r-coordinate vanishes at the end of the spike, while the spacetime R coordinate goes to infinity.

One can probe this correspondence further by studying the dynamics of the DBI spike soliton [1][5][6][7][8]. In [5] it was shown that for $p = 3, 4$ the equation governing oscillations of the spike in directions transverse to the Dp-brane world-volume is identical to that for certain oscillations of a test-string threading the horizon of a Dp-brane spacetime. In addition to these "overall transverse" modes, the test-string also has "relative transverse" oscillations in directions transverse to the string but tangent to the Dp-brane world-volume. Our focus in this paper will be on finding an identification of these relative transverse modes with degrees of freedom of the DBI spike soliton for the case of the D3-brane. This task is more complicated than in the overall transverse case, because the relevant perturbations of the spike soliton involve the coupled gauge and scalar field dynamics of the $3+1$-dimensional DBI system in a nontrivial way, which has no a priori analogue in the $1+1$-dimensional test-string system.

Carrying out the identification requires a detailed understanding of both the test-string and the coupled world-volume gauge and scalar field dynamics. As we work through this material we highlight a number of interesting aspects of both the test-string and DBI spike soliton systems. For example, in section (2) we note the existence for the test-string of a map, bearing a striking resemblence to T-duality, that takes overall transverse modes $X_\perp$into relative transverse modes $Y_{||}$ and vice-versa. The map implies equality of the absorption coefficients for the two types of modes. Another curious feature of the test-string system is an $R \to 1/R$ symmetry that exchanges the near horizon and asymptotically flat regions [8]. In the DBI system, we show that such a symmetry continues to hold for all angular momentum modes and exchanges the flat part of the brane with the region down the spike.

We give low frequency approximations to the absorption coefficients for all modes of both systems. For the DBI system this includes modes carrying nonzero angular momentum. We note that, although suppressed by an angular momentum barrier, the transmission of these modes down the spike is nonzero. Therefore, even though the spike gradually takes on characteristics of a fundamental string, it remains effectively 3-dimensional all the way out to infinity. The presence of the higher angular momentum modes illustrates a basic difference between the DBI spike soliton and spacetime test-string systems.

Our main result is an understanding of how the relative transverse modes are manifest in the DBI system. Physically, an oscillation of the spike soliton parallel to the brane looks like a linearly oscillating charge. Therefore, we should expect to see dipole radiation in both the electromagnetic and scalar fields [8]. We find that this is indeed the case and further that the moduli scalars describing the center of mass position of the spike soliton satisfy the same equation as the test-string relative transverse modes.

2. Fluctuating Test String in a Dp-brane Background

We start by deriving the equations of motion for fluctuations of a test-string in a Dp-brane background. The string metric for the Dp-brane is given by

$$\begin{aligned} ds^2 &= H^{-1/2}(-dt^2 + \ldots + dx_p^2) + H^{1/2}(dR^2 + R^2 d\Omega_{8-p}^2) \equiv g_{\alpha\beta} dx^\alpha dx^\beta. \\ H &= 1 + \left(\frac{\mu}{R}\right)^{7-p}, \qquad R^2 = x_{p+1}^2 + \ldots + x_9^2. \end{aligned} \tag{2.1}$$

The Nambu-Goto area action for the test string is simply $S = \int d^2\sigma \sqrt{-\det G_{AB}}$, where $G_{AB} = \partial_A x^\alpha \partial_B x^\beta g_{\alpha\beta}$ is the induced metric on the worldsheet. Note that this string

does not couple to the RR gauge field of the Dp-brane. A static string stretching radially outward in the x_9 direction from the Dp-brane horizon at $R = 0$ solves the equations of motion and we study perturbations around it. Choosing static gauge $\sigma^0 = t$ and $\sigma^1 = x^9$ (implying $\sigma^1 = R$ for the radial string), to second order in small fluctuations we have

$$-det\, G_{AB} \simeq +1 - \left\{(\partial_t x^k)^2 - H^{-1}(\partial_R x^k)^2\right\} - \left\{H(\partial_t x^\mu)^2 - (\partial_R x^\mu)^2\right\}, \tag{2.2}$$

where $k, = 1, .., p$ and $\mu = p+1, ..., 8$. Denoting the overall transverse modes x^μ by $X_\perp$ and the relative transverse modes by $Y_{||}$, the equations of motion for these modes are then given by

$$\partial_R^2 X_\perp - H\partial_t^2 X_\perp = 0, \tag{2.3}$$

$$\partial_R(\frac{1}{H}\partial_R Y_{||}) - \partial_t^2 Y_{||} = 0. \tag{2.4}$$

