Development and Implementation of Algebraic Formulas for Calculation in Trapezoids

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МАТЕМАТИЧЕСКОЕ ОБРАЗОВАНИЕ: СОВРЕМЕННОЕ СОСТОЯНИЕ И ПЕРСПЕКТИВЫ

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DEVELOPMENT AND IMPLEMENTATION OF ALGEBRAIC FORMULAS FOR CALCULATION IN TRAPEZOIDS

Complex geometrical problems are problems whose process of solution requires a search, and standard solution algorithms for which do not exist. Such problems are called nonstandard problems.

Even outstanding students have trouble solving many of these problems, and these are usually not included in the school curriculum, even not at the highest levels. There is therefore a need to find other, non-geometrical, methods of solution which might turn non-standard problems into standard ones.

For some geometrical shapes there exist formulas which express the connection between the different elements of these shapes. For instance, for a triangle there are formulas which connect the sides with the medians, the heights and the angle bisectors. Also, there exists a set of formulas that connect the sides and the diagonals of a quadrilateral inscribed in a circle. If we consider these sets of formulas, we shall find that they allow us to calculate the parts (elements) of a given quadrilateral from the given ones.

In the present article we shall focus on problems which reveal connections between the elements of an arbitrary trapezoid. These problems differ from each other by the set of given elements, by the drawings and the required auxiliary constructions, and therefore also have different geometric solutions. We shall offer a set of formulas (not necessarily minimal), which connects the different elements of an arbitrary trapezoid. This set of formulas shall allow an algebraic calculation (i.e., without the use of drawings and auxiliary constructions) of the missing elements based on the given elements of the trapezoid, thus transforming non-standard problems of an arbitrary trapezoid into standard problems of solving a set of quadratic equations. This approach shall allow us: (a) to incorporate new problems in school studies which were not integrated in the curriculum due to their complexity, and (b) to transform some 5th (highest) — level problems into problems which are accessible to 4th level students as well.

It is important to note that all the proofs of the formulas in the system are based on the use of the Pythagorean theorem, the properties parallelograms, Heron’s formula for the area of a triangle and the properties of similar triangles. These are proofs which do not go beyond the scope of the geometric material studied at school.

In addition, the algebraic solution of trapezoid problems does not require the ability to solve sets of equations higher than of the second degree (i.e., a level attained at the 10th grade), therefore the set of formulas can be implemented in the learning process.

1 Hadamard J, «Lecons de geometrie elementare», paragraphs, 128—130.
2 Ibid., exercise 139.
3 The elements of a trapezoid are: the sides, the diagonals and their parts, the midlines and their parts, as well as the altitude.
of a trapezoid for school students as part of the regular studies, extra lessons and enrichment classes.

As we already mentioned, the system of formulas of a random trapezoid is not minimal, and it allows flexibility in selecting the formulas, thus simplifying the process of solution.

The formulas for the connection between the different parts of an arbitrary trapezoid.

Shown is an arbitrary trapezoid $ABCD$ ($AB \parallel CD$, $CD > AB$) (Fig. 1),

![Fig. 1]

Fig. 1

To simplify the notation we denote:

a) The bases: $AB$ by $a$, $CD$ by $b$

b) The sides: $BC$ by $c$, $AD$ by $d$

c) The diagonals: $AC$ by $m$, $BD$ by $n$,

and the parts of the diagonals: $AE$ by $m_1$, $CE$ by $m_2$, $BE$ by $n_1$, and $DE$ by $n_2$

d) The midlines: of the sides $FG$ by $f$, of the bases $JK$ by $l$, of the diagonals $HI$ by $g$, and the parts of $JK$: $EJ$ by $l_1$, $KE$ by $l_2$

e) The altitude $JP$ (or $AQ$ etc.) by $h$ (Fig. 2).

