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Tensor operators III, some fundamental tensor operator identities

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Tensor operators III: Some fundamental tensor operator identities

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Let \( s \geq 2 \) and let \( a = (a_1, a_2, \ldots, a_s) \), where \( s, a_i \in \mathbb{Z} \) and \( a_1 > a_2 > \cdots > a_s > 1 \). Let \( 1 < t < s \). The method of undetermined coefficients was used to investigate natural transformations \( \Omega: \Lambda^a \rightarrow \Lambda^{a_1} \otimes \Lambda^{a_2} \otimes \cdots \otimes \Lambda^{a_s} \) and \( \Omega_\gamma: \Lambda^a \rightarrow \Lambda^{(a_1, a_2, \ldots, a_s)} \otimes \Lambda^n_\gamma \). Necessary and sufficient conditions were found on the coefficients defining \( \Omega \) and sufficient conditions on the coefficients defining \( \Omega_\gamma \). Explicit formulas for \( \Omega_1 \) and \( \Omega_\gamma \) are given herein.

1. INTRODUCTION

The theory of 3j and 6j symbols developed by physicists in connection with the quantum theory of angular momentum is a remarkably detailed theory of how tensor products of finite-dimensional representations of SU(2) decompose. This theory has recently had an unexpected application to the theory of three manifolds, via beautiful constructions amounting to q deformations of the 3j and 6j symbols, discovered by Kirillov, Reshetikhin, Turaev, Viro, and others.1-6 These recent mathematical developments add an even greater interest to the work (beginning in Ref. 7) of Biedenharn and his school to develop a theory of “tensor operators” with “Racah-Wigner algebra” that generalizes to arbitrary SU(\( n \)) the classical results for SU(2).

In Ref. 8, an attempt was made to establish a precise mathematical foundation for some of the concepts that are intuitively (but perhaps not always rigorously) sketched by the physicists in the work on SU(\( n \)) just cited. In Ref. 8 four specific tests were outlined for judging the success of this attempt to provide a rigorous definition for the concept of “coherent tensor operator.” That is, four specific assertions \( W_1, W_2, W_3, W_4 \) were proposed that provide a necessary condition that the concepts of Ref. 8 indeed embody the concepts in the work of Biedenharn's school. These four assertions have been verified for coherent tensor operators for SU(2) and for SU(3) in Ref. 9. They remain open for SU(\( n \)) for \( n > 3 \), and they make sense for all connected compact Lie groups. The case SU(3) already represents, we believe, a significant extension (perhaps ready for q deformation) of the Racah–Wigner formalism of 3j and 6j symbols.

What seems to us to be the major obstacle to further progress in this direction is the need to establish analogues for SU(\( n \)) of the identities for SU(3) given in Lemmas 4.4 and 4.6 of Ref. 8 together with the consequent “straightening rules” for coherent tensor operators over SU(3) presented in Lemma 4.7 of Ref. 8. What is accomplished in the present article is a first step (as we hope) in this direction; namely, we obtain (in a different formalism) explicit extensions from SU(3) to SU(\( n \)) of a subset of the identities just mentioned, roughly the natural extensions of the identities of Lemmas 4.4 and 4.6 of Ref. 8.

We work in the context of the representation functors \( \Lambda^a \) introduced in Ref. 10. The referee has suggested that we review this “shape algebra” construction. We sketch the construction here only for the general linear group, which is all we need for our present purposes, remarking however that similar constructions have been developed for the other classical groups in Refs. 11 and 12 and for \( G_2 \) in Ref. 13.
Let $E$ be an $n$-dimensional vector space over the field $\mathbb{C}$ of complex numbers. The "shape algebra" $\Lambda^+E$ is a $\mathbb{C}$ algebra constructed in a basis free way from $E$. It is the direct sum of precisely one copy of each finite-dimensional irreducible polynomial representation $\Lambda^aE$ of the general linear group $GL(E)$

$$\Lambda^+E = \bigoplus_{a \in A} \Lambda^aE. \quad (1.1)$$

There are a number of popular choices for the set $A$ of all labels of such irreducible representations of $GL(E)$; for the purposes of the present construction it is convenient to choose $A$ to be the set of all "partitions" $\alpha = (a_1,a_2,...,a_r)$ (with $a_1 \geq a_2 \geq ... \geq a_r > 1$) such that $\max \alpha = a_1 \leq n = \dim E$. The set of partitions forms an additive semigroup, where the sum $\alpha + \beta$ of $\alpha$ and $\beta = (b_1,b_2,...,b_s)$ is defined to be the partition obtained by arranging the natural numbers $a_1,...,a_r,b_1,...,b_s$ in nonincreasing order.

The algebra $\Lambda^+E$ is defined, by generators and relations, as follows: it is the commutative, associative $\mathbb{C}$ algebra generated by the $2^n$-dimensional complex vector space $\Lambda E$ (the usual exterior algebra on $E$) with relations as explained in Eq. (1.3) below. We denote by $\cdot$ the multiplication (which is commutative) on $\Lambda^+E$. (This multiplication in $\Lambda^+E$ is to be carefully distinguished from the exterior product $\wedge$ in the algebra $\Lambda E$.) Thus, a typical element of the shape algebra $\Lambda^+E$ is a $\mathbb{C}$-linear combination of elements of the form

$$\omega = (e_1 \wedge \cdots \wedge e_a) \cdot (f_1 \wedge \cdots \wedge f_b) \cdots (h_1 \wedge \cdots \wedge h_c), \quad (1.2)$$

where the $e$'s, $f$'s, and $h$'s are in $E$. Since $\Lambda^+E$ is commutative, we may assume without loss of generality that in Eq. (1.2), $a > b > ... > c$; thus $\alpha = (a,b,...,c)$ is a partition in the sense defined above. We then define $\Lambda^\alpha E$ to be the $\mathbb{C}$-linear span of such "shapes of degree $\alpha$" as in Eq. (1.2).

Note that $\Lambda^aE \cdot \Lambda^bE \subseteq \Lambda^{a+b}E$.

To complete this sketch of the shape algebra, it remains only to explain the defining relations among the expressions (1.2). They are given by the following important "Young symmetry conditions":

If $n \geq a > b > 1$, and if the elements $e_1,...,e_{a+1},f_1,...,f_{b-1}$ all lie in $E$, then

$$\sum_{\lambda=1}^{a+1} (-1)^\lambda (e_1 \wedge \cdots \wedge \hat{e}_\lambda \wedge \cdots \wedge e_{a+1}) \cdot (e_1 \wedge f_1 \wedge \cdots \wedge f_{b-1}) = 0. \quad (1.3)$$

[In Eq. (1.3) as elsewhere we use the symbol $\hat{\cdot}$ to indicate the deletion of an element in a sequence.] Note that the defining relations (1.3) for $\Lambda^+E$ are compatible with the gradation (1.1).

For a more detailed discussion of this method of constructing the irreducible representations of $GL(E)$, see Refs. 10 and 14. Since our representation spaces $\Lambda^aE$ are spaces of "shapes" rather than spaces of tensors, one could call transformations between the representations "shape operators." We have, however, decided to keep the traditional name of "tensor operators." The advantages of the shape construction for the study of coherent tensor operators are discussed in Ref. 8.

This shape algebra formalism will be utilized below in setting up a somewhat elaborate formalism for studying tensor products $\otimes$ of irreducible representations of $GL(E)$. The formalism, which is explained in detail in Secs. II and V below, is one which we expect will play a role in future work.

Both Secs. III and IV are technical, related, respectively, to the determination of a sign and the Young symmetry properties of a certain expression. Section V concerns the key concept (for this article) of "exchange tables" and their associated "exchange sums" (Definitions 5.1 and 5.3), and Sec. VI discusses some properties of these. In Secs. VII and VIII we put all this together to obtain sets of equations whose solutions give explicit formulas for certain natural
II. NOTATION

Let \( s > 1, s \in \mathbb{Z} \). Let \( s = \{1, \ldots, s\} \). Let \( \alpha = \langle a_1, a_2, \ldots, a_s \rangle \), where \( a_i \in \mathbb{Z} \) and \( a_1 > a_2 > \cdots > a_s > 1 \). Let \( R_i = \{ x_1^{(i)}, x_2^{(i)}, \ldots, x_{a_i}^{(i)} \} \), for \( 1 \leq i \leq s \). Let \( R = \bigcup_{i=1}^{s} R_i \).

**Definition 2.1:** For \( P \subseteq R \) and \( 1 \leq i \leq s \), define

\[
P(i) = P \cap R_i,
\]

so that \( P = \bigcup_{i=1}^{s} P(i) \).