Below we will need the energy densities $E_\perp, E_{||}$ and energy fluxes $F_\perp, F_{||}$ for these wave equations which satisfy the conservation law $\partial_t E + \partial_R F = 0$. These are given by

$$E_\perp = \frac{1}{2}\left(H\dot{X}_\perp^2 + (\partial_R X_\perp)^2\right), \qquad F_\perp = -\dot{X}_\perp \partial_R X_\perp \tag{2.5}$$

$$E_{||} = \frac{1}{2}\left(\dot{Y}_{||}^2 + \frac{1}{H}(\partial_R Y_{||})^2\right), \qquad F_{||} = -\frac{1}{H}\dot{Y}_{||}\partial_R Y_{||}. \tag{2.6}$$

2.1. An Analogue of T-Duality

By definition, oscillations of the end point of a fundamental string on a Dp-brane satisfy Neumann boundary conditions for directions along the brane and Dirichlet boundary conditions for perpendicular components. The two types of oscillations are interchanged by the action of T-duality, which also changes the dimension of the brane.

A priori, the overall and relative transverse modes of the test string $X_\perp$ and $Y_{||}$ are geometrically distinct, probing different components of the background spacetime metric. Indeed, as will see below, when equations (2.3) and (2.4) are put in standard scattering form, the potentials are qualitatively different. Surprisingly however, there exists a simple map relating the two types of modes that bears a striking resemblance to T-duality. Specifically, making the substitutions

$$\partial_R Y_{||} = H(R)\partial_t X_\perp, \qquad \partial_t Y_{||} = \partial_R X_\perp \tag{2.7}$$

brings equation (2.3) into the form of equation (2.4). This implies, for example, that if $X_\perp$ is a solution of (2.3) with time dependence $e^{-i\omega t}$, then $Y_{||} = i/\omega \partial_R X_\perp$ solves (2.4). Note

that the map (2.7) interchanges Dirichlet and Neumann boundary conditions. Also note that, although T-duality changes the dimension of the background brane, because of the required translation invariance of the lower dimensional brane, the metric function $H(R)$ remains unchanged.

The relation (2.7) between overall and relative transverse modes implies that the absorption coefficients for the two types of modes are identical. If $X_\perp$ and $Y_{||}$ are related by (2.7), then the corresponding energy fluxes $F_\perp$ and $F_{||}$ defined above are equal. The dimensionless absorption coefficient is the ratio of the flux absorbed into the horizon to the flux incident from infinity

$$\sigma = \frac{F_{hor}}{F_\infty^{in}} \tag{2.8}$$

Hence the absorption coefficients $\sigma_\perp$ and $\sigma_{||}$ are equal. Note that this is an exact statement about the governing equations (2.3) and (2.4). However, one must keep in mind that the derivation of these equations from the area action assumes $|\partial_R X| \sim \omega A \ll 1$, where A is the amplitude of the wave.

2.2. Absorption Coefficients for $X_\perp$ and $Y_{||}$

We now specialize to the case of a test-string ending on a D3-brane and compute scattering coefficients $\sigma_\perp = \sigma_{||} \equiv \sigma$ for overall and relative transverse waves to be absorbed into, or reflected back from, the horizon [1]. In order to bring equations (2.3) and (2.4) into standard scattering form, we change to a tortoise coordinate R_* defined by $dR_* = \sqrt{H(R)}dR$ and also rescale the mode wavefunctions according to $\psi_\perp = H^{+1/4}X_\perp$ and $\chi_{||} = H^{-1/4}Y_{||}$. For Fourier modes with time dependence $e^{\pm i\omega t}$, we then have

$$\partial_{R_*}^2 \psi_\perp + (\omega^2 - V_\perp(R_*))\psi_\perp = 0 \tag{2.9}$$

$$\partial_{R_*}^2 \chi_{||} + (\omega^2 - V_{||}(R_*))\chi_{||} = 0 \tag{2.10}$$

where the overall and relative transverse potentials are given by

$$\begin{aligned} V_\perp &= H^{-1/4}\partial_{R_*}^2(H^{1/4}) = \frac{5\mu^4}{R^6}H^{-3} \\ V_{||} &= H^{+1/4}\partial_{R_*}^2(H^{-1/4}) = \left(\frac{-5\mu^4}{R^6} + \frac{2\mu^8}{R^{10}}\right)H^{-3} \end{aligned} \tag{2.11}$$

The tortoise coordinate R_* has the asymptotic behavior that $R_* \sim R$ for $R \to \infty$, while $R_* \sim -\mu^2/R$ for $R \to 0$. The overall transverse potential $V_\perp$ can be seen to fall off rapidly,

like R_*^{-6}, for both $R_* \to \pm\infty$. Solutions to the overall transverse equation are then well approximated by plane waves $\psi_\perp \sim \exp(\pm i\omega R_*)$ in both asymptotic regions. Therefore, at long wavelengths, $V_\perp$ can be approximated by a delta-function potential. Performing the matching gives the dimensionless absorption coefficient (2.8) [1]

$$\sigma = 4(\mu\omega)^2 \tag{2.12}$$

We have demonstrated by means of the 'T-Duality' discussed above that this is also the absorption coefficient for the relative transverse waves.

