Modeling and Optimization of Link Traffic Flow (Paper #08-2129)

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Abstract

Congestion in networks greatly reduces efficiency and production. Systems with limited capacities or resources require analysis in order to ensure optimal results, which may be in terms of cost, data transmitted or vehicles discharged. A network is like an interconnected web and if one section or link is not performing optimally, the network may not be operating efficiently. The objective of this research is to maximize link traffic throughput in the long run to alleviate congestion. The approach is to model the changes in link traffic states as a discrete Markov chain, from mathematical theory, due to its random or stochastic nature. This link traffic flow model can be used for varying stochastic processes with corresponding performance measures, for example, a service rate. While this paper focuses on vehicular traffic flow, other disciplines are invited to collaborate on the study of link traffic flow in similar stochastic systems. This research presents a novel blocking probability distribution to account for congestion based on the $M/G/c/c$ state-dependent queuing model. The objective function with the blocking probability was optimized and the results were compared with a simulation model. The optimal solution to the objective function is a flow at which throughput on the link is maximized for the long run. Under a Vehicle infrastructure Integration (VII) scenario, this model may serve as the basis of link flow control in an effort to achieve the maximum link throughput in the long run.

INTRODUCTION

Congestion is a problem that is ubiquitous in systems, decreasing output and efficiency. While the components of systems or networks vary, efficient production is desirable. Often, congestion is due to competition for limited resources or capacities. As an example, think of a conventional bucket, the capacity of the bucket is restricted to the height of its walls. If any microscopic section of the wall of the bucket were absent or shorter than the rest, its contents would spill out to the height of the shortest section. Similarly, the congestion in networks may be based on the performance of a single section or link in the system.

Whether you are considering data packets through an internet path or vehicles along a section of roadway, their basic link performance is restrained by a limited capacity. Various link traffic flow models have been used in computer networks with limited bandwidth, energy conservation in wireless devices and in noise prediction in the field of acoustics. Once link performance is optimized (height of bucket section optimized), then the optimization can be expanded to a series of links or an entire network to reduce congestion. Given the Federal Highway Administration’s (FHWA) estimate that vehicle miles of travel increased 89 percent while lane-miles of highways increased only 5 percent between 1980 and 2003 [1], congestion in transportation has become an important issue affecting most Americans, both directly and indirectly through opportunity costs and rising food prices.

This research intends to provide a link traffic flow optimization model that may be applicable to transportation as well as computer and industrial networks. Some assumptions made are the stochastic nature of traffic flow, Markovian traffic states and a non-linear decreasing service rate. Based on the expected traffic state from discrete Markov chain theory, an objective function for the optimal throughput was created with a new blocking probability distribution from stochastic traffic $M/G/c/c$ state-dependent queuing theory [2-5]. The proposed blocking probability formulation is unique to the best of the authors’ knowledge. The objective function with the blocking probability penalizes unstable flows. The optimal solution is a proposed flow at which the link throughput would be maximized based upon the present conditions.

LITERATURE REVIEW

This section presents a review of previous literature relevant to the proposed model of link traffic flow optimization. Additionally, the foundations of the proposed model are highlighted for their merit.
Short-term traffic forecasting can be a useful design tool to accommodate expected future demand. There have been numerous methods presented, compared and expanded upon for short-term traffic forecasting which include random walk, historical average, Markov chain, time-series, joint distribution recalculation, neural networks, genetic algorithms, Bayesian networks, nonparametric regression and local linear regression among others. Despite the range of methods, Markov chain’s simplicity in deployment and computation has proven robust for short-term traffic flow prediction [6-9]. Markov chain is a mathematical theory for random processes, thoroughly defined in the next section. For numerous categories of traffic states, Markov chain can be computationally demanding, for $n$ number of traffic states, the transition matrix contains $n \times n$ probabilities. Yet if the traffic states can be grouped into small categories as carried out in this research, the likelihood that traffic will transfer to another state can be easily generated.