**Definition 2.2:** The number \( \text{inv}(A, B) \) of *inversions* of an ordered pair \((A, B)\) of disjoint subsets of \( R \) is defined via Eqs. (2.2) and (2.3).

\[
\text{inv}(A, B) = \#\{ (x^{(i)}_m, x^{(i)}_n) \in A \times B : m > n \}
\]

if \( A \subseteq R_i \) and \( B \subseteq R_i \),

\[
\text{inv}(A, B) = \sum_{i=1}^{s} \text{inv}(A(i), B(i)).
\]

**Definition 2.3:** For \( P = (P_1, \ldots, P_r) \) where the \( P_j \) are pairwise disjoint subsets of \( R \), define \( \text{inv}(P) \) by the following formula:

\[
\text{inv}(P) = \sum_{1 \leq i < j \leq r} \text{inv}(P_i P_j).
\]

**Definition 2.4:** For \( P \subseteq R \), define \( \omega(P) \) via Eqs. (2.5), (2.6), (2.7).

\[
\omega(\emptyset) = 1,
\]

\[
\omega(P) = x^{(i)}_{m_1} \wedge x^{(i)}_{m_2} \wedge \cdots \wedge x^{(i)}_{m_t}
\]

if \( P = \{ x^{(i)}_{m_1}, x^{(i)}_{m_2}, \ldots, x^{(i)}_{m_t} \} \subseteq R_i \) with \( m_1 < m_2 < \cdots < m_t \),

\[
\omega(P) = \omega(P(1)) \wedge \omega(P(2)) \wedge \cdots \wedge \omega(P(s)).
\]

**Definition 2.5:** For \( P = (P_1, \ldots, P_r) \) where the \( P_j \) are pairwise disjoint subsets of \( R \), and \( 1 \leq i \leq r \), define \( \omega(P) \) and \( \omega_i(P) \) as follows:

\[
\omega(P) = (-1)^{\text{inv}(P)} \omega(P_1) \boxtimes \omega(P_2) \boxtimes \cdots \boxtimes \omega(P_r).
\]

\[
\omega_i(P) = (-1)^{\text{inv}(P)} \omega(P_1) \cdot \omega(P_2) \cdots \omega(P_i) \cdots \omega(P_r) \rho \omega(P_i).
\]

**Definition 2.6:** For \( A \subseteq R \) and \( x, y \in R \), define \([x, y]A\) to be the set obtained from \( A \) by replacing \( x \) by \( y \) and \( y \) by \( x \). Formally,

\[
[x, y]A = \begin{cases} 
A \cup \{x\} - \{y\}, & \text{if } x \in A, \; y \in A \\
A \cup \{y\} - \{x\}, & \text{if } x \notin A, \; y \notin A \\
A, & \text{otherwise.}
\end{cases}
\]
Definition 2.7: For \( P = (P_1, \ldots, P_r) \) such that \( P_j \subseteq R \), and given \( x, y \in R \), define \([x, y]_P\) as follows:

\[
[x, y]_P = ([x, y]_P) = (\ldots, [x, y]_P) (\ldots).
\]

(2.11)

Definition 2.8: Let \( x, y, x_1, \ldots, x_r \) be distinct elements of \( R \), and let \( \omega = x_1 \wedge x_2 \wedge \cdots \wedge x_r \). We make the following three definitions.

\[
[x, y]_\omega = \omega,
\]

(2.12)

\[
[x, x_i]_\omega = [x, x_i]_\omega = x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_r \quad \text{for} \quad 1 \leq i \leq r,
\]

(2.13)

where \( x \) has replaced \( x_i \) in the \( i \)th position of \( \omega \).

\[
[x, x_j]_\omega = [x, x_j]_\omega = x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_r \quad \text{for} \quad 1 \leq i < j \leq r,
\]

(2.14)

where the positions of \( x_i \) and \( x_j \) have been interchanged in \( \omega \).

Definition 2.9: For \( P = (P_1, \ldots, P_r) \) where the \( P_j \) are pairwise disjoint subsets of \( R \), \( x, y \in R \), and \( 1 \leq t < r \), we make the following two definitions.

\[
[x, y]_P(P) = (-1)^{\text{inv}(P)}([x, y]_P(P_1)) \otimes \cdots \otimes ([x, y]_P(P_r)).
\]

(2.15)

\[
[x, y]_P(P) = (-1)^{\text{inv}(P)}([x, y]_P(P_1)) \cdots \overline{[x, y]_P(P)} \cdots ([x, y]_P(P)) \otimes [x, y]_P(P).
\]

(2.16)

Definition 2.10: Let \( r \geq 1 \), \( r \in \mathbb{Z} \), and let \( T : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \). Define \( \epsilon(I, I'; J, J'; T) \) for \( I, I' \in \mathbb{Z}, J, J' \in \mathbb{Z} \) by Eqs. (2.17), (2.18), and (2.19).

\[
\epsilon(I, I'; J, J'; T) = 1, \quad \text{if} \quad I = I' \quad \text{or} \quad J = J'.
\]

(2.17)

\[
\epsilon(I, I'; J, J'; T) = \sum_{J < J' \leq r} T(I, J) + \sum_{J < r < J'} T(I', J) + \sum_{r < I < I'} T(I, J) + \sum_{r < I < I'} T(I, J'),
\]

(2.18)

if \( I < I' \) and \( J < J' \).

Symbolically,

\[
\epsilon = \begin{cases} \text{I} & \text{J} \\ \text{I'} & \text{J'} \end{cases}.
\]

\[
\epsilon(I, I'; J, J'; T) = \epsilon(I', I; J, J'; T) = \epsilon(I, I'; J', J; T).
\]

(2.19)

[Property (2.19) asserts that \( \epsilon \) depends only on the unordered pairs \((I, I')\) and \((J, J')\).]

Definition 2.11: For \( P = (P_1, P_2, \ldots, P_r) \) where \( P_j \subseteq R \), we define

\[
\epsilon(I, I'; J, J'; P) = \epsilon(I, I'; J, J'; T),
\]

(2.20)

where \( T(i, j) = \#P_j(i) \).

Lemma 2.12: If \( i, j, k \) are distinct and \( J, J' \) are distinct, then
\[ (-1)^{\epsilon(i,j,j';T)}(-1)^{\epsilon(k,j,j';T)} = (-1)^{\epsilon(i,k,j';T)}. \]  

(2.21)

**Proof:** Using Eq. (2.19) reduce to the case \( i < j < k \). Details omitted. Q.E.D.

### III. A SIGN COMPUTATION

**Proposition 3.1:** Let \( P = (P_1, P_2, \ldots, P_r) \) where the \( P_j \) are pairwise disjoint subsets of \( R \), let \( xeP_j \cap R_f \) and let \( y \not\in P_f \cap R_{f'} \) where \( x \not= y \). Let \( 1 \leq t \leq r \). Then

\[ [x,y] \omega(P) = (-1)^{\epsilon(I,J,J';P)} \omega([x,y]P) \]  

(3.1)

and

\[ [x,y] \omega_j(P) = (-1)^{\epsilon(I,J,J';P)} \omega_j([x,y]P). \]  

(3.2)

**Proof:** We present only the proof of Eq. (3.1), the proof of Eq. (3.2) being identical. Since Eq. (3.1) is compatible with the identity \([x,y] \omega(P) = [y,x] \omega(P)\), we may and do assume that \( I < I' \).

Let \( T:Z \times R \to Z \) be defined by the formula \( T(i,j) = \#P_i(j) \).

Let \( x = x_m^{(I)} \) and \( y = x_n^{(I')} \) with \( I < I' \) be as in the statement of Proposition 3.1, and let \( \tilde{P} = [x,y]P \). We must apply \([x,y]\) to

\[ \omega(P) = (-1)^{\text{inv}(P)} \omega(P_1) \otimes \cdots \otimes \omega(P_r). \]

Adopting the notation \( \omega_j = [x,y] \omega(P_j) \) for \( 1 \leq j \leq r \), we have

\[ [x,y] \omega(P) = (-1)^{\text{inv}(P)} \omega_1 \otimes \cdots \otimes \omega_r. \]  

(3.3)

Note first that

\[ \omega_j = \omega(\tilde{P}_j), \quad \text{if} \quad j \neq J \quad \text{and} \quad j \neq J'. \]  

(3.4)

The rest of the proof is a separate analysis of five special cases.

**Case 1:** \( I < I' \) and \( J < J' \). Define four numbers \( a, b, c, d \) as follows: \( a = \) the number of \( x_m^{(I)} \) in \( P_J(I) \) with \( m' > m \); \( b = \) the number of \( x_m^{(I)} \) in \( P_J(I) \) with \( m' > m \); \( c = \) the number of \( x_n^{(I')} \) in \( P_J(I') \) with \( n' < n \); \( d = \) the number of \( x_n^{(I')} \) in \( P_J(I') \) with \( n' < n \).

The element \( \omega_j \) arises from

\[ \omega(P_J) = \omega(P_J(1)) \wedge \cdots \wedge \omega(P_J(I)) \wedge \cdots \wedge \omega(P_J(I')) \wedge \cdots \wedge \omega(P_J(s)) \]

by replacing \( x_m^{(I')} \) in \( \omega(P_J(I)) \) by \( x_n^{(I')} \); and similarly \( \omega_j \) arises from

\[ \omega(P_J) = \omega(P_J(1)) \wedge \cdots \wedge \omega(P_J(I)) \wedge \cdots \wedge \omega(P_J(I')) \wedge \cdots \wedge \omega(P_J(s)) \]

by replacing \( x_n^{(I')} \) in \( \omega(P_J(I')) \) by \( x_m^{(I)} \). If we rewrite \( \omega_j \) by moving the element \( x_n^{(I')} \) over the \( a + \sum_{i < l < r} T(i,j) + c \) variables to its proper place in \( \omega(P_J(I')) \), we obtain

\[ \omega_j = (-1)^{a+c+\sum_{i < l < r} T(i,j)} \omega(\tilde{P}_j), \]  

(3.5)

and similarly

\[ \omega_j = (-1)^{b+d+\sum_{i < l < r} T(i,j')} \omega(\tilde{P}_j). \]  

(3.6)

We next claim that
To prove this, note that $\text{inv}(P) + \text{inv}(\tilde{P})$ is congruent mod 2 to $\#A + \#\tilde{A}$ where $A$ consists of the inversions for $P$ that are not such for $\tilde{P}$ and $\tilde{A}$ consists of the inversions for $\tilde{P}$ that are not such for $P$. The union $A \cup \tilde{A}$ of these two disjoint sets consists of precisely the following six disjoint sets of elements: (i) the $2b$ pairs $(x_m^{(I)}, x_m^{(I')})$ with $x_m^{(I)} \in P_I(I)$ and $m > m'$ (lying in $A$); (ii) the $2a$ pairs $(x_n^{(I)}, x_n^{(I')})$ with $x_n^{(I)} \in P_J(I)$ and $m' > m$ (lying in $A'$); (iii) the $\Sigma_{j < j'} T(I, j)$ pairs $(x_m^{(I)}, x_m^{(I')})$ with $x_m^{(I)} \in P_J(I)$ and $j < j'$ (lying in $A$ or $\tilde{A}$ according as $m > m'$ or $m < m'$); (iv) the $2c$ pairs $(x_n^{(I)}, x_n^{(I')})$ with $x_n^{(I)} \in P_J(I')$ and $n > n'$ (lying in $A'$); (v) the $2d$ pairs $(x_m^{(I)}, x_m^{(I')})$ with $x_m^{(I)} \in P_{J'}(I')$ and $n < n'$ (lying in $A$); (vi) the $\Sigma_{j < j'} T(I', j)$ pairs $(x_n^{(I)}, x_n^{(I')})$ with $x_n^{(I)} \in P_{J'}(I')$ and $J < j < J'$ (lying in $A$ or $\tilde{A}$ according as $n < n'$ or $n > n'$).