The near horizon limits of $X_\perp$ and $Y_{||}$ are suggestive of $3+1$ dimensional physics. Near the horizon the tortoise coordinate $R_* \to -\infty$ and we have

$$(\partial^2_{R_*} + \omega^2)\psi_\perp \simeq 0, \qquad (\partial^2_{R_*} - \frac{2}{R_*^2} + \omega^2)\chi_{||} \simeq 0. \tag{2.13}$$

The mode wavefunctions in this limit are then

$$X_\perp \simeq \frac{1}{R_*} e^{-i\omega R_*}, \qquad Y_{||} \simeq R_* e^{-i\omega R_*}\left(1 - \frac{i}{\omega R_*}\right) \tag{2.14}$$

As the overall transverse wave $X_\perp$ approaches the horizon, it has the characteristic $1/R_*$ decay of the spherically symmetric mode of a scalar field carrying energy off to infinity in a $3+1$ dimensional space. As we will discuss below, this matches the behavior of the lowest angular momentum mode for overall transverse oscillations of the $3+1$ DBI spike soliton. We also note that, the equation for $Y_{||}$, if regarded as the equation for a radial function in $3+1$ dimensions, has a dipole nature. It contains the $l(l+1)/R_*^2$ term with $l = 1$. The divergent asymptotic behavior of the relative transverse waves $Y_{||}$ in (2.14) is confusing, but note that the energy flux computed from (2.6) remains finite. This behavior turns out to be appropriate so that $Y_{||}$ may be realized in the DBI spike system as an $l = 1$ oscillatory mode radiating energy to infinity along the D3-brane.

2.3. A Curious $R \to 1/R$ Symmetry

We complete this section by noting an interesting symmetry of the overall transverse wave equation (2.3) [8]. One can check that if $X_\perp(R)$ solves (2.3), then so does

$$\tilde{X}_\perp(R) = R\, X_\perp(\frac{\mu^2}{R}) \tag{2.15}$$

In terms of the tortoise coordinate, this property arises from the $R_* \to -R_*$ reflection symmetry of the scattering potential $V_\perp(R_*)$. The mapping (2.15) relates the behavior of oscillations of the test string in the AdS near horizon region to behavior in the asymptotically flat region. We will return to this symmetry below when analyzing the DBI modes, where (2.3) is generalized to contain all angular momentum modes, but keeps this symmetry.

3. Fluctuations of the DBI Spike Soliton

The DBI action for the world-volume degrees of freedom of a D3-brane in a flat background and corresponding equations of motion are given in static gauge by

$$S = \int d^4x \sqrt{-det\,(\eta_{\mu\nu} + F_{\mu\nu})}, \qquad \left(\frac{1}{\eta - F^2}\right)_\lambda^{\ \nu} \partial_\nu F_\mu^{\ \lambda} = 0, \tag{3.1}$$

where the Greek indices μ, ν run from $0, \ldots, 9$. For the D3-brane we divide these into two sets, directions $\alpha, \beta = 0, 1, 2, 3$ tangent to the brane and directions $a, b = 4, \ldots, 9$ transverse to the brane. We then have

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \qquad F_{\alpha b} = -\partial_\alpha \phi_b, \qquad F_{ab} = 0, \tag{3.2}$$

where A_α is the world-volume gauge field and ϕ_a are the overall transverse scalars.

As shown in [1][2], the DBI theory has BPS soliton solutions that represent a collection of spikes projecting outward from the brane. Taking the spikes to point in the ϕ_9 direction, the solitonic gauge and scalar fields are related via $F_\alpha^{\ 0} = \pm\partial_\alpha \phi_9$, and the equations of motion reduce to $\nabla^2 \phi_9 = 0$. For a single spike at the $3+1$ dimensional origin, we have $\phi_9 = \pm q/r$.

The spike soliton represents a fundamental string ending at an electric charge on the D3-brane. Strictly speaking the string meets the D3-brane only at $r = 0$, $\phi_9 = \infty$ [9]. The spike itself has properties which interpolate between those of the D3-brane proper and those of the string. For example, as we will see below, the spike remains effectively $3+1$ dimensional even as it narrows near the origin. On the other hand, when coupled to the bulk spacetime fields, the spike acts as a source for the components of the NS anti-symmetric tensor field appropriate to a fundamental string [4].