Traffic flow optimization has also been presented in a wealth of papers. While some have made use of the Markov decision process in their traffic flow optimization [10, 11], others have used M/G/c/c state-dependent queuing theory to model single link vehicular traffic flow in order to perform optimization [2-5]. From M/G/c/c state-dependent queuing theory, this paper proposes a new blocking probability distribution based on the discrete Markov chain traffic predictions. The blocking probability has been successful in random systems whose travel speed decreases non-linearly as a function of the traffic density in the system, which is the case for vehicular and pedestrian traffic flows. The objective in this research is to maximize throughput. The throughput equation comes from queuing theory performance measures and is defined as a function of the input flow rate and a penalty or risk function. The penalty function accounts for the loss of smooth traffic flow due to congestion. Here discrete first-order Markov chain theory from traffic forecasting is combined with M/G/c/c state-dependent queuing theory in order to maximize throughput on a link while minimizing the risk of traffic instability. However, this paper is not the only research that combines Markov chain theory and throughput analysis [12].

THEORY

This research requires some background information on the theories used. To facilitate subsequent discussion, these theories are briefly introduced here. Advanced readers who are familiar with discrete Markov chain and M/G/c/c queuing theories may skip this section and proceed to the next section.

Discrete Markov Chain Theory

From mathematics, discrete Markov chain theory produces conditional or transition probabilities of a finite number of future events without requiring any historical knowledge of the data, only the present state. Markov chain is applicable to time discrete, stochastic processes. The Markovian property is given by,

$$ p(S(t+1) = A|S(0), S(1), S(2), ..., S(t)) = p(S(t+1) = A|S(t)) $$

where

- $S(t+1)$ = the state immediately following the present state after the next time interval
- $S(t)$ = the present state
- $S(0), ..., =$ the states prior to $S(t)$
- $A$ = the state that $S(t)$ will be in at time $t=(t+1)$

The one-step transition probability is a probability that a future state will occur immediately following the present state. The one-step transition probabilities are defined as,

$$ p_{ij} = \frac{n}{N} $$
\[
\text{for } i, j = (1, 2, \ldots, a)
\]

where

- \( p_{ij} = \text{the probability that State } j \text{ will immediately follow State } i \)
- \( n = \text{the number of times State } j \text{ immediately follows State } i \)
- \( N = \text{the number of times State } i \text{ transitions to any state including repetition of State } i \)
- \( a = \text{the number of defined traffic states} \)

The one-step transition probability matrix represents all the possible state changes between traffic states, where each cell in the matrix corresponds to the probability of a present and future state pair \((p_{ij})\). The size of the transition matrix is \( a \times a \), where \( a \) is the number of defined states.

**Modeling Vehicular Traffic Flow Using Discrete Markov Chain Theory**

It is easily observed that the current and previous traffic states can show the trend in the next interval. Various prediction methods use historical data to model successive conditions, for example, weather and climate forecasts. The future state of traffic in the subsequent interval has strong but not deterministic or fixed relationship to the current or very recent states. This relationship can be considered a conditional probability. That is to say, the next traffic state obeys a probability distribution and the probability of the next state is determined by the current and immediately preceding states. However, a general assumption is that states too far back in the past will not give any more information than the current and immediate states. If a series has the attribute that, given the current and N-1 preceding states, the future state is independent of the states prior to the present state, the series can be called a Markov chain with N-order [8]. In this paper, Markov chain is employed to predict the expected traffic states with a proper probability distribution since fluctuating historical data for vehicular traffic flow cannot accurately forecast congestion due to its stochastic nature [7-9].

**Traffic States Classification**

There are three parameters that characterize traffic states: speed \((v)\), flow \((q)\), and density \((k)\). With an elapsed time change, these three parameters also change. As a result, traffic states change continuously as a function of time. If \( w \) is defined as a variable to represent a traffic state, \( w \) can be written as \( w = (q, k, v) \). Additionally, speed \((v)\) and flow \((q)\) can be represented as a function of density \((k)\) by the traffic flow fundamental relationship \( q = kv \).

Now the representation of traffic states can be reduced to one variable \( w(t) = k(t) \). During each time interval, there is a specific corresponding traffic state that can be represented by its density. Speed or flow was not chosen to singly describe the traffic states because both speed and flow are multivalued functions in the fundamental diagrams. Some logic behind this step is that traffic density is often how drivers perceive congestion. If a driver feels that there may be an uncomfortable number of vehicles in his line of sight, the driver will moderate his speed appropriately for safety and ease of driving.