It is clear that total number modulo 2 of all the inversions described in (i)–(vi) equals the right hand side of Eq. (3.7).

It follows from Eqs. (3.3)–(3.7) that

$$[x, y]_{\omega}(P) = (-1)^{\epsilon_{I, J'; J, I'; P}} \text{inv}(\tilde{P}) \text{inv}(P) \oplus \cdots \oplus \omega(\tilde{P}) = (-1)^{\epsilon_{I, J'; J, I'; P}} \omega(\tilde{P}),$$

which is Eq. (3.1) for the case $I < I'$ and $J < J'$.

Case 3: $I = I'$ and $J < J'$. Similar to Case 1. We omit the details.

Case 2: $I < I'$ and $J > J'$. Similar to Case 1. We omit the details.

Case 4: $I > I'$ and $J < J'$. Similar to Case 1. We omit the details.

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(x_p^{(I)}x_q^{(I)}) with x_p^{(I)} \in P_j(I), p strictly between m and n, and q = \min\{m,n\} (lying in A or \tilde{A} according as q = n or q = m); (iv) the d pairs (x_p^{(I)}x_q^{(I)}) with x_p^{(I)} \in P_j(I), p strictly between m and n, and q = \max\{m,n\} (lying in A or \tilde{A} according as q = m or q = n); (v) the 1 pair (x_m^{(I)}x_n^{(I)}) (lying in A or \tilde{A} according as m > n or m < n); (vi) the \sum_{j<j'} T(I,j) pairs (x_p^{(I)}x_m^{(I)}) with x_p^{(I)} \in P_j(I) and J < j < J' (lying in A or \tilde{A} according as m > p or m < p); (vii) the \sum_{j<j'} T(I,j) pairs (x_p^{(I)}x_n^{(I)}) with x_p^{(I)} \in P_j(I) and J < j < J' (lying in A or \tilde{A} according as n < p or n > p). It is clear that total number modulo 2 of all the inversions described in (i)–(vii) equals the right hand side of Eq. (3.10).

Now Eq. (3.1) for the case I = I' and J < J' is an easy consequence of Eqs. (3.3), (3.4), and (3.8)–(3.10).

Case 4: I = I' and J > J'. Very similar to Case 3. We omit the details.

Case 5: I < I' and J = J'. Since \omega_J arises from \omega(P_J) by the interchange of two distinct elements (x_m^{(I)} and x_n^{(I)}), \omega_J = -\omega(P_J). Since P = \tilde{P}, we have

\omega_J = -\omega(\tilde{P}_J)

and

\text{inv}(P) = \text{inv}(\tilde{P}).

Now Eq. (3.1) for the case I < I' and J = J' follows easily from Eqs. (3.3), (3.4), (3.11), and (3.12). Q.E.D.

IV. AN APPLICATION OF YOUNG SYMMETRY

Definition 4.1: Let P = (P_1, P_2, ..., P_r) where the P_j are pairwise disjoint subsets of R. Given 1 < J_1 \neq J_2 < r and x \in P_{J_1}, we define S(x; J_1, J_2)P = (P_1, P_2, ..., P_r) as follows:

P'_{J_1} = P_{J_1} \setminus \{x\}, P'_{J_2} = P_{J_2} \cup \{x\}.

P'_j = P_j, \text{ if } j \neq J_1, J_2.

[Informally, S(x; J_1, J_2)P is obtained from P by transferring x from P_{J_1} to P_{J_2}.]

Lemma 4.2: Let P = (P_1, P_2, ..., P_r) where the P_j are pairwise disjoint subsets of R. Let 1 < J \neq J' < r, 1 < I, I' < s, and x \in P_I(I). Then

\begin{equation}
(-1)^{\epsilon(I,I',J,J';S(x;J,J'))P} = \begin{cases} 
-(-1)^{\epsilon(I,I',J,J';P)} & \text{if } I \neq I' \\
-1, & \text{if } I = I'.
\end{cases}
\end{equation}

Proof: Omitted. Q.E.D.

Proposition 4.3: Let P = (P_1, P_2, ..., P_r) where the P_j are pairwise disjoint subsets of R. Suppose that J, J', and t are distinct, 1 < J, J', t < r, and that \#P_J - 1 > \#P_{J'} + 1. Let 1 < k < s be such that P_k \cap R_t \neq \emptyset. Then

\begin{equation}
\sum_{x \in P_k(k)} \omega_{(x; J, J')} P = -\sum_{i=1}^{s} \sum_{x \in P_i(i)} (-1)^{\epsilon(i,k; J,J';P)} \omega_{S(x; J,J') P}.
\end{equation}

Proof: Let x \in P_k(k). Using Young symmetry (1.3) and Proposition 3.1, we compute as follows:
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\[ \omega \{ S(y;J,J') P \} = \sum_{x \in P_{J-J'}} \omega \{ S(y;J,J') P \} \]

\[ = \sum_{i=1}^{s} \sum_{x \in (P_{J-J'}) \cap R_i} [x,y] \omega \{ S(y;J,J') P \} \]

\[ = \sum_{i=1}^{s} \sum_{x \in (P_{J-J'}) \cap R_i} (-1)^{d(k;J,J';s(y;J,J')P)} \omega \{ S(y;J,J') P \}. \]

An application of Lemma 4.2 concludes the proof. Q.E.D.

V. EXCHANGE TABLES

Definition 5.1: An \( \alpha \) table is a map \( T: \mathbb{S} \times \mathbb{S} \to \mathbb{Z} \) such that

\[ \sum_{j=1}^{s} T(i,j) = a_i \sum_{j=1}^{s} T(j,i), \quad \text{for } i=1,2,\ldots,s. \]  

(5.1)

An \( \alpha \)-exchange table is an \( \alpha \)-table \( T \) such that \( T(i,j) > 0 \) for all \( (i,j) \in \mathbb{S} \times \mathbb{S} \).

We denote by \( \mathcal{F}(\alpha) \) the set of all \( \alpha \)-exchange tables.

Definition 5.2: A \( T \) partition of \( R \) for \( T \in \mathcal{F}(\alpha) \) is an \( s \)-tuple \( P = (P_1,\ldots,P_s) \) of subsets \( P_j \) of \( R \) satisfying Eqs. (5.2) and (5.3).

\( R \) is the disjoint union of \( P_1,P_2,\ldots,P_s \).  

(5.2)

\[ P_j(i) = T(i,j), \quad 1 \leq i,j \leq s. \]  

(5.3)

We denote by \( \mathcal{P}(T) \) the set of \( T \) partitions of \( R \). We will make the convention that \( \mathcal{P}(T) = \emptyset \) for all \( \alpha \)-tables \( T \) that take negative values.

Definition 5.3: For \( T \in \mathcal{F}(\alpha) \) and \( 1 \leq i < s \), define \( |T| \) and \( |T|_i \) as follows:

\[ |T| = \sum_{P \in \mathcal{P}(T)} \omega(P). \]  

(5.4)

\[ |T|_i = \sum_{P \in \mathcal{P}(T)} \omega_i(P). \]  

(5.5)

We make the convention that \( |T| = 0 \) and \( |T|_i = 0 \) for all \( \alpha \)-tables \( T \) that take negative values.

Lemma 5.4: Let \( T \in \mathcal{F}(\alpha) \) be an \( \alpha \)-exchange table, and let \( 1 \leq i < s \). Then \( |T| \) and \( |T|_i \) are alternating in the elements of \( R_{k_i} \) for every \( i \in \mathbb{S} \).

Proof: Let \( x, y \in R_{k_i} \), \( x \neq y \). Proposition 3.1 implies that \( [x,y] \omega(P) = -\omega([x,y]P) \) for all \( P \in \mathcal{P}(T) \). Therefore

\[ [x,y] |T| = \sum_{P \in \mathcal{P}(T)} [x,y] \omega(P) = - \sum_{P \in \mathcal{P}(T)} \omega([x,y]P) = |T|, \]

which shows that \( |T| \) is alternating. The proof that \( |T|_i \) is alternating is similar. Q.E.D.

Example: \( \alpha = (3,2,1) \).

\[ R_1 = \{ x_1, x_2, x_3 \}, \quad R_2 = \{ y_1, y_2 \}, \quad R_3 = \{ z \}, \quad R = R_1 \cup R_2 \cup R_3. \]
TABLE I. The twelve $a$-exchange tables $T_a$ for $a=(3,2,1)$.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>0</th>
<th>0</th>
<th>3</th>
<th>0</th>
<th>0</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$=0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_2$=1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_3$=1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_4$=0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_5$=0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$T_6$=0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_7$=0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_8$=0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_9$=0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$T_{10}=2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$T_{11}=1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$T_{12}=2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE II. The six $T$-partitions $P_a$ for $T = 1 1 0$.

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 = P_2$</td>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$z$</td>
</tr>
<tr>
<td>$P_3 = P_4$</td>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$z$</td>
</tr>
<tr>
<td>$P_5 = P_6$</td>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$z$</td>
</tr>
</tbody>
</table>

$\omega(P_1) = x_1 \wedge x_2 \wedge y_1 \oplus x_3 \wedge y_2 \oplus z$

$\omega(P_2) = -x_1 \wedge x_3 \wedge y_1 \oplus x_2 \wedge y_2 \oplus z$

$\omega(P_3) = x_2 \wedge x_3 \wedge y_1 \oplus x_1 \wedge y_2 \oplus z$

$\omega(P_4) = -x_1 \wedge x_2 \wedge y_2 \oplus x_3 \wedge y_1 \oplus z$

$\omega(P_5) = -x_1 \wedge x_3 \wedge y_2 \oplus x_2 \wedge y_1 \oplus z$

$\omega(P_6) = -x_2 \wedge x_3 \wedge y_2 \oplus x_1 \wedge y_1 \oplus z$
\( \mathcal{T}(\alpha) \) contains the 12 elements \( T_p \), \( 1 \leq i \leq 12 \), of Table I. \( \mathcal{P}(T_3) \) contains the 6 elements \( P_p \), \( 1 \leq i \leq 6 \), of Table II.