Here we will catalogue all the fluctuating modes of the DBI spike soliton and identify the particular modes which correspond to the oscillations of the test string described in section 2. Note that in the DBI coordinates above $r \to 0$ is going down the spike to what corresponds to the asymptotically flat region in the spacetime ($R \to \infty$). Let $F_\mu^{\ \nu} = \bar{F}_\mu^{\ \nu} + \delta F_\mu^{\ \nu}$, where $\bar{F}_\mu^{\ \nu}$ is the unperturbed spike solution. The perturbed equations of motion [5],

$$\bar{B}_\lambda^{\ \nu} \partial_\nu (\delta F_\mu^{\ \lambda}) + \left(\bar{B}\left[\bar{F}\delta F + \delta F \bar{F}\right]\bar{B}\right)_\lambda^{\ \nu} \partial_\nu \bar{F}_\mu^{\ \lambda} = 0, \qquad \bar{B}_\lambda^{\ \nu} = \left(\frac{1}{\eta - \bar{F}^2}\right)_\lambda^{\ \nu} \tag{3.3}$$

yield a system of four equations corresponding to different values for the free index $\mu = 0, k, a, 9$, where $k = 1, 2, 3$ and $a = 4, \ldots, 8$. The overall transverse modes ϕ_a decouple from the other fields and satisfy the equation

$$\nabla^2 \phi_a - \tilde{H}(r) \partial_0^2 \phi_a = 0, \tag{3.4}$$

where $\tilde{H}(r) = 1 + q^2/r^4$. The remaining fields give a coupled Maxwell-scalar field system, coupled through the background spike soliton $\phi_9 = \pm q/r$

$$\begin{aligned} \nabla \cdot \vec{E} \mp (\tilde{H}(r) - 1)\partial_0^2 \psi + \vec{\nabla} \tilde{H}(r) \cdot (\vec{E} \mp \vec{\nabla}\psi) &= 0, \\ \vec{\nabla} \times \vec{B} - \tilde{H}(r) \partial_0 \vec{E} \pm (\tilde{H}(r) - 1)\partial_0 \vec{\nabla}\psi &= 0, \\ \nabla^2 \psi - \tilde{H}(r) \partial_0^2 \psi \pm \vec{\nabla} \tilde{H}(r) \cdot (\vec{E} \mp \vec{\nabla}\psi) &= 0, \end{aligned} \tag{3.5}$$

where $E_k = \delta F_{0k}$, $\delta F_{kl} = -\epsilon_{klm} B^m$, $\delta F_{kA} = -\partial_k \phi_A$, and $\delta F_{k9} = -\partial_k \psi$. Note that for $\tilde{H}(r) = 1$ ($q = 0$), these equations reduce to the decoupled Maxwell and transverse scalar equations.

In order to untangle equations (3.5) for the coupled gauge and scalar field system, it is useful to define the combination $\vec{v} = \vec{E} \mp \vec{\nabla}\psi$. Fourier transform in time, or alternatively consider only an overall time dependence $e^{-i\omega t}$. In terms of $\vec{v}$, the equations of motion and the Bianchi identities may be rewritten as the set (see also [7])

$$\nabla^2 \vec{v} + \omega^2 \tilde{H}(r) \vec{v} = 0, \tag{3.6a}$$

$$\vec{\nabla} \cdot \vec{v} = \pm \omega^2 \psi, \tag{3.6b}$$

$$\nabla^2 \psi + \omega^2 \tilde{H}(r) \psi = \mp \vec{v} \cdot \vec{\nabla} \tilde{H}(r) \tag{3.6c}$$

$$i\omega \vec{B} = \vec{\nabla} \times \vec{v} \tag{3.6d}$$

$$\vec{\nabla} \times \vec{B} = -i\omega \left(\vec{\nabla}\psi + \tilde{H}(r) \vec{v} \right) \tag{3.6e}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{3.6f}$$

Note that the wave equation for each cartesian component v^k is the same as the equation (3.4) for the transverse scalars ϕ_A. Given a solution of (3.6a) for $\vec{v}$, solutions for the remaining fields may be determined in the following simple way. Let $\vec{v}$ be a solution to (3.6a), then ψ is given by the right hand side of the Gauss' law constraint (3.6b). The dynamical equation (3.6c) for ψ is then satisfied as a consequence of (3.6a). Given $\vec{v}$, the world-volume magnetic field $\vec{B}$ is determined by the dynamical equation (3.6d). Equation

(3.6f) is then identically satisfied, while equation (3.6e) follows from the equations above. Finally, the world-volume electric field is found from via the definition $\vec{E} = \vec{v} \pm \vec{\nabla}\psi$.