List of the relation formulas:

1. $(a) f = \frac{a+b}{2}$; $(b) g = \frac{b-a}{2}$

(c) $\frac{g}{f} = \frac{b-a}{a+b}$

2. $m^2 + n^2 = c^2 + d^2 + 2ab$

3. $(m^2 - n^2) \frac{g}{f} = c^2 - d^2$ or from 1(c):

$(m^2 - n^2) \frac{g}{f} = c^2 - d^2$

4. $m^2 + n^2 = 2(f^2 + l^2)$

5. $c^2 + d^2 = 2(g^2 + l^2)$

6. $|m^2 - n^2| = 4f\sqrt{l^2 - h^2}$

7. $|c^2 - d^2| = 4g\sqrt{l^2 - h^2}$

8.(a)

$$n_2 - n_1 = 2 \cdot EI = m_2 - m_1 = 2 \cdot EH = \frac{b-a}{m}$$

(b) $l_2 - l_1 = \frac{2 \cdot EL}{l} = \frac{b-a}{a+b}$

(c) $\frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{l_1}{l_2} = \frac{a}{b}$

9. $S_{ABCD} = \frac{1}{2} (a+b) \cdot h = f \cdot h$
2. Proofs of the formulas

Note: we shall limit ourselves to proving those formulas of the system above, which do not appear in textbooks in geometry, i.e., formulas 2—8 a, b.

Proof of formula 2

We make use of the following auxiliary construction (see Fig. 3):

\[ AR \parallel BD, AO \parallel BC \]
\( (R, O \in CD) \)

Based on our notation chosen above: \( AR = n, AO = c, CO = a, DR = a, \)
in addition, we denote \( OQ \) by \( x \) and \( QD \) by \( y \).

We express the square of the altitude \( AQ \) four times using the sides of the following four triangles (using the Pythagorean theorem):

(1) In \( \triangle AOOQ \): \( AO^2 = c^2 - x^2 \)
(2) In \( \triangle ADOQ \): \( AO^2 = d^2 - y^2 \)
(3) In \( \triangle ACQ \): \( AO^2 = m^2 - CO^2 \)
(4) In \( \triangle ARQ \): \( AO^2 = n^2 - QO^2 \)

By adding the equalities (1) and (2), (3) and (4) term-by-term we have:
\[ 2AO^2 = c^2 + d^2 - x^2 - y^2 \]
and hence:
\[ m^2 + n^2 - CO^2 - QO^2 = c^2 + d^2 - x^2 - y^2 \]

Since \( CO = a + x \) (*) and \( QO = a + y \) (**),
we obtain after substitution:
\[ m^2 + n^2 - (a + x)^2 - (a + y)^2 = c^2 + d^2 - x^2 - y^2 \]
or:
\[ m^2 + n^2 = c^2 + d^2 + 2a^2 + 2a(x + y)^2 \]
also, there holds:
\[ x + y = OD = CR - CO - RD = (a + b) - a - a = b - a \]
\[ x + y = b - a \ (***) \]
Therefore:
\[ m^2 + n^2 = c^2 + d^2 + 2a^2 + 2a(b - a) \]
And finally:
\[ m^2 + n^2 = c^2 + d^2 + 2ab \]

Proof of formula 3

We subtract the equalities (1) and (2), (3) and (4) from the previous proof term-by-term, and we obtain respectively:
\[ m^2 - CO^2 - n^2 + QO^2 = 0 \]
and \( c^2 - x^2 - d^2 + y^2 = 0 \)
We work with each of these equalities separately and we bring them to the following form using the equalities (*), (**), and (***):

\[
\begin{align*}
m^2 - n^2 & = (a + x)^2 - (a + y)^2 & c^2 - d^2 & = (x - y)(x + y) \\
m^2 - n^2 & = (x - y)(2a + x + y) & c^2 - d^2 & = (x - y)(b - a) \\
m^2 - n^2 & = (x - y)(a + b) & \end{align*}
\]

We divide the two equalities term-by-term and obtain:

\[
\frac{c^2 - d^2}{m^2 - n^2} = \frac{(x - y)(b - a)}{(x - y)(a + b)}
\]

or

\[
c^2 - d^2 = (m^2 - n^2) \frac{b - a}{a + b}.
\]