**VI. THREE IMPORTANT LEMMAS**

**Definition 6.1:** For \( T, I, I', J, J' \in \mathcal{S} \), we define \( S(I, I'; J, J') T \) to be the map \( T' : \mathcal{S} \times \mathcal{S} \to \mathcal{Z} \) constructed as follows:

\[
\text{If } I = I' \text{ or } J = J', \text{ then } T' = T. \quad (6.1)
\]

\[
\text{If } I \neq I' \text{ and } J \neq J', \text{ then }
\]

\[
T'(i, j) = T(i, j) \quad \text{unless } i \in \{I, I'\} \text{ and } j \in \{J, J'\},
\]

\[T'(I, J) = T(I, J) - 1, \quad T'(I, J') = T(I, J') + 1,\]

\[T'(I', J) = T(I', J) + 1, \quad T'(I', J') = T(I', J') - 1. \quad (6.2)\]

Note that if \( T \) is an \( \alpha \) table, then \( S(I, I'; J, J') T \) is also an \( \alpha \) table. And if \( T \) is an \( \alpha \)-exchange table, then \( S(I, I'; J, J') T \) is an \( \alpha \)-exchange table if and only if \( T' \) takes only non-negative values, i.e., if and only if \( I = I' \) or \( J = J' \) or \( [T(I, J) > 0 \text{ and } T(I', J') > 0] \).

Note that

\[S(I', I; J', J) T = S(I, I'; J, J') T. \quad (6.3)\]

\[S(I, I'; J', J) S(I, I'; J, J') T = T. \quad (6.4)\]

**Definition 6.2:** For \( T \in \mathcal{T}(\alpha) \), and \( i, j \in \mathcal{S} \) define the sets \( \mathcal{P}_{ij} \) as follows:

\[\mathcal{P}_{ij}(T) = \{P \in \mathcal{P}(T) : x_{ij} P \neq 0 \}. \quad (6.5)\]

We make the convention that \( \mathcal{P}_{ij}(T) = \emptyset \) for all \( \alpha \)-tables \( T \) that take negative values.

Note that for \( T \in \mathcal{T}(\alpha) \), \( \mathcal{P}_{ij}(T) = \emptyset \) if and only if \( T(i, j) = 0 \).

**Definition 6.3:** We define \( |T|_{ij} \) and \( |T|_{i} \) for \( T \in \mathcal{T}(\alpha) \), \( 1 \leq i \leq s \), and \( (i, j) \in \mathcal{S} \times \mathcal{S} \) as follows:

\[|T|_{ij} = \sum_{P \in \mathcal{P}_{ij}(T)} \omega(P). \quad (6.6)\]

\[|T|_{i} = \sum_{P \in \mathcal{P}_{i}(T)} \omega(P). \quad (6.7)\]

Our conventions imply that \( |T|_{ij} = 0 \) and \( |T|_{i} = 0 \) for all \( \alpha \)-tables \( T \) that take negative values.

Observe that for each \( i \in \mathcal{S} \) the set \( \mathcal{P}(T) \) of \( T \) partitions of \( \mathcal{R} \) is the disjoint union of the \( s \) disjoint subsets \( \mathcal{P}_{ij}(T) \), \( 1 \leq j \leq s \). It is therefore evident that

\[|T| = \sum_{j=1}^{s} |T|_{ij}, \quad 1 \leq i \leq s, \quad (6.8)\]

and

\[|T|_{i} = \sum_{j=1}^{s} |T|_{ij}, \quad 1 \leq i \leq s. \quad (6.9)\]
Definition 6.4: For $T \in \mathcal{T}(\alpha)$, $(u,v) \leq (s-1)$, and $1 \leq l \leq s$, define $F_{w_0}^I(T)$ and $F_{w_0}^{l_i}(T)$.

$$F_{w_0}^I(T) = \sum_{P \in \mathcal{P}_{l+1,\mu}(\alpha)} \left[ x, x_{l+1} \right] \omega(P).$$  

(6.10)

$$F_{w_0}^{l_i}(T) = \sum_{P \in \mathcal{P}_{l+1,\mu}(\alpha)} \left[ x, x_{l+1} \right] \omega(P).$$  

(6.11)

Note that the sums defining $F_{w_0}^I(T)$ and $F_{w_0}^{l_i}(T)$, are nonempty if and only if $T(l,u) > 0$ and $T(l+1,u) > 0$.

Lemma 6.5: For all $T \in \mathcal{T}(\alpha)$ and all $(u,v) \leq (s-1)$, we have the two Eqs. (6.12) and (6.13), where $T' = S(l,l+1;u,v)T$.

$$F_{w_0}^I(T) = (-1)^{\varepsilon(l,l+1;u,v;u,v)} T' (l,v) | T' |^{l+1,u}.$$  

(6.12)

$$F_{w_0}^{l_i}(T) = (-1)^{\varepsilon(l,l+1;u,v;u,v)} T' (l,v) | T' |^{l+1,u}.$$  

(6.13)

Proof: The proofs of Eqs. (6.12) and (6.13) are identical, so we write out only the first. Let $T' = S(l,l+1;u,v)T$ and observe that

$$( -1)^{\varepsilon(l,l+1;u,v;u,v)} = ( -1)^{\varepsilon(l,l+1;u,v;u,v)}. $$

Now compute, using Proposition 3.1

$$F_{w_0}^I(T) = \sum_{P \in \mathcal{P}_{l+1,\mu}(\alpha)} \left[ x, x_{l+1} \right] \omega(P).$$

(6.10)

$$= \sum_{P \in \mathcal{P}_{l+1,\mu}(\alpha)} \left[ x, x_{l+1} \right] \omega([x, x_{l+1}] P).$$

$$= \sum_{P \in \mathcal{P}_{l+1,\mu}(\alpha)} (-1)^{\varepsilon(l,l+1;u,v;[x, x_{l+1}] P)} \omega(P).$$

$$= ( -1)^{\varepsilon(l,l+1;u,v;u,v)} T' (l,v) | T' |^{l+1,u}.$$  

(6.12)

$$= ( -1)^{\varepsilon(l,l+1;u,v;u,v)} T' (l,v) | T' |^{l+1,u}. $$

Q.E.D.

Definition 6.6: We fix once for all an index $t$, $1 \leq t \leq s$. We define the $(s-1)$-tuple $(e_1, \ldots, e_{s-1})$ of integers $e_i$ by the following two conditions:

$$1 \leq e_1 < \cdots < e_{s-1} \leq s.$$  

(6.14)

$$\{e_1, \ldots, e_{s-1}, t\} = \{1, 2, \ldots, s\}.$$  

(6.15)
Thus the $e_i$ label the elements of $s$ that are complementary to $t$.

**Definition 6.7:** A map $T : s \times s \to \mathbb{Z}$ is $t$ normal if and only if $T(i, e_j) = 0$ for all $(i, j)$ such that $i < j$. We will denote by $T(\alpha)$ the set of $t$-normal maps in $T(\alpha)$.

**Definition 6.8:** For $T : s \times s \to \mathbb{Z}$ and $I, J, J' \subseteq s$, we define $S(I, J, J')$ to be the map $T' : s \times s \to \mathbb{Z}$ constructed as follows:

If $J = J'$, then $T' = T$. 

If $J \neq J'$, then

$$T'(i, j) = T(i, j), \quad \text{if } i \notin I \text{ or } j \notin J \cup J'.$$

$$T'(I, J) = T(I, J) - 1, \quad T'(I, J') = T(I, J') + 1.$$  

**Lemma 6.9:** Let $T \in T(\alpha)$, $l < s - 1$, $u \in \{e_1, \ldots, e_l\}$, and $v \in \{e_{l+1}, \ldots, e_s\}$ be such that $T(l, u) > 0$ and $T(l + 1, v) > 0$. Let

$$T' = S(l, l + 1; u, v) T \in T(\alpha).$$

(a) If $v \in \{e_1, \ldots, e_l\}$, then $T'$ is $t$ normal. If $v = e_l + 1$, then $l < s - 2$ and $T'$ is not $t$ normal.

(b) If $(u, v) = (l, e_{l+1})$ or $(u, v) = (e_k, e_{l+1})$ where $1 < k < l$, then

$$|T'|_{l+1,u} = \sum_{i=l+1}^{s} (-1)^{\epsilon(i, l, u; T')} |T'|_{l+1, u} + (-1)^{\epsilon(l, l+1, e; T')} |T'_{l+1}|_{l+1, u},$$

where $T'_i = S(l, i; v, e_i) T'$ is $t$ normal for $i > l + 1$.

(c) If $(u, v) = (e_l, e_{l+1})$, then

$$|T'|_{l+1,u} = \sum_{i=l+1}^{s} (-1)^{\epsilon(i, l, u; T')} |T'|_{l+1, u} + (-1)^{\epsilon(l, l+1, e; T')} |T'_{l+1}|_{l+1, u},$$

where $T'_i = S(l; v, e_i) T'$ is $t$ normal for $i > l + 1$.

**Proof:** (a) Note first that $e_{l+1}$ is only defined if $l < s - 2$. If $v = e_{l+1}$, then $T'(l, e_{l+1}) = 1 > 0$, showing that $T'$ is not $t$ normal. Conversely, if $v \in \{e_1, \ldots, e_l\}$, then none of the points $(l, u)$, $(l, v)$, $(l + 1, u)$, $(l + 1, v)$ is of the form $(i, e_j)$ with $i < j$. Therefore $T'(i, e_j) = T(i, e_j) = 0$ for all $i < j$, which proves that $T'$ is $t$ normal.

(b) The map $T'$ fails to be $t$ normal only because $T'(l, e_{l+1}) = 1 > 0$. So it is clear that $T'_i$ is $t$ normal for $i > l + 1$.