The full set of independent solutions to the system (3.6) can then be enumerated as follows. Each cartesian component of $\vec{v}$ can be expanded as $v^k = \sum A^k_{lm} Y_{lm}(\Omega) P_l(r)$, where the functions $P_l(r)$ satisfy a radial equation which follows from (3.6a). The scalar ψ and the magnetic field $\vec{B}$ are then determined as described above in terms of $P_l(r)$ and are indexed by angular momentum mode l. As we will see below, the multipole composition of the full solution for a given value of l is actually of a mixed nature.

Both the overall transverse equation (3.4) and the relative transverse system (3.6) have a conserved energy and energy flux vector defined via $\frac{d\mathcal{E}}{dt} + \vec{\nabla} \cdot \vec{\mathcal{F}} = 0$. For the overall transverse modes ϕ_a we have

$$\mathcal{E}_\perp = \frac{1}{2}\left(\tilde{H}(r)\dot{\phi}^2 + \vec{\nabla}\phi \cdot \vec{\nabla}\phi\right), \qquad \mathcal{F}_\perp = -\dot{\phi}\vec{\nabla}\phi, \tag{3.7}$$

while the relative transverse equations give

$$\begin{aligned} \mathcal{E}_{||} &= \frac{1}{2}\left(\dot{\psi}^2 + \vec{B}^2 + \tilde{H}(r)\vec{v}^2\right) + (\vec{\nabla}\psi)^2 + \tilde{H}(r)\vec{v} \cdot \vec{\nabla}\psi, \\ \vec{\mathcal{F}}_{||} &= \vec{\mathcal{F}}_{\psi\psi} + \vec{\mathcal{F}}_{\psi v} + \vec{\mathcal{F}}_{EB}, \\ \vec{\mathcal{F}}_{\psi\psi} &= -\dot{\psi}\vec{\nabla}\psi, \qquad \vec{\mathcal{F}}_{\psi v} = -(\tilde{H}(r) - 1)\dot{\psi}\vec{v}, \qquad \vec{\mathcal{F}}_{EB} = \vec{E} \times \vec{B}. \end{aligned} \tag{3.8}$$

3.1. Overall Transverse Fluctuations

In order to write down the most general solution to the overall transverse wave equation (3.4), we decompose the scalars ϕ in terms of spherical harmonics, $\phi = \sum A_{lm} Y_{lm}(\theta, \varphi) P_l(r)$, where A_{lm} are constant coefficients. The radial equation following from (3.4) is then given by

$$\frac{1}{r^2}\partial_r(r^2 \partial_r P_l) + \left(\omega^2 \tilde{H}(r) - \frac{l(l+1)}{r^2}\right) P_l = 0, \tag{3.9}$$

which as $r \to \infty$ reduces to the usual Helmholtz radial equation. If we work instead in terms of the radial coordinate $R = q/r$, then this becomes

$$\partial_R^2 P_l + \left(\omega^2 \tilde{H}(R) - \frac{l(l+1)}{R^2}\right) P_l = 0, \qquad R = \frac{q}{r}. \tag{3.10}$$

For $l = 0$, as noted in [5], this is the same as the equation (2.3) governing overall transverse oscillations of a test-string in a D3-brane background, provided that the parameter q^2 that

specifies the charge of the spike soliton is identified with the parameter μ^4 from the D3-brane metric. The quantity μ^4 is proportional to the ADM mass per unit volume of the D3-brane spacetime (2.1), which is in turn proportional to the D3-brane tension T_3. The DBI charge q is proportional the the tension of the attached fundamental string T_1. The equality $q^2 = \mu^4$ then requires $T_3 \propto T_1^2$, a result discussed in [10] that follows from duality arguments involving alternate dimensional reductions of M-branes.

We are also interested in the $l > 0$ modes of the transverse scalars. While the $l = 0$ modes of the overall transverse DBI scalars can be indentified with overall transverse modes of the test-string, the higher l modes present in the DBI system have no obvious analogue amongst excitations of the test-string. From equation (3.9) one might think that the higher l modes are suppressed as they propagate down the spike towards $r = 0$. However, a closer analysis shows that, although there is the usual angular momentum suppression, $l > 0$ modes can in fact propagate out to infinity along the spike.