**Proof of formula 4**

We use the following auxiliary construction (see Fig. 4):

- \( JY \parallel BD, JX \parallel AC \)
- \( X, Y \in CD \)

From the properties of opposite sides in a parallelogram it is easy to see that:

\[
JX = m, JY = n, XY = a + b,
\]

and hence:

\[
KY = XK = \frac{a + b}{2}.
\]

In the right-angled triangle \( JXP \) we have: \( XP^2 = JX^2 - JP^2 \)

Or: \((*)\) \( XP^2 = m^2 - h^2 \)

On the other hand: \( XP = XK + KP \),

and hence, using the notation from section 2: \( XP^2 = \left( \frac{a + b}{2} + KP \right)^2 \).

Finally: \((**)*\) \( XP^2 = (f + KP)^2 \)

And from equalities \((*)\) and \((**)\) it follows that: \((***) m^2 - h^2 = (f + KP)^2 \)

In a similar manner, in the right angled triangle \( JYP \) we have:

\[
PY^2 = JY^2 - JP^2 \text{ or } PY^2 = n^2 - h^2,
\]

On the other hand we have: \( PY = KY - KP \),

And therefore \( PY^2 = \left( \frac{a + b}{2} - KP \right)^2 \) or \( PY^2 = (f - KP)^2 \)
and finally: \((***)\) \(n^2 - h^2 = (f - KP)^2\).
Now we add \((***)\) to \((***)\) and obtain: \(m^2 + n^2 - 2h^2 = (f + KP)^2 + (f - KP)^2\)
and hence: \(m^2 + n^2 = 2(f^2 + h^2 + KP^2)\),
or: \(m^2 + n^2 = 2(f^2 + P)\)

**Proof of formula 5**

From formulas 2 and 4 it follows that: \(c^2 + d^2 + 2ab = 2f^2 + 2l^2\).

hence: \(c^2 + d^2 = \frac{(a+b)^2}{2} - 2ab + 2l^2\)

\[c^2 + d^2 = \frac{(b-a)^2}{2} + 2l^2\]

\[c^2 + d^2 = 2g^2 + 2l^2\], and finally: \(c^2 + d^2 = 2(g^2 + P)\)

**Proof of formula 6**

We use the same auxiliary construction as in the proof of formula 4 (see Fig.4). It is easy to show that the triangle \(JXY\) and the trapezoid \(ABCD\) have the same area (among other things due to the congruence of the two pairs of triangles), in other words: \((*)\) \(S_{JXY} = S_{ABCD}\).

To find the area of the triangle \(JXY\) we use Heron’s formula:

Half the perimeter of triangle \(JXY\) is: \(\frac{JX + JY + XY}{2}\),
or using our notation: \(\frac{m + n + (a + b)}{2}\), therefore:

\[S_{JXY} = \frac{m + n + (a + b)}{2} \cdot \frac{(m + n + (a + b) - m)}{2} \cdot \frac{(m + n + (a + b) - n)}{2} \cdot \frac{(m + n + (a + b) - (a + b))}{2} = \]

\[= \frac{(m + n + a + b)(n + a + b) + m - n + a + b)}{2} \cdot \frac{(m + n - a + b)}{2} \cdot \frac{(m + n - a + b)}{2} = \]

\[= \left(\frac{m + n}{2} - \frac{a + b}{2}\right)^2 \left(\frac{a + b}{2} - \frac{m - n}{2}\right)^2 = \left(\frac{m + n}{2} - f^2\right)\left(f^2 - \frac{m - n}{2}\right)^2\]

since \(S_{ABCD} = fh\), from the equality of the areas \((*)\), it follows that:
\[
\left( \frac{(m+n)^2}{2} - f^2 \right) \cdot \left( f^2 - \left( \frac{m-n}{2} \right)^2 \right) = f^2 h^2
\]

and hence:
\[
\frac{(m^2 + 2mn + n^2 - 4f^2)}{4} \cdot \frac{(4f^2 - m^2 + 2mn - n^2)}{4} = f^2 h^2
\]

\[
8f^2 m^2 - m^4 + 2m^2 n^2 + 8f^2 n^2 - n^4 - 16f^4 = 16f^2 h^2
\]

\[
8f^2 \left( m^2 + n^2 \right) - \left( m^2 - n^2 \right)^2 - 16f^4 - 16f^2 h^2 = 0
\]