Let

$$T'' = S(l; u, e_l) T',$$

so that

$$T'_i = S(l; v, e_i) T'', \quad \text{for } i > l + 1.$$  

Observe, cf. Lemma 4.2, that

$$(-1)^{\epsilon(i, l, e; T'')} = -(-1)^{\epsilon(l, l+1, e; T'_i)}, \quad \text{for } i > l + 1.$$  

Now compute using Proposition 4.3.
This completes the proof of (b).

(c) The proof of (c) is very similar to that of (b). Define $T''$ as in the proof of (b) and initiate the same computation

$$|T''|_{l+1,u} = \sum_{i=l+1}^{s} \sum_{x \in P_{i}(l)} -|\epsilon(|l,e_{P}^{u}|)\omega(S(x;e_{P}v))P|.$$ 

We must now treat the terms $i=l+1$ and $i>l+1$ differently.

If $i>l+1$, we have, exactly as in the proof of (b),

$$\sum_{x \in P_{i+1}(l+1)} -\omega(S(x;e_{P}v))P = T''(i,v) |T''|_{l+1,u}. \tag{1}$$

But if $i=l+1$, we get

$$\sum_{x \in P_{l+1}(l+1)} -\omega(S(x;e_{P}v))P = \sum_{x \in P_{l+1}(l+1) - \{x_{l+1}^{l+1}\}} -\omega(S(x_{l+1}^{l+1};e_{P}v))P + \omega(S(x_{l+1};e_{P}v))P + \omega(S(x_{l+1}^{l+1};e_{P}v))P$$

$$= \sum_{x \in P_{l+1}(l+1)} -\omega(P) + \sum_{x \in P_{l+1}(l+1)} -\omega(x) = T''_{l+1}(l+1,v) |T''_{l+1}|_{l+1,u} + |T''_{l+1}|_{l+1,u}. \tag{2}$$

Q.E.D.

**Lemma 6.10:** Let $T \in \mathcal{F}(\alpha)$. Let $1<l<s-2$ and suppose that $T(l,e_{l})=0$. Then
where $T_i = S(l+1,i;e_{l+1}; e)T$ is $t$ normal for $i > l+1$.

**Proof:** Calculate as follows, using Young symmetry (1.3) and Proposition 3.1.

\[
|T|^l+1,q+1 = \sum_{P \in \mathcal{P}_{l+1,q+1}(T)} \omega_i(P)
\]

\[
= \sum_{P \in \mathcal{P}_{l+1,q+1}(T)} \sum_{x \in P_{q+1}} [x, x_i^{(l+1)}] \omega_i(P)
\]

\[
= \sum_{P \in \mathcal{P}_{l+1,q+1}(T)} \sum_{x \in P_{q+1}(i)} [x, x_i^{(l+1)}] \omega_i(P)
\]

\[
= \sum_{i = l+1}^s \sum_{P \in \mathcal{P}_{l+1,q+1}(T)} (-1)^{\varepsilon(l+1,i;e_{l+1}; T)} \omega_i([x, x_i^{(l+1)}] P)
\]

\[
= \sum_{i = l+1}^s (-1)^{\varepsilon(l+1,i;e_{l+1}; T)} \sum_{P \in \mathcal{P}_{l+1,q+1}(T_i)} \omega_i(P)
\]

\[
= \sum_{i = l+1}^s (-1)^{\varepsilon(l+1,i;e_{l+1}; T)} T_i(i, e_{l+1}) |T|^l+1,q+1_i.
\]

The proof of Lemma 6.10 is now completed with the simple remark that for $l+1 < i < s$,

\[
(-1)^{\varepsilon(l+1,i;e_{l+1}; T)} = (-1)^{\varepsilon(l+1,i;e_{l+1}; T_i)}.
\]

Q.E.D.

**VII. THE NATURAL TRANSFORMATION $\Omega$**

Let

\[
\Omega = \sum_{T \in \mathcal{F}(a)} A_T |T|,
\]

where our goal is to determine scalars $A_T$ such that the assignment

\[
\omega(R_1) \cdot \omega(R_2) \cdots \omega(R_s) \mapsto \Omega
\]

(7.2)

gives a well-defined natural transformation

\[
\Omega : \Lambda^a \to \Lambda^{a_1} \otimes \Lambda^{a_2} \otimes \cdots \otimes \Lambda^{a_s}.
\]
This means, in the formalism described in Sec. I, that substitution of vectors from the vector space $E$ for the symbols $x_j^{(l)}$ implicit in Eq. (7.2) gives a well-defined $GL(E)$-equivariant transformation from the irreducible representation $\Lambda^aE$ to the tensor product $\Lambda^aE \otimes \Lambda^bE \otimes \cdots \otimes \Lambda^sE$ of irreducibles.

In view of Lemma 5.4, in order for Eq. (7.2) to be well-defined it is necessary and sufficient that $\Omega$ respect the Young symmetry relations (1.3), which can be written as

$$\Omega = \sum_{x \in R_l} [x, x_1^{(l+1)}] \Omega, \quad \text{for} \quad 1 < l < s - 1. \quad (7.3)$$

By convention we set $A_T = 0$ for all $\alpha$-tables $T$ that take negative values.

Lemma 7.1: For $1 < l < s - 1,$

$$\sum_{x \in R_l} [x, x_1^{(l+1)}] \Omega = \sum_{T \in \mathcal{T}(\alpha)} \sum_{(u, v) \in \mathcal{S}_1 \times \mathcal{S}_2} A_T F_{uv}(T) = \sum_{T \in \mathcal{T}(\alpha)} \sum_{u \in \mathcal{S}_1} B_u(T) |T|^{l+1, u}, \quad (7.4)$$

where

$$B_u(T) = \sum_{v \in \mathcal{S}_2} (-1)^{\varepsilon(l+1; u, v; T)} T(l, v) A_{S(l+1; u, v; T)}. \quad (7.5)$$

Proof: It results easily by using Eq. (6.12) to rewrite $F_{uv}(T),$ and then collecting terms that are multiples of $|T|^{l+1, u}$ for each $T$ and $u.$

Theorem 7.2: Let $1 < l < s - 1.$ Then Eqs. (7.6) and (7.7) are equivalent.

$$\Omega = \sum_{x \in R_l} [x, x_1^{(l+1)}] \Omega. \quad (7.6)$$

$$A_T = B_u(T) \quad (7.7)$$

for all $T \in \mathcal{T}(\alpha)$ and $u \in \mathcal{S}_1$ such that $T(l+1, u) > 0.$

Proof: Because the variables in $R$ are linearly independent, the set $\{\omega(P) | P \in \mathcal{P}(T), T \in \mathcal{T}(\alpha)\}$ is linearly independent. It follows that for $T \in \mathcal{T}(\alpha)$ and $u \in \mathcal{S}_1,$ $|T|^{l+1, u} = 0$ if and only if $T(l+1, u) = 0.$ Moreover, the nonzero elements among the $|T|^{l+1, u}, T \in \mathcal{T}(\alpha),$ and $u \in \mathcal{S}_1,$ are linearly independent. Theorem 7.2 therefore results from equating coefficients of $|T|^{l+1, u}$ on the two sides of equation $\Omega = \sum_{x \in R_l} [x, x_1^{(l+1)}] \Omega,$ the left side of which is, by Eqs. (7.1) and (6.8),

$$\Omega = \sum_{T \in \mathcal{T}(\alpha)} \sum_{u \in \mathcal{S}_1} A_T |T|^{l+1, u}$$

and the right side of which is given by Eq. (7.4).

Corollary: The two assertions (7.8) and (7.9) are equivalent for a collection of scalars $A_T,$ $T \in \mathcal{T}(\alpha).$

$$\Omega = \sum_{x \in R_l} [x, x_1^{(l+1)}] \Omega, \quad \text{for} \quad 1 < l < s - 1. \quad (7.8)$$

$$(1 + T(l, u)) A_T = \sum_{\nu = 1}^{s} (-1)^{\varepsilon(l+1; u, v; T)} T(l, v) A_{S(l+1; u, v; T)} \quad (7.9)$$

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for all triples \((T,l,u)\) where \(T \in \mathcal{T}(\alpha)\), \(1 < l < s - 1\), \(1 < u < s\), and such that \(T(l+1,u) > 0\).

**Proof:** This is an explicit version of Theorem 7.2, making use of the equality \((-1)^{\epsilon(l+1;u;k,T)} = -1\).

Q.E.D.

The Eqs. (7.7) are by no means independent. For \(T \in \mathcal{T}(\alpha)\), \(1 < l < s - 1\), and \(u \in \mathbb{S}\) such that \(T(l+1,u) > 0\), let

\[
\Phi_{l,u,T} = A_T - B'_u(T).
\]

Then it is easy to verify that if \(w \neq u\) and \(T(l,w) > 0\), then

\[
\Phi_{l,w,T'} = \frac{1}{\epsilon(l+1;w;u)} \Phi_{l,u,T'}
\]

where \(T' = S(l,l+1;w,u) T \in \mathcal{T}(\alpha)\) and \(T'(l+1,w) > 0\).

**Examples:** A solution of Eq. (7.9) for \(\alpha = (m,n)\), \(m > n > 1\), is given by

\[
A_{m-r,n-r} = (m-r)! \text{ for } 0 < r < n.
\]

A solution of Eq. (7.9) for \(\alpha = (3,2,1)\) is given by Table III.

We have not found a simple explicit solution to the recurrence relations (7.9) for \(A_T\) for general \(\alpha\).

**VIII. THE NATURAL TRANSFORMATION \(\Omega_t\)**

Let \(1 < t < s\), and let

\[
\Omega_t = \sum_{T \in \mathcal{T}(\alpha)} A'_T |T|_t,
\]

where our goal is to determine scalars \(A'_T\) such that the assignment

\[
\omega(R_1) \cdot \omega(R_2) \cdots \omega(R_t) \mapsto \Omega_t
\]

gives a well-defined natural transformation

\[
\Omega_t : \Lambda^\alpha \rightarrow \Lambda^{(a_1, \ldots, a_t, \ldots, a_s)} \otimes \Lambda^a.
\]

[Note that the sum defining \(\Omega_t\) runs over only the \(t\)-normal elements of \(\mathcal{T}(\alpha)\).]