Moreover, the time and density ranges must be partitioned into discrete units for use of discrete Markov chain. This research assumes a time interval of 5 minutes and a density interval of 12.5 veh/km/ln. The upper limit of the density range is the jam density. Jam density is considered the density at which the vehicles are bumper-to-bumper and completely stopped [14]. Average jam density values reported vary between 185-250 vehicles per mile per lane (vpmpl) [14]. This has been translated to 140 veh/km/ln.

In order to more clearly illustrate the traffic states, assume that density ranges from 0 to 150 veh/km/ln. This continuous range of values is divided into discrete segments over the range by a basic unit of graduation. Using \( \Delta k = 12.5 \text{ veh/km/ln} \) as the basic unit to divide up the density range, the resulting critical points for density
are 0, 12.5, 25, 37.5, 50, ..., 150 veh/km/ln. Despite jam density’s definition as the upper limit of the density range, the last state allows for a density greater than 140 veh/km/ln. This gap is to allow the model to account for any instantaneous occurrences of densities greater than the assumed jam density. The arrangement of the combinations of traffic density states generates a finite transition matrix to represent all the possible state changes between traffic states, where each cell in the matrix represents the probability of a present and future density pair. The size of the transition matrix is $a \times a$, where $a$ is the number of traffic states. Therefore, twelve traffic states are defined in this paper and summarized in Table 1.

<table>
<thead>
<tr>
<th>Traffic States Classification</th>
<th>Density Interval $k_i$ (veh/km/ln)</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>$0 \leq k_i \leq 12.5$</td>
</tr>
<tr>
<td>State 2</td>
<td>$12.5 \leq k_i \leq 25$</td>
</tr>
<tr>
<td>State 3</td>
<td>$25 \leq k_i \leq 37.5$</td>
</tr>
<tr>
<td>State 4</td>
<td>$37.5 \leq k_i \leq 50$</td>
</tr>
<tr>
<td>State 5</td>
<td>$50 \leq k_i \leq 62.5$</td>
</tr>
<tr>
<td>State 6</td>
<td>$62.5 \leq k_i \leq 75$</td>
</tr>
<tr>
<td>State 7</td>
<td>$75 \leq k_i \leq 87.5$</td>
</tr>
<tr>
<td>State 8</td>
<td>$87.5 \leq k_i \leq 100$</td>
</tr>
<tr>
<td>State 9</td>
<td>$100 \leq k_i \leq 112.5$</td>
</tr>
<tr>
<td>State 10</td>
<td>$112.5 \leq k_i \leq 125$</td>
</tr>
<tr>
<td>State 11</td>
<td>$125 \leq k_i \leq 137.5$</td>
</tr>
<tr>
<td>State 12</td>
<td>$137.5 \leq k_i \leq 150$</td>
</tr>
</tbody>
</table>

**M/G/c/c Queuing Theory**

The M/G/c/c state-dependent queuing model accounts for flows with congestion and capacity constraints for the circulation space. The acronym M/G/c/c stands for Markovian arrival process, General state-dependent service rates, $c$ parallel servers and a total capacity of $c$ [3]. A server is a channel or lane through which traffic can flow. The service rate is defined as the ratio of the average travel speed for a certain number of customers on the link to the average travel speed of a single occupant, $f(n) = \frac{V_n}{V_1}$. The service rate is a decreasing non-linear function in the M/G/c/c state-dependent queuing model [5]. As the density of the units entering the circulation space increases, the average travel speed of the units decreases as seen in the fundamental relationship [13], yet the empirical curves for vehicular traffic flow strongly suggest that the exponential model may provide a more accurate approximation for the average travel speed with variations in vehicular density [2]. However, there is a linear model and an exponential model to describe the relationship between density and mean travel speed. The linear model is defined as,

$$V_n = \frac{A}{C}(C + 1 - n)$$

where

- $V_n$ = the average travel speed for $n$ units in the circulation space
- $A = V_1$ = the average travel speed of a single occupant
- $C$ = the capacity of the circulation space
- $n$ = the number of units utilizing the circulation space

The exponential model is described by,

$$V_n = A \exp\left[-\left(\frac{n-1}{\beta}\right)^\gamma\right]$$

where
\( \gamma = \text{scale and shape parameter, } \gamma = \frac{\ln(V_a/A)}{\ln(V_b/A)} / \frac{a-1}{b-1} \)