In the last equality we replace the sum (from formula 4):
\[
8f^2 \left( 2f^2 + 2l^2 \right) - 16f^4 - 16f^2 h^2 = \left( m^2 - n^2 \right)^2
\]

\[
16f^2 \left( l^2 - h^2 \right) = \left( m^2 - n^2 \right)^2
\]

and finally we obtain:
\[
\left| m^2 - n^2 \right| = 4f \sqrt{l^2 - h^2}.
\]

**Proof of formula 7**

In formula 6 we replace the difference \( m^2 - n^2 \) by the expression \( \frac{a+b}{b-a} (c^2 - d^2) \)

(Their equality is a result of formula 3):
\[
\left| \frac{a+b}{b-a} \cdot (c^2 - d^2) \right| = 4f \sqrt{l^2 - h^2}.
\]

Therefore, since \( \frac{a+b}{b-a} > 0 \), we obtain:
\[
\frac{a+b}{b-a} \cdot | c^2 - d^2 | = 4f \sqrt{l^2 - h^2},
\]

or:
\[
| c^2 - d^2 | = 4 \cdot \frac{b-a}{a+b} \cdot \frac{a+b}{2} \cdot \frac{1}{\sqrt{l^2 - h^2}}
\]

and finally:
\[
| c^2 - d^2 | = 4g \sqrt{l^2 - h^2}
\]
Proof of formulas 8a and 8b

First we prove that \(2 \cdot EN = m_2 - m_1\)
\[EH = CE - CH = m_2 - \frac{CA}{2} = \]
\[= m_2 - \frac{m_1 + m_2}{2} = \frac{m_2 - m_1}{2}\]

In Figure 5 it is given that \(AR || BD, AM || JK\)

From the similarity of the triangles \(EHI\) and \(ACR\) it follows that:
\[
\frac{EH}{AC} = \frac{HI}{CR},
\]
in addition, according to the notation \(AC = m, HI = \frac{b-a}{2}\), and \(CR = a+b\).

After substitution we obtain:
\[
\frac{EH}{m} = \frac{2}{a+b},
\]
and therefore:
\[(*) \quad EH = \frac{m(b-a)}{2(a+b)}, \text{ and finally } \frac{2 \cdot EH}{m} = \frac{b-a}{a+b}.
\]

From the similarity of the triangles \(EHL\) and \(ACM\) it follows that \(\frac{EH}{AC} = \frac{EL}{AM}\).

Therefore from (*) and since \(AM = l\), we obtain after substitution:
\[
\frac{m(b-a)}{2(a+b)} = \frac{EL}{l}, \text{ and hence } \frac{2 \cdot EL}{l} = \frac{b-a}{a+b}.
\]

3. Examples for the algebraic solution in an arbitrary trapezoid

Example 1

In a trapezoid it is given that: The lengths of the sides are \(6\sqrt{2}\) cm and \(2\sqrt{7}\) cm, the lengths of the diagonals are 13 cm and 9 cm.

Find the lengths of the bases and the midline of the bases.

Solution: We denote the side whose length is \(6\sqrt{2}\) by \(c\), and the side whose length is \(2\sqrt{7}\) by \(d\). Since \(c > d\), we also have \(m > n\) (because from formula 3, the differences \(c^2 - d^2\) and \(m^2 - n^2\) have the same sign).
Therefore \( m = 13 \) and \( n = 9 \).