In view of Lemma 5.4, in order for Eq. (8.2) to be well-defined it is necessary and sufficient that \(\Omega_t\) respect the Young symmetry relations (1.3), which can be written as

\[
\Omega_t = \sum_{x \in \mathbb{R}_t} [x,x]^{(l+1)} \Omega_p \text{ for } 1 < l < s - 1.
\]

By convention we set \(A'_T = 0\) for all \(\alpha\)-tables \(T\) that take negative values. We will never encounter a symbol \(A'_T\) where \(T \notin \mathcal{T}(\alpha)\) but \(T \in \mathcal{T}(\alpha)_t\).

**Lemma 8.1:** Let \(t \in \mathbb{S}\) and \(1 < l < s - 1\). Then

**TABLE III.** A solution of Eq. (7.9), for \(\alpha = (3,2,1)\). The exchange tables \(T_{\in \mathcal{T}(\alpha)}\) are as given in Table I.

<table>
<thead>
<tr>
<th>(l)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A'_T)</td>
<td>1</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{6})</td>
<td>(-\frac{1}{6})</td>
<td>(-\frac{1}{6})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{12})</td>
<td>0</td>
</tr>
</tbody>
</table>

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where $\xi^l_u(T)$ is defined for $T \in \mathcal{F}(\alpha)$, and $u \in \{e_1, \ldots, e_p\}$ by Eq. (8.5).

$$\xi^l_u(T) = \sum_{\alpha \in \{e_1, \ldots, e_p\}} M^l_u(T, \alpha) + \sum_{l=1+1}^{s} N^l_u(T, i),$$

with

$$M^l_u(T, \alpha) = (-1)^{\alpha(l+1)} T(\alpha, \alpha, \alpha, e_{l+1}) A^l_{S(l, l+1; \alpha), \alpha}$$

and

$$N^l_u(T, i) = \left\{ \begin{array}{ll}
(-1)^{l+1} T(i, e_{l+1}) A^l_{S(l, l+1; e_{l+1}), T} & \text{if } l < s - 2 \\
0 & \text{if } l = s - 1.
\end{array} \right.$$

Proof: We will express $\Omega_t$ and $\sum_{x \in R_1} [x, x^i_{l+1}] \Omega_t$ separately as linear combinations of $|T|^{l+1, u}_t$.

We begin with $\Omega_t$. We have

$$\Omega_t = \sum_{T \in \mathcal{F}(\alpha)} \sum_{\alpha \in \{e_1, \ldots, e_p\}} A^l_T |T|^{l+1, u}_t + B^l + C^l,$$

where

$$B^l = C^l = 0, \text{ if } l = s - 1.$$ 

$$B^l = \sum_{T \in \mathcal{F}(\alpha), T(\alpha, \alpha) > 0} A^l_T |T|^{l+1, e_{l+1}}_t, \text{ if } 1 < l < s - 2.$$ 

$$C^l = \sum_{T \in \mathcal{F}(\alpha), T(\alpha, \alpha) = 0} A^l_T |T|^{l+1, e_{l+1}^{(i)}}_t, \text{ if } 1 < l < s - 2.$$ 

We claim that

$$C^l = \sum_{T \in \mathcal{F}(\alpha)} \sum_{\alpha \in \{e_1, \ldots, e_p\}} \sum_{i=l+1}^{s} -N^l_u(T, i) |T|^{l+1, u}_t, \text{ if } 1 < l < s - 2.$$ 

Indeed, re-expressing $|T|^{l+1, e_{l+1}^{(i)}}_t$ in Eq. (8.11) via Eq. (6.20) and then grouping terms with a common factor of $|T|^{l+1, u}_t$ yields
\[ C' = \sum_{T \in \mathbb{F}(\alpha), \quad T(\ell, \ell) = 0} \sum_{i = \ell + 1}^{s} \left\{ (-1)^{e(l+1, k \ell, \ell+1; T)} T(i, e_{l+1}) \right\} T_{t+1}^{l+1, e_{l}} \]

where \( T_{i} = S(l+1, i; e_{l+1}, e_{l})T \)

\[ = \sum_{T \in \mathbb{F}(\alpha), \quad T(\ell, \ell) = 0} \sum_{i = \ell + 1}^{s} \left\{ (-1)^{e(l+1, k \ell, \ell+1; T)} T(i, e_{l+1}) A_{T}^{l}(l+1, k \ell, \ell+1) T \right\} T_{t+1}^{l+1, e_{l}}. \] (8.13)

Next, verify the simple Eqs. (8.14) and (8.15).

\[ S(l+1, i; e_{l+1}, e_{l}) T = S(l, i+1; e_{l+1}, u) S(l, e_{l} e_{l+1}) T, \quad \text{if} \quad u = e_{l} \quad \text{and} \quad i > l + 1. \] (8.14)

\[ (-1)^{e(l+1, k \ell, \ell+1; T)} = (-1)^{e(l+1, u e_{l+1}; T)} S(l, e_{l} e_{l+1}) T (-1)^{e(l, k \ell, \ell+1; T)}, \quad \text{if} \quad u = e_{l} \quad \text{and} \quad i > l + 1. \] (8.15)

Substitution of Eqs. (8.14) and (8.15) into Eq. (8.13) gives

\[ C' = \sum_{T \in \mathbb{F}(\alpha), \quad T(\ell, \ell) = 0} \sum_{i = \ell + 1}^{s} -N_{i}^{l}(T, i) T_{t+1}^{l+1, e_{l}}. \]

Formula (8.12) follows upon observing that \( N_{i}^{l}(T, i) = 0 \) if \( T(l, e_{l}) = 0, \quad u \in \{ e_{1}, \ldots, e_{l-1}, i \} \), and \( l+1 < i < s \), because then \( S(l, l+1; e_{l+1}, u) S(l, i; e_{l+1}, e_{l}) T \) takes a negative value at \( (l, e_{l}) \).

Next we use Lemmas 6.5 and 6.9 to rewrite the expression

\[ \sum_{x, x_{l+1}} [x, x_{l+1}] \Omega_{i} = \sum_{T \in \mathbb{F}(\alpha), \quad T(\ell, \ell) = 0} \sum_{u \in \{ e_{1}, \ldots, e_{l+1}, i \}} A_{T}^{l}(T, u) \] (8.16)

(where \( v = e_{l+1} \) is permitted only if \( l < s - 2 \)).

More precisely, we break up Eq. (8.16) into two pieces (8.17) and (8.18) that are analyzed separately. An application of Eq. (6.13) followed by a straightforward collecting of terms yields

\[ \sum_{T \in \mathbb{F}(\alpha), \quad T(\ell, \ell) = 0} \sum_{u \in \{ e_{1}, \ldots, e_{l+1}, i \}} A_{T}^{l}(T, u) = \sum_{T \in \mathbb{F}(\alpha), \quad T(\ell, \ell) = 0} \sum_{u \in \{ e_{1}, \ldots, e_{l+1}, i \}} M_{u}^{l}(T, u) T_{t+1}^{l+1, u}. \] (8.17)

And if \( 1 < l < s - 2 \), then an application of Eqs. (6.13), (6.18), and (6.19) followed by a collecting of terms yields
\[ \sum_{T \in \mathcal{T}(\alpha)} \sum_{u \in \{e_1, \ldots, e_{\ell}\}} A^i_T F_{uv}(T)_t \]

\[ = \begin{cases} 
\sum_{T \in \mathcal{T}(\alpha)} \left( A^i_T | T|^{i+1, e_{l+1}} + \sum_{i=l+1}^{s} N^i_T(T, 1| T|^{i+1, u}) \right), & \text{if } 1 < l < s - 2 \text{ and } T(l, e_l) > 0 \\
0, & \text{if } l = s - 1 \text{ or } T(l, e_l) = 0.
\end{cases} \]

(8.18)

Now Lemma 8.1 follows immediately by combining Eqs. (8.8), (8.9), (8.10), (8.12), (8.16), (8.17), and (8.18). Q.E.D.

Theorem 8.2: Let \( \varepsilon \in \mathbb{Z} \) and let \( 1 < l < s - 1 \). Then Eq. (8.20) implies Eq. (8.19).

\[ \Omega_T = \sum_{x \in R_j} [x, x^{(l+1)}] \Omega_x \]  

(8.19)

\[ A^i_T = \Omega_x, \quad \text{for all } T \in \mathcal{T}(\alpha), \text{and } u \in \{i, e_1, \ldots, e_{\ell}\} \]  

(8.20)

such that \( T(l+1, u) > 0 \).

**Proof:** Immediate from Lemma 8.1. Q.E.D.

Note that the elements \( | T|^{i+1, u} \) such that \( T(l+1, u) > 0 \) occurring in Eq. (8.4) may not be linearly independent, which is why we have not reversed the implication Eq. (8.20) \( \Rightarrow \) Eq. (8.19).

**IX. THE NATURAL TRANSFORMATION \( \Omega_1 \)**

Throughout Sec. IX we set \( \ell = 1 \). Therefore

\[ e_j = j + 1, \quad 1 < j < s - 1. \]  

(9.1)

**Definition 9.1:** A map \( T : \mathcal{S} \times \mathcal{S} \to \mathbb{Z} \) is 1-special if and only if \( T(i, e_j) = 0 \) for all \((i, j)\) such that \( i \neq j \) and \( i \neq j + 1 \).