\( \beta = \text{scale and shape parameter, } \beta = \frac{a - 1}{\ln(A/V_a)^{\gamma}} = \frac{b - 1}{\ln(A/V_b)^{\gamma}} \)

and

\( a, b = \text{number of vehicles based on the empirical data} \)

\( V_a = \text{average travel speed at density } a, \text{ approximated from empirical data} \)

\( V_b = \text{average travel speed at density } b, \text{ approximated from empirical data} \)

In a one-directional link, assume that there is an arrival rate at which the units arrive to the link and which is independent of the number of units on the link. Also, assume that there is a service rate on the link that depends on the number of units utilizing the link. The M/G/c/c steady state probabilities are given by,

\[
P_n = \left\{ \frac{(\lambda \vartheta)^n}{n! f(n) f(n-1) \ldots f(2) f(1)} \right\} P_0 \]

for \( n = 1, 2, \ldots, C \)

and \( P_0 \) is the empty system probability described by,

\[
P_0^{-1} = 1 + \sum_{i=1}^{C} \left\{ \frac{(\lambda \vartheta)^i}{i! f(i) f(i-1) \ldots f(2) f(1)} \right\}
\]

where

\( \lambda = \text{the arrival rate} \)

\( \vartheta = \text{the mean service requirement, } \vartheta = L/A, L \) is the length of the circulation space

\( C = \text{the capacity of the circulation space, } C = kLN, k \) is the capacity of the circulation space per square unit, \( N \) is the number of service channels

\( f(n) = \text{service rate} \)

From the exponential model of average travel speed \( (V_n) \), the service rate \( f(n) \) is given by,

\[
f(n) = \frac{V_n}{V_1} = \frac{A}{V_1} \exp \left[ -\left( \frac{n-1}{\beta} \right)^{\gamma} \right] = \exp \left[ -\left( \frac{n-1}{\beta} \right)^{\gamma} \right]
\]

From the linear model of the average travel speed \( (V_n) \), the service rate \( f(n) \) is defined as,

\[
f(n) = \frac{V_n}{V_1} = \frac{A}{CV_1} \left( C + 1 - n \right) = \frac{C + 1 - n}{C}
\]

Furthermore, the linear state probabilities are expressed by,

\[
P_n = \left\{ \frac{(\lambda L/A)^n}{\prod_{j=1}^{n} j[(C+1-j)/C]} \right\} P_0
\]

and

\[
P_0^{-1} = 1 + \sum_{j=1}^{C} \left\{ \frac{(\lambda L/A)^j}{\prod_{j=1}^{j} j[(C+1-j)/C]} \right\}
\]

From the exponential model of average travel speed \( (V_n) \), the service rate \( f(n) \) is given by,
\[ f(n) = \frac{V_n}{V_1} = \frac{A}{V_1} \exp \left[ -\left( \frac{n-1}{\beta} \right)^\gamma \right] = \exp \left[ -\left( \frac{n-1}{\beta} \right)^\gamma \right] \]

Similarly, the exponential state probabilities are described by,

\[ P_n = \left( \frac{\lambda L}{A} \right)^n / \prod_{j=1}^{n} j \exp \left[ -\left( \frac{j-1}{\beta} \right)^\gamma \right] P_0 \]

and

\[ P_0^{-1} = 1 + \sum_{i=1}^{c} \left( \frac{\lambda L}{A} \right)^i / \prod_{j=1}^{i} j \exp \left[ -\left( \frac{j-1}{\beta} \right)^\gamma \right] \]

From the M/G/c/c steady state probabilities, several performance measures can be derived.

\[ P_b = P_n (n = C) \]

\[ \theta = \lambda(1 - P_b) \]

where

- \( P_b = \) blocking probability, the steady state probability when the number of units on the link is at capacity
- \( \theta = \) throughput, the rate of vehicles leaving the link
- \( \lambda = \) the arrival rate to the link

**Modeling Vehicular Traffic Flow Using M/G/c/c Queuing Theory**

As anyone may observe, the speed of traffic on a roadway is greatly influenced by the density of vehicles on a link. As the density of the vehicles on the link increases, the average travel speed of the vehicles on the link decreases. M/G/c/c queuing theory has been successfully used to model vehicular and pedestrian traffic flow for decreasing non-linear service rates. The empirical data of this research in Figure 1 shows the same trend.