We substitute the lengths of the sides and the diagonals in formulas 2 and 3 in the list, and obtain:

\[
\begin{align*}
169 + 81 &= 72 + 28 + 2ab \\
72 - 28 &= \frac{b - a}{a + b} \cdot (169 - 81)
\end{align*}
\]

or \( \begin{cases} ab = 75 \\ b - a = 1 \\ a + b = 2 \end{cases} \)

from the equation of the system we obtain: \( b = 3a \).

We substitute \( b \) in the first formula and obtain: \( 3a^2 = 75 \),

and hence: \( a = 5 \text{ cm} \), \( b = 15 \text{ cm} \).

After substituting the lengths of the segments \( m, n \) and \( f \) \( \left( f = \frac{5 + 15}{2} = 10 \right) \) in formula 4, we obtain: \( 13^2 + 9^2 = 2(10^2 + P) \) or \( P = 25 \). Therefore: \( l = 5 \text{ cm} \).

**Example 2**

In a trapezoid it is given that: The lengths of the bases are \( a = 7 \text{ cm} \), \( b = 14 \text{ cm} \).
The length of the midline of the bases is \( l = 4.5 \text{ cm} \), the length of the altitude is \( h = \frac{12\sqrt{5}}{7} \text{ cm} \).

Find the lengths of the sides and the diagonals in the trapezoid.

**Solution:** From the datum it follows that \( f = \frac{a + b}{2} = 10.5 \), \( g = \frac{b - a}{2} = 3.5 \).

After substituting the known segments in formulas 4, 5, 6 and 7 we obtain the following two systems: (*) \( \begin{cases} c^2 + d^2 = 65 \\ c^2 - d^2 = 33 \end{cases} \) and (**) \( \begin{cases} m^2 + n^2 = 261 \\ m^2 - n^2 = 99 \end{cases} \).

Assuming \( c > d \), we obtain from system (*) \( 2c^2 = 98 \Rightarrow c^2 = 49 \Rightarrow c = 7 \text{ cm} \),

and also: \( d^2 = 16 \Rightarrow d = 4 \text{ cm} \).

From the inequality \( m > n \) and from system (**) we obtain:

\[ 2m^2 = 360 \Rightarrow m^2 = 180 \Rightarrow m = 6\sqrt{5} \text{ cm} \]

and finally: \( n^2 = 81 \Rightarrow n = 9 \text{ cm} \).

4. Examples for problems in calculation of an arbitrary trapezoid which can be solved by the algebraic method

Note: To write down the problems:

a) We denote all the parts of the trapezoid as specified in section 2,

b) All the values (the lengths of the segments and the area of the trapezoid) shall be measured using the same units.
1) Given: $a = 3\sqrt{3.4} , b = 10\sqrt{3.4} , c = 14, d = 7$.
Calculate $m$ and $n$ [answer: $m = 19, n = 2\sqrt{22}$]

2) Given: $a = 2, b = 14, m = 13, n = 7$.
Calculate $c$ and $d$ [answer: $c = 3\sqrt{14}, d = 6$]

3) Given: $a = 3, b = 9, l = 2\sqrt{21}, d = \sqrt{65}$.
Calculate $c, m$ and $n$ [answer: $c = 11, m = 4\sqrt{11}, n = 8$]

4) Given: $f = 30, g = 15, l = 15, m = 39$.
Calculate $n, c$ and $d$ [answer: $n = 27, c = 18\sqrt{2}, d = 6\sqrt{7}$]

5) Given: $c = 78, d = 32.5, l = 42.25, \frac{g}{f} = \frac{5}{8}$
Calculate $a, b$ and $S$ [answer: $a = 25.35, b = 109.85, S = 2028$]

6) Given: $m = 12, n = 9, f = 7.5, d = 7.8$.
Calculate $a, b$ and $c$ [answer: $a = 2.4, b = 12.6, c = 7.2\sqrt{2}$]

7) Given: $d = 3, m = \sqrt{73}, f = 6, l = \sqrt{13}$.
Calculate $h, a, b$ and $c$ [answer: $h = 3, a = 4, b = 8, c = 5$]