If \( T \) is 1-special, then \( T \) is of the form

\[ T = \begin{array}{ccc}
\ast & \ast & 0 & 0 & 0 \\
\ast & \ast & \ast & 0 & \cdots \\
\ast & 0 & \ast & \ast & 0 \\
& \cdots & \cdots & \cdots & \cdots \\
\ast & 0 & 0 & 0 & \ast & \ast \\
\ast & 0 & 0 & 0 & 0 & \ast 
\end{array} \]

**Theorem 9.2:** A well-defined natural transformation

\[ \Omega_1 : \Lambda^\alpha \to \Lambda^{(a_2, \ldots, a_s)} \otimes \Lambda^{a_1} \]

can be defined via Eqs. (8.1) and (8.2) by choosing \( A^i_T \) as in Eq. (9.2).

\[ A^i_T = \begin{cases} 
(-1)^{j-1} \sum_{j=1}^{i} T(i, j) T(i+1, j) \prod_{i=1}^{s} T(i, 1), & \text{if } T \in \mathcal{T}(\alpha)_1 \text{ is 1-special} \\
0, & \text{if } T \in \mathcal{T}(\alpha)_1 \text{ is not 1-special}
\end{cases} \]  

(9.2)
By Theorem 8.2 it will suffice to prove that formula (9.2) gives a solution to the Eqs. (8.20) for \( t=1, 1<i<s-1 \). That will be proved by a direct substitution of the formula (9.2) for \( A_T^I \) into the Eqs. (8.20). The computation is facilitated by two lemmas. For the rest of Sec. IX we assign \( A_T^I \) the value given by Eq. (9.2).

**Lemma 9.3:** Suppose that \( t=1 \) and \( T \in \mathcal{F}(\alpha)_1 \), and let \( 1<i<s-1 \). If \( u, \nu \in \{ t, e_1, \ldots, e_l \} \), \( T(l+1, u) > 0 \), and \( N_u^I(T, \nu)_1 \neq 0 \), then either

\[
(u, \nu) \in \{(t, e_l), (t, t), (e_l, t), (e_l, e_l)\}
\]

and \( T \) is \( 1 \)-special or

\[
l>2, \quad (u, \nu) \in \{(e_{l-1}, t), (e_{l-1}, e_l)\},
\]

and \( T \) fails to be \( 1 \)-special only through the single equation \( T(l+1, e_{l-1}) = 1 > 0 \).

**Proof:**

Case 1: Suppose that \( u=v \). Then

\[
N_u^I(T, \nu)_1 = -T(l, u)A_T^I,
\]

which equals 0 if \( T \) is not \( 1 \)-special. Since \( T \) is therefore \( 1 \)-special, the condition \( T(l+1, u) > 0 \) implies that \( u \in \{ t, e_1, \ldots, e_l \} \), and by hypothesis \( u \neq e_{l+1} \). Therefore \( u = v = t \) or \( u = v = e_l \).

Case 2: Suppose that \( u \neq v \). Then

\[
N_u^I(T, \nu)_1 = T(l, u)S(l, l+1; v, u)T \neq 0
\]

Therefore the hypothesis \( N_u^I(T, \nu)_1 \neq 0 \) implies that \( T(l, u) > 0 \) and \( S(l, l+1; v, u)T \) is \( 1 \)-special.

The computation

\[
S(l, l+1; v, u)T(l, u) = T(l, u) + 1 > 0,
\]

shows that \( S(l, l+1; v, u)T \) fails to be \( 1 \)-special if \( u \in \{ t, e_1, \ldots, e_l \} \). Similarly,

\[
S(l, l+1; v, u)T(l+1, u) = T(l+1, u) + 1 > 0,
\]

shows that \( S(l, l+1; v, u)T \) fails to be \( 1 \)-special if \( u \in \{ t, e_1, e_{l+1} \} \). By hypothesis \( u \neq e_{l+1} \), so we have proved that

\[
(u, \nu) \in \{(t, e_l), (t, t), (e_l, t), (e_l, e_l)\}.
\]

Suppose that \( (u, \nu) \in \{(t, e_l), (e_l, t)\} \). Then \( S(l, l+1; v, u)T(i, e_l) = T(i, e_l) \) for any \( i \neq l \) or \( i=1 \) and \( j \neq l \). Since \( S(l, l+1; v, u)T \) is \( 1 \)-special, it follows that \( T \) is \( 1 \)-special.

Now suppose that \( i>2 \) (so that \( e_{i-1} \) is defined) and that \( (u, \nu) \in \{(e_{i-1}, t), (e_{i-1}, e_l)\} \). Because \( S(l, l+1; v, u)T \) is \( 1 \)-special, \( 0 = S(l, l+1; v, u)T(i, e_l) = T(i, e_l) \) if \( i \neq j \) and \( i \neq j+1 \) and \( (i, j) \neq (l+1, l-1) \), which shows that \( T \) satisfies the conditions for \( 1 \)-speciality except for the vanishing of \( T(l+1, e_{i-1}) \). Finally, because \( S(l, l+1; v, u)T \) is \( 1 \)-special, \( 0 = S(l, l+1; v, u)T(l+1, e_{l-1}) = T(l+1, e_{l-1}) - 1 \), which shows that \( T(l+1, e_{l-1}) = 1 \). Q.E.D.

**Lemma 9.4:** Let \( t=1 \) and \( T \in \mathcal{F}(\alpha)_1 \), and let \( 1<i<s-2 \). Suppose that \( u \in \{ t, e_1, \ldots, e_l \} \), \( l+1<i<s \), \( T(l+1, u) > 0 \), and \( N_u^I(T, i)_1 \neq 0 \). Then \( i=l+1 \) and either \( u \in \{ t, e_l \} \) and \( T \) is \( 1 \)-special or \( i>2 \), \( u = e_{l-1} \), and \( T \) fails to be \( 1 \)-special only through the single equation \( T(l+1, e_{l-1}) = 1 > 0 \).

**Proof:** In all cases \( N_u^I(T, i)_1 \) is a multiple (possibly 0) of

\[
T(i, e_{l+1})A_T^I_{S(i, l+1; e_{l+1}, u)}S(l, l+1; e_{l+1})T.
\]
Therefore the hypothesis \( N_u(T,i) \neq 0 \) implies that

\[
S(l,l+1;e_{l+1},u)S(l,i;e,\epsilon_{l+1})T \quad \text{is 1-special.} \tag{9.3}
\]

If \( i > l + 1 \), then

\[
S(l,l+1;e_{l+1},u)S(l,i;e,\epsilon_{l+1})T(i,e_i) = T(i,e_i) + 1 > 0,
\]

which contradicts Eq. (9.3). Hence \( i = l + 1 \). And if \( u \in \{ e_1, \ldots, e_{l-2} \} \), then

\[
S(l,l+1;e_{l+1},u)S(l,l+1;e,\epsilon_{l+1})T(l,u) = T(l,u) + 1 > 0,
\]

which contradicts Eq. (9.3) with \( i = l + 1 \). We have shown that

\[
i = l + 1 \quad \text{and} \quad u \in \{ t, e_{l-1}, e_l \}.
\]

If \( u = e_l \), then \( S(l,l+1;e_{l+1},u)S(l,i;e,\epsilon_{l+1})T = T \), so Eq. (9.3) implies that \( T \) is 1-special. If \( u = t \), then \( S(l,l+1;e_{l+1},u)S(l,i;e,\epsilon_{l+1})T(k,e_j) = T(k,e_j) \) if \( (k,j) \neq (l,l) \) and \( (k,j) \neq (l+1,l) \), so once again Eq. (9.3) implies that \( T \) is 1-special.

Now suppose that \( l \geq 2 \) (so that \( e_{l-1} \) is defined) and that \( u = e_{l-1} \). By Eq. (9.3),

\[
0 = S(l,l+1;e_{l+1},u)S(l,i;e,\epsilon_{l+1})T(k,e_j) = T(k,e_j)
\]

if \( k \neq j \) and \( k \neq j+1 \) and \( (k,j) \neq (l+1,l-1) \), which shows that \( T \) satisfies the conditions for 1-specialty except for the vanishing of \( T(l+1,e_{l-1}) \). Finally, by Eq. (9.3),

\[
0 = S(l,l+1;e_{l+1},u)S(l,i;e,\epsilon_{l+1})T(l+1,e_{l-1}) - T(l+1,e_{l-1}) - 1,
\]

which shows that \( T(l+1,e_{l-1}) = 1 \). Q.E.D.

**Proof of Theorem 9.2:** We prove that \( A_T \), as in Eq. (9.2), satisfies the Eqs. (8.20) for \( i = 1, 1 < l < s - 1 \). By Lemmas 9.3 and 9.4 the only nontrivial Eqs. (8.20) to be checked are Eqs. (9.4) and (9.5)

\[
A_T = \xi_i(T), \tag{9.4}
\]

where \( 1 < l < s - 1, u \in \{ t, e_l \} \), \( T \in \mathcal{T} \) is 1-special, and \( T(l+1,u) > 0 \).

\[
A_T = \xi_{e_{l-1}}(T), \tag{9.5}
\]

where \( 2 < l < s - 1 \) and \( T \in \mathcal{T} \) fails to be 1-special only through the single equation \( T(l+1,e_{l-1}) = 1 \).

We will make these equations more explicit, then leave the final verifications to the reader. **Equation (9.4), \( u = t, 1 < l < s - 1 \), \( T \) is 1-special, and \( T(l+1,t) > 0 \).**

We have to verify

\[
A_T = M_l(T,t) + M_l(T,e_l) + N_l(T,l+1).
\]

**Equation (9.4), \( u = e_l, 1 < l < s - 1 \), \( T \) is 1-special and \( T(l+1,e_l) > 0 \).**

We have to verify

\[
A_T = M_l(T,e_l) + M_l(T,t) + N_l(T,l+1).
\]

**Equation (9.5): 2 < l < s - 1, and \( T \) is assumed 1-special except for the single equation \( T(l+1,e_{l-1}) = 1 \).**
We have to verify
\[ A^I_T = M^I_{i-1}(T, t)_1 + M^I_{i-1}(T, e_i)_1 + N^I_{i-1}(T, l + 1)_1. \]

Note that since \( T \) is not 1-special, \( A^I_T = 0 \). Q.E.D.