Two additional forms of the radial equation are useful for this analysis. In terms of the rescaled radial function $F_l = rP_l$ equation (3.9) becomes

$$\partial_r^2 F_l + \left(\omega^2 H(r) - \frac{l(l+1)}{r^2}\right) F_l = 0, \qquad F_l = rP_l. \tag{3.11}$$

Finally, making use of the D3-brane spacetime tortoise coordinate R_* defined via $dR_* = \sqrt{H(R)}dR$ and rescaling the radial function according to $P_l = H^{-1/4}\phi_l$ gives

$$\partial_{R_*}^2 \phi_l + \left(\omega^2 - \frac{l(l+1)}{H(R)R^2} - V_\perp\right) \phi_l = 0, \tag{3.12}$$

where R is regarded as a function of R_* and the potential $V_\perp$ is given in equation (2.9). Equation (3.12), like its $l = 0$ case (2.9), is invariant under the reflection $R_* \to -R_*$, which interchanges the flat part of the brane near $r = \infty$ with the region far down the spike near $r = 0$. Both the scattering potential $V_\perp$ and the angular momentum barrier in (3.12) are symmetric about the mouth of the spike. An l-mode with a given initial amplitude may be started at either end, and the subsequent propagation is the same in either case. In appendix A below, we show that the dimensionless absorption coefficient for modes of arbitrary angular momentum l is given in the low energy limit by

$$\sigma_l \equiv \frac{|c_l|^2}{|a_l|^2} \simeq \frac{4(\omega\sqrt{q})^{4l+2}}{L^2,} \tag{3.13}$$

where L is a numerical constant given in the appendix. We see that although the absorption of $l > 0$ modes is indeed supressed by higher powers of the frequency ω, there is no absolute

barrier to transmission down the spike. The spike remains functionally $3+1$ dimensional all the way out to infinity.

We are left with two questions. What is the analogue, if any, of the $l \neq 0$ DBI modes in the test string picture? And, does the DBI spike soliton actually include a 1+1 dimensional string, or just the attachment point of the string? One possibility is that to include the string, one must add the string action to the DBI 3-brane action, and find solutions of the combined system. The string would then be attached at the end of the spike. Conversely, the test string in the D3-brane spacetime approximation does not include the smooth transition mouth region, which then misses the higher l modes.

Finally, these DBI modes also display the symmetry that for the test-string relates the physics in the near horizon anti-deSitter region to that in the asymptotically flat region. In the DBI system, the relation is between the region down the spike and the flat region of the brane. Note that (3.11) and (3.10) are the same equation, so that there is a kind of $r \to q/r$ symmetry in the transverse scalar wave equation. Precisely, if $F_l(r)$ is a solution to (3.11), then $P_l(r) = rF_l(q/r)$ is also a solution. This generalizes the symmetry of the spacetime totally transverse modes (2.15)to all l.

3.2. Relative Transverse Modes

There is no direct match among the DBI world-volume fields for the test-string relative transverse degrees of freedom $Y_{||}$. However, the physical picture is clear [8]. The string ends in an electric charge on the D3-brane world-volume, and relative transverse oscillations of the string result in oscillations of the end point. Since the oscillating end point carries both electric and scalar charge, we expect to get both scalar and electromagnetic radiation.

In the overall transverse case, only the $l = 0$ mode of the world-volume scalars corresponded to the test-string degrees of freedom, and this will also be the case for the relative transverse modes. As discussed above, a solution to the relative transverse system of equations (3.6) can be specified in terms of a solution of (3.6a) for $\vec{v}$. Solutions to the system specified by the $l = 0$ modes of $\vec{v}$, we will see, match the relative transverse modes of the test-string. We will also see that these modes arise via oscillations of the moduli of the spike soliton corresponding to its center of mass position.

To illustrate more generally the behavior of the Maxwell-scalar system (3.6), we will give three examples, having $l = 0, 1, 2$ and absorption coefficients that go like $(\omega\sqrt{q})^{4l+2}$ as $\omega\sqrt{q} \to 0$. These describe the analogue of the spacetime relative transverse oscillations, a dilation of the charge source, and an oscillating electric dipole respectively.

There are three independent $l = 0$ modes corresponding to the different components of $\vec{v}$ (see also the discussion in [8]). Taking, for example, a wave polarized in the z-direction, we have

$$v^x = v^y = 0, \qquad v^z = P_0(r). \tag{3.14}$$

Following the steps described above, the other world-volume fields are then given by

$$\begin{aligned} \psi &= \frac{1}{\omega^2} P_0'(r) \cos\theta, \\ E^{\hat{\theta}} &= -\left(P_0 + \frac{1}{\omega^2 r} P_0' \right) \sin\theta, \qquad E^{\hat{r}} = E^{\hat{\phi}} = 0, \\ B^{\hat{\phi}} &= \frac{i}{\omega} P_0' \sin\theta, \qquad B^{\hat{r}} = B^{\hat{\theta}} = 0, \end{aligned} \tag{3.15}$$

where hatted indices indicate components in an orthonormal frame. We can now compute the flux of radiation $\vec{\mathcal{F}}_{||}$ for this solution using the expressions in (3.8). The two contributions $\vec{\mathcal{F}}_{\psi\psi}$ and $\vec{\mathcal{F}}_{\psi v}$ each individually diverge as r approaches zero, down the spike. However, their sum is finite. We find