8) Given: $c = 2, d = 8, l_1 = 3.2, l_2 = 4.8$.
Calculate $a, b, m$ and $n$ [answer: $a = 24, b = 36, m = 2\sqrt{286}, n = 28$]

Calculate $h$ and $l$ [answer: $h = 14\frac{14}{29}, l = 14.5$]

10) Given: $m_1 = 2.75, m_2 = 8.25, n_2 = 6.75, l_1 = 1.5$.
Calculate $c, d$, and $S$ [answer: $c = 0.5\sqrt{249}, d = 6.5, S = 8\sqrt{35}$]

11) Given: $f = 8, g = 2, l = 3 , m = 8.125$.
Calculate $c$ and $d$ [answer: $c = 8, d = \sqrt{34}$]
1) Given: \( a = 3\sqrt{3.4} \), \( b = 10\sqrt{3.4} \), \( c = 14 \), \( d = 7 \).
Calculate \( m \) and \( n \) [answer: \( m = 19 \), \( n = 2\sqrt{22} \)]

2) Given: \( a = 2 \), \( b = 14 \), \( m = 13 \), \( n = 7 \).
Calculate \( c \) and \( d \) [answer: \( c = 3\sqrt{14} \), \( d = 6 \)]

3) Given: \( a = 3 \), \( b = 9 \), \( l = 2\sqrt{21} \), \( d = \sqrt{65} \).
Calculate \( c \), \( m \) and \( n \) [answer: \( c = 11 \), \( m = 4\sqrt{11} \), \( n = 8 \)]

4) Given: \( f = 30 \), \( g = 15 \), \( l = 15 \), \( m = 39 \).
Calculate \( n \), \( c \) and \( d \) [answer: \( n = 27 \), \( c = 18\sqrt{2} \), \( d = 6\sqrt{7} \)]

5) Given: \( c = 78 \), \( d = 32.5 \), \( l = 42.25 \), \( \frac{g}{f} = \frac{5}{8} \).
Calculate \( a \), \( b \) and \( S \) [answer: \( a = 25.35 \), \( b = 109.85 \), \( S = 2028 \)]

6) Given: \( m = 12 \), \( n = 9 \), \( f = 7.5 \), \( d = 7.8 \).
Calculate \( a \), \( b \) and \( c \) [answer: \( a = 2.4 \), \( b = 12.6 \), \( c = 7.2\sqrt{2} \)]

7) Given: \( d = 3 \), \( m = \sqrt{73} \), \( f = 6 \), \( l = \sqrt{13} \).
Calculate \( h \), \( a \), \( b \) and \( c \) [answer: \( h = 3 \), \( a = 4 \), \( b = 8 \), \( c = 5 \)]

8) Given: \( c = 2 \), \( d = 8 \), \( l_1 = 3.2 \), \( l_2 = 4.8 \).
Calculate \( a \), \( b \), \( m \) and \( n \) [answer: \( a = 24 \), \( b = 36 \), \( m = 2\sqrt{286} \), \( n = 28 \)]

9) Given: \( m = 21 \), \( n = 20 \), \( S = 210 \).
Calculate \( h \) and \( l \) [answer: \( h = 14\frac{14}{29} \), \( l = 14.5 \)]

10) Given: \( m_1 = 2.75 \), \( m_2 = 8.25 \), \( n_2 = 6.75 \), \( l_1 = 1.5 \).
Calculate \( c \), \( d \), and \( S \) [answer: \( c = 0.5\sqrt{249} \), \( d = 6.5 \), \( S = 8\sqrt{35} \)]

11) Given: \( f = 8 \), \( g = 2 \), \( l = 3 \), \( m = 8.125 \).
Calculate \( c \) and \( d \) [answer: \( c = 8 \), \( d = \sqrt{34} \)]
Given: \( \frac{a}{b} = \frac{1}{7} \), \( l = \sqrt{61} \), \( d = 8 \), \( EH = 4.875 \) (see Fig. 1)

Calculate \( a, b, c \) and \( n \) [answer: \( a = 2, b = 14, c = \sqrt{130}, n = 9 \)]