X. THE NATURAL TRANSFORMATION \( \Omega_s \)

Throughout Sec. X we set \( t = s \). Therefore
\[ e_j = j, \quad 1 < j < s - 1. \tag{10.1} \]

**Definition 10.1:** For an \( s \)-normal map \( T \colon \mathbb{X}_s \to \mathbb{Z} \) define \( \sigma(T) \) and \( D^s_T \) by the formulas (10.2) and (10.3).

\[ \sigma(T) = \sum_{1 < j < i < s} T(i, j) \left( T(i, i) + \sum_{i < k < s} T(k, j) \right). \tag{10.2} \]

\[ D^s_T = \left\{ \prod_{i=1}^{s-1} (T(i, i) + s - 1 - j)! \left( \sum_{k=i+1}^{s} T(k, i) \right)! \right\}, \text{ if } T \text{ assumes no negative values} \]
\[ 0, \text{ otherwise.} \tag{10.3} \]

**Theorem 10.2:** A well-defined natural transformation
\[ \Omega_s : \Lambda^\alpha \to \Lambda^{(a_1, \ldots, a_{s-1})} \otimes \Lambda^{a_s} \]

can be defined via Eqs. (8.1) and (8.2) by choosing \( A^I_T \) as in Eq. (10.4).
\[ A^I_T = (-1)^{\sigma(T)} D^s_T, \quad \text{for } T \in \mathcal{T}(\alpha)_s. \tag{10.4} \]

By Theorem 8.2 it will suffice to prove that formula (10.4) gives a solution to the Eqs. (8.20) for \( t = s, 1 < l < s - 1 \). Before giving the proof of that, which is a direct substitution of the formula (10.4) for \( A^I_T \) into the Eqs. (8.20), we address some questions of sign.

**Lemma 10.3:** Let \( T : \mathbb{X}_s \to \mathbb{Z} \) be \( s \)-normal. Let \( l \) satisfy \( 1 < l < s - 1 \), and let \( u, v \in \{e_1, \ldots, e_s\} \). Then
\[ (-1)^{e(l+1; u, v)_T} (-1)^{\sigma(s(l+1; u, v)_T)} = \begin{cases} +(-1)^{\sigma(T)}, & u \neq v \text{ and } e \notin \{u, v\} \\ -(-1)^{\sigma(T)}, & u = v \text{ or } e \in \{u, v\}. \end{cases} \tag{10.5} \]

**Proof:** The proof is a tedious but nonetheless straightforward analysis of the five separate cases which we list below [with the signs asserted in Eq. (10.5)].

- **Case 1:** \( u = v \); sign is \(-1\).
- **Case 2:** \( u \neq v, u, v \in \{e_1, ..., e_{l-1}\} \); sign is \(-1\).
- **Case 3:** \( u \neq v, \exists \{e_u, v\}, e \in \{u, v\} \); sign is \(+1\).
- **Case 4:** \( u \neq v, \subseteq \{e_u, v\}, e \notin \{u, v\} \); sign is \(-1\).
- **Case 5:** \( \{u, v\} = \{e_s\} \); sign is \(+1\). Q.E.D.
\(-1\) \varepsilon(l+1, u, e_{l+1}, S(l, e, e_{l+1}, T)) \varepsilon(l, e, e_{l+1}, T, T) \varepsilon(S(l, l+1, u, e, e_{l+1}, T))

\[ \begin{cases} +(-1)^\sigma(T), & \text{if } [i=l+1 \text{ and } u=e_i] \text{ or } [i>l+1 \text{ and } u\neq e_i] \\ -(-1)^\sigma(T), & \text{if } [i>l+1 \text{ and } u\neq e_i] \text{ or } [i=l+1 \text{ and } u\neq e_i]. \end{cases} \] (10.6)

**Proof:** The proof is a tedious but nonetheless straightforward analysis of the six separate cases which we list below [with the signs asserted in Eq. (10.6)].

1. **Case 1:** \(u \in \{e_1, \ldots, e_{l-1}\}, i=l+1; \text{ sign is } -1\).
2. **Case 2:** \(u \in \{e_1, \ldots, e_{l-1}\}, i>l+1; \text{ sign is } +1\).
3. **Case 3:** \(u = e_i, i=Z+1; \text{ sign is } +1\).
4. **Case 4:** \(u = e_i, i>l+1; \text{ sign is } -1\).
5. **Case 5:** \(u = s, i=Z+1; \text{ sign is } -1\).
6. **Case 6:** \(u = s, i>l+1; \text{ sign is } +1\). Q.E.D.

**Lemma 10.5:** Let \(T(a)\) and let \(l<s-1\). If \(T(l,l)=0\), then \(T(l+1,u)=0\) for \(u \in \{e_1, \ldots, e_{l+1}, s\} = \{1, \ldots, l+1, s\}\).

**Proof:** Definition 6.7 and the hypothesis \(T(l,l)=0\) imply that \(T(i,j)=0\) for \((i,j)\) such that \(1<i<l \text{ and } l<j<s-1\). Hence, by Definition 5.1, we have

\[
\sum_{i=1}^{l} a_i = \sum_{i=1}^{l} T(i,s) + \sum_{i=1}^{l-1} T(i,j) = \sum_{i=1}^{l-1} \left( \sum_{j=1}^{l} T(i,j) \right) + \sum_{j=1}^{l-1} T(i,s) < \sum_{j=1}^{l-1} \sum_{j=1}^{l} a_j + a_s.
\]

Therefore \(a_i < a_s\). Since \(a_i > a_{i-1} > \cdots > a_s\), we can conclude that \(a_i = a_s\). It follows that

\[
\sum_{j=1}^{l-1} \left( a_j - \sum_{i=1}^{l} T(i,j) \right) + \left( a_s - \sum_{i=1}^{l} T(i,s) \right) = 0
\]

and hence that

\[
a_j = \sum_{i=1}^{l} T(i,j), \text{ for } j \in \{1, \ldots, l-1, s\}.
\]

Therefore

\[
T(i,j) = 0, \text{ for } l+1<i<s, \text{ and } j \in \{1, \ldots, l-1, s\}. \text{ Q.E.D.}
\]

**Proof of Theorem 10.2:** Assign \(A'_l\) the value given by Eq. (10.4). We will verify the equation \(g'_u(T)(s) = A'_s\) for \(T \in \mathcal{T}(\alpha)_s\), \(1<s-1\), \(u \in \{e_1, \ldots, e_{l+1}\}\) such that \(T(l+1,u) > 0\). To do this we will compute \(g'_u(T)(s)\) explicitly.

Let \(T \in \mathcal{T}(\alpha)_s\) and let \(1<s-1\). Introduce quantities \(X, Y,\) and \(Z\) as follows.

\[
X = \sum_{v=1}^{l-1} T(l,v), \quad Y = \sum_{k=l+1}^{s} T(k,l), \quad Z = T(l,l) + s - l - 1.
\]

Observe that for \(T \in \mathcal{T}(\alpha)_s\),

\[
X + T(l,s) = Y.
\]

The computation breaks into two cases.

1. **Case 1:** \(1<s-1\), \(u = e_l = l\), \(T \in \mathcal{T}(\alpha)_s\) and \(T(l+1,l) > 0\).
Observe that the hypothesis $T(l+1,l) > 0$ implies that $Y > 0$. Now using Eqs. (8.6), (8.7), (10.3), (10.4), (10.5), and (10.6) we evaluate five terms whose sum is, by definition, $\xi^l_u(T)_s$:

$$\sum_{v=1}^{l-1} M^l_u(T,v)_s = (-1)^{\sigma(T)} D^*_T \cdot X(T(l,l) + s - l)/Y.$$  

$$M^l_u(T,l)_s = (-1)^{\sigma(T)} D^*_T \cdot T(l,l).$$  

$$M^l_u(T,s)_s = (-1)^{\sigma(T)} D^*_T \cdot T(l,s)(T(l,l) + s - l)/Y.$$  

$$N^l_u(T,l+1)_s = \begin{cases} (-1)^{\sigma(T)} D^*_T \cdot T(l+1,l+1), & \text{if } l < s-1 \\ 0, & \text{if } l = s-1. \end{cases}$$

$$\sum_{i=l+2}^{i} N^l_u(T,i)_s = \begin{cases} (-1)^{\sigma(T)} D^*_T \cdot (T(l+1,l+1) + s - l - 1), & \text{if } l < s-1 \\ 0, & \text{if } l = s-1. \end{cases}$$

The preceding five terms clearly add to $(-1)^{\sigma(T)} D^*_T = A^l_T$, so we have $\xi^l_u(T)_s = A^l_T$.

**Case 2**: $1 < l < s-1$, $u \in \{e_1, \ldots, e_{l-1}, l\} = \{1, \ldots, l-1, s\}$, $T \in \mathcal{T}(\alpha)_s$, and $T(l+1,u) > 0$.

Observe that the hypothesis $T(l+1,u) > 0$ implies, by Lemma 10.5, that $T(l,l) > 0$. It follows that $Z > 0$.

Now using Eqs. (8.6), (8.7), (10.3), (10.4), (10.5), and (10.6) we evaluate five terms whose sum is, by definition, $\xi^l_u(T)_s$:

$$\sum_{v=1}^{l-1} M^l_u(T,v)_s = (-1)^{\sigma(T)} D^*_T \cdot X.$$

$$M^l_u(T,l)_s = (-1)^{\sigma(T)} D^*_T \cdot T(l,l)(Y+1)/Z.$$  

$$M^l_u(T,s)_s = (-1)^{\sigma(T)} D^*_T \cdot T(l,s).$$

$$N^l_u(T,l+1)_s = \begin{cases} (-1)^{\sigma(T)} D^*_T \cdot T(l+1,l+1)(Y+1)/Z, & \text{if } l < s-1 \\ 0, & \text{if } l = s-1. \end{cases}$$

$$\sum_{i=l+2}^{i} N^l_u(T,i)_s = \begin{cases} (-1)^{\sigma(T)} D^*_T \cdot (T(l+1,l+1) + s - l - 1)(Y+1)/Z, & \text{if } l < s-1 \\ 0, & \text{if } l = s-1. \end{cases}$$

The preceding five terms clearly add to $(-1)^{\sigma(T)} D^*_T = A^l_T$, so we have $\xi^l_u(T)_s = A^l_T$.

**Q.E.D.**

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