$$\begin{aligned} \mathcal{F}^r_{\psi\psi} + \mathcal{F}^r_{\psi v} &= -\frac{i}{2\omega} \cos^2\theta (P_0^* P_0' - P_0^{*\prime} P_0) \\ \mathcal{F}^r_{EB} &= -\frac{i}{2\omega} \sin^2\theta (P_0^* P_0' - P_0^{*\prime} P_0) \\ \mathcal{F}^r_{||} &= -\frac{i}{2\omega} (P_0^* P_0' - P_0^{*\prime} P_0) \end{aligned} \tag{3.16}$$

which shows that while radiation in the individual fields exhibits a dipole pattern, the total radiated energy flux is isotropic. We also see explicitly that the total radiated flux is identical to that in the totally transverse case, as expected because the solution for the various fields is given in terms of the single function P. This further implies that the absorption coefficient for the $l = 0$ relative transverse DBI excitation is equal to the absorption coefficient for the test string relative transverse mode, using the identity of the two totally transverse systems, and the equality $\sigma_\perp = \sigma_{||}$ for the test string absorbtion coefficients. In fact we have now established on of our main results, the equality of all the absorption coefficients for the $l = 0$ modes of the DBI system with the modes of the test-string

$$\sigma_{DBI||} = \sigma_{DBI\perp} = \sigma_{ST||} = \sigma_{ST\perp} \approx 4(\omega\sqrt{q})^2, \tag{3.17}$$

where the last relation follows from equation (3.13) and equating $q^2 = \mu^4$ as discussed above.

The dynamical degrees of freedom of the test string and DBI relative transverse modes appear quite different. For the test string, there are simply the three scalars $Y_{||}^k$, while in the DBI system the dynamics are described by the coupled $\vec{E}$, $\vec{B}$ and ψ fields. To see how the degrees of freedom $Y_{||}^k$ arise in the DBI system [8], consider perturbations of the spike soliton in which the moduli of the spike vary,

$$\phi_9 = \frac{q}{|\vec{x} - \Delta\vec{x}(t, x^i)|}. \tag{3.18}$$

The scalar field ψ is then given by

$$\psi \equiv \delta\phi_9 = \frac{q\vec{x} \cdot \Delta\vec{x}}{|\vec{x}|^3}. \tag{3.19}$$

We note here that in fact the spike position moduli Δx^i satisfy the same equation (2.4) as the relative transverse modes of the test-string. For example, choosing $\Delta x = \Delta y = 0$ yields $\psi = \frac{q\Delta z cos\theta}{r^2}$ which in turn implies that v^k is of the form assumed in (3.14). Therefore, Δz is given by the function $\Delta z = \frac{r^2}{\omega^2 q} P_0'(r)$. Two further differentiations then yield the result that Δz indeed satisfies equation (2.4). Note finally that $\vec{v}$ and $\vec{B}$ can be found using $P_0 = -\frac{q}{r^2} H(r)\Delta z'$, so that the full set of fields is determined by the the behavior of the spike modulus Δz.

3.3. Higher l Modes

We close this section by presenting two higher l solutions to the relative transverse DBI system (3.6). The first is the spherically symmetric mode of $\vec{v}$, which is actually the $l = 1$ mode of the relative transverse system. Take $v^k = \frac{x^k}{r} P_1(r)$ which is equivalent to $v^r = P_1, v^\theta = v^\phi = 0$. The remaining fields are then determined to be

$$\begin{aligned} \psi &= \frac{1}{\omega^2}\left(P_1' + \frac{2}{r}P_1\right), \qquad E^r = -\frac{q^2}{r^4}P_1, \\ E^\theta &= E^\phi = 0, \qquad B^k = 0. \end{aligned} \tag{3.20}$$

We see that this mode is a spherically symmetric oscillation of the charged region. There is therefore no electromagnetic radiation, however there is radiation in the scalar field ψ with a total flux having the same form as (3.16)

$$F_{tot}^r = -\frac{i}{2\omega}(P_1^* P_1' - P_1^{*\prime} P_1) \tag{3.21}$$

Using (3.21) and the expression for the absorption coefficients in (3.13) we have for this mode $\sigma_1 = \frac{4}{9}(\omega\sqrt{q})^6$.