![Non-Linear Service Rate from Empirical Data](image)
MATHEMATICAL MODEL

General Concept of Modeling

This model can be used to maximize throughput on a link given the present conditions. From the present traffic state, the probability that traffic will transition to any other state is derived from Markov chain theory. In this model, density is used to represent the traffic states. Using a weighted average of the one-step transition probabilities for a particular traffic state, the expected traffic density in the next time interval can be calculated. The expected number of vehicles on the link is given by the expected traffic density, the length and the number of lanes of the link. The expected number of vehicles is used in exponential model of the blocking probability. Next, the blocking probability is used in the objective function to maximize throughput of the link. The logic is summarized in Table 2.

Table 2 Model Development Flow Chart

<table>
<thead>
<tr>
<th>PRESENT TRAFFIC STATE q(t)</th>
<th>↓</th>
</tr>
</thead>
<tbody>
<tr>
<td>q(t) → Pr [q(t+1)=1,2,…,12]</td>
<td>↓</td>
</tr>
<tr>
<td>E[q(t+1)] = E[k]</td>
<td>↓</td>
</tr>
<tr>
<td>n = E[k]·l·N</td>
<td>↓</td>
</tr>
<tr>
<td>Exponential P_b = f(n)</td>
<td>↓</td>
</tr>
<tr>
<td>θ = q(t) (1-P_b)</td>
<td>↓</td>
</tr>
</tbody>
</table>

Figure 2 shows the basic operation of the objective function. Here the objective function has been broken down into its components, \( \theta = q(t) - [q(t)P_b] \). The first term of the rewritten equation is the input flow. The second term is the penalty function. In the figure, the red dashed line represents the input flow, the blue solid curve indicates the suggested throughput corresponding to the input flow and the green dash-dot-dash curve denotes the penalty or risk associated with any level of congestion. From the objective function, \( \theta = q(t) - [q(t)P_b] \), it is obvious that the throughput will be less than the input volume, even by a negligibly small amount, unless the traffic meets the boundary conditions. To maximize the throughput, this research seeks the peak of the blue solid curve.
The Link Traffic Flow Optimization Model

As previously stated, this research aims to maximize the throughput ($\theta$) of a link in the long run to reduce congestion. The throughput includes the present flow ($q(t)$) on the link and accounts for congestion with a corresponding blocking probability ($P_b$). The objective function is:

Maximize: \[ \theta = q(t)(1 - P_b) \]
Subject to: \[ 0 \leq q(t) \leq q_{\text{max}} \]
\[ 0 \leq P_b \leq 1 \]

where
\[ \theta = \text{throughput on the link} \]
\[ q(t) = \text{present flow} \]
\[ q_{\text{max}} = \text{maximum possible flow} \]
\[ P_b = \text{probability of congestion} \]

From queuing theory, the blocking probability is the state-dependent probability as a function of the mean service requirement and the relative service rate when the number of vehicles on the road is at its capacity [2]. The proposed blocking probability distribution has been defined as a function of a similar relative service rate \( \left( \frac{v_n}{v_f} \right) \), the number of vehicles on the link and the number of vehicles on the link at capacity. Here the free flow speed replaces the previous average travel speed of a lone occupant, which was defined as the posted speed limit in [2]. It may be unrealistic to assume that drivers remain under the posted speed limit. The proposed blocking probability distribution is centered on the Markov chain transition probabilities instead of the M/G/c/c steady state probabilities.

**Blocking Probability Distribution Formulation**

The blocking probability is given by

\[ P_b = n \left( 1 - \frac{v_n}{v_f} \right) \]

where
\[ n = \text{expected number of vehicles on the link}, \quad n = k_E \cdot L \cdot N, \quad \text{where } k_E = \text{expected future density}, \quad L = \text{length of the link}, \quad L = 0.55 \text{ km and } N = \text{number of lanes on the link}, \quad N = 4 \]
\[ v_n = \text{average travel speed on the link for } n \text{ vehicles} \]
\[ v_f = \text{free flow speed}, \quad v_f = 115 \text{ km/h} \]
\[ n_c = \text{number of vehicles on the