Our final example is very similiar to the oscillating dipole in standard E&M (see e.g, [11]). Take $\vec{v}$ to be a generalization of a dipole[3]

$$v^r = 2cos\theta P_2(r), \qquad v^\theta = \frac{1}{r}P_2(r), \qquad v^\phi = 0 \tag{3.22}$$

This is equivalent to taking the cartesian components v^k proportional to the radial function $P_2(r)$ times linear combinations of the spherical harmonics $Y_{20}, Y_{2,\pm 1}$. The scalar field ψ then has a dipole form, there is dipole radiation in the electromagnetic fields, and the radial component of $\vec{E}$ is asymptotically of the same form as in [11]:

$$\begin{aligned}
\psi &= \frac{2}{\omega^2}\cos\theta\left(P_2' + \frac{3}{r}P_2\right) \\
E^{\hat{\theta}} &= \sin\theta[P_2 - \frac{2}{\omega^2 r}(P_2' + \frac{3}{r}P_2)] \\
B^{\hat{\phi}} &= -i\frac{\sin\theta}{\omega}(P_2' + \frac{3}{r}P_2) \\
E^{\hat{r}} &= \frac{2cos\theta}{\omega^2}\left(\frac{1}{r}P_2' + \frac{3}{r^2}P_2 - \frac{\omega^2 q^2}{r^4}P_2\right)
\end{aligned} \tag{3.23}$$

In the limit $r \to \infty$, $E^{\hat{r}}$ then has the usual dipole form

$$E^{\hat{r}} \simeq \frac{2\cos\theta}{r^2}e^{i\omega r}. \tag{3.24}$$

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Appendix A. Scattering Coefficients

In this appendix we derive the low energy absorption coefficients for all modes of the DBI system. Recall that the radial equation (3.10) governs both overall transverse oscillations and the oscillations of each component v^k in the relative transverse DBI systems

[3] Recall that this is all in fourier space, so there are also implicit oscillating factors

(as well as the overall transverse test-string modes). The radial equation (3.10) is put in standard scattering form in (3.12) by using the tortoise coordinate $dR_* = \sqrt{H(R)}dR$ and the rescaled wavefunction $\phi = H^{1/4}P$. Note that this implies that the region down the spike is $R_*, R \to \infty$, while the flat part of the D3-brane is $R_* \to -\infty$, $R \to 0$ and that the angular momentum term in (3.12) falls off like $l(l+1)/R_*^2$ as $R_* \to \pm\infty$. The potential $V_\perp$ falls off like $1/R_*^6$ at either end, implying that solutions to (3.12) are well approximated in terms of Bessel functions at the two ends. We can then write

$$\begin{aligned} R_* \to \infty; \qquad & \phi_l(R_*) \simeq \omega R_* \left(a_l h_l^{(2)}(\omega R_*) + s_l h_l^{(1)}(\omega R_*) \right) \equiv \phi_{spike}, \\ R_* \to -\infty; \qquad & \phi(R_*) \simeq \omega R_* c_l h_l^{(2)}(\omega R_*) \equiv \phi_{out}. \end{aligned} \tag{A.1}$$

The strategy for calculating the absorption coefficients for the different l modes is to expand $P_{l,spike}$ and $P_{l,out}$ for small $|R_*|$ and patch them together using the $\omega = 0$ solution valid in a middle region. Recall that $\psi = H^{-1/4}\phi$. Using the asymptotic forms of the Hankel functions and the toroise coordinate we then get the expansions

$$\begin{aligned} P_{l,spike} &\simeq \frac{(a_l + s_l)}{(2l+1)!!}(\omega R)^{l+1} + i\frac{(a_l - s_l)(2l-1)!!}{(\omega R)^l} \\ P_{l,out} &\simeq -\omega\sqrt{q}\left\{ \frac{1}{(2l+1)!!}\left(-\frac{\omega q}{R}\right)^l + i(2l-1)!!\left(-\frac{R}{\omega q}\right)^{l+1} \right\} c_l, \end{aligned} \tag{A.2}$$

where $(2l+1)!! \equiv (2l+1)(2l-1)\cdots(5)(3)(1)$. For $\omega = 0$ we have

$$\partial_R^2 P_{l,middle} - \frac{l(l+1)}{R^2} P_{l,middle} = 0, \tag{A.3}$$

which is solved by

$$\psi_{l,middle} = \alpha_l R^{l+1} + \beta_l \frac{1}{R^l}. \tag{A.4}$$

Matching coefficients across the three regions yields the relations

$$\begin{aligned} a_l + s_l &= i(-1)^l (\omega\sqrt{q})^{-(2l+1)} L c_l \\ a_l - s_l &= i(-1)^l (\omega\sqrt{q})^{2l+1} L c_l, \end{aligned} \tag{A.5}$$

where $L = (2l+1)!!(2l-1)!!$. For $\omega\sqrt{q} \to 0$ this gives the dimensionless absorption coefficient σ_l

$$\sigma_l \equiv \frac{|c_l|^2}{|a_l|^2} \simeq \frac{4(\omega\sqrt{q})^{4l+2}}{L^2}. \tag{A.6}$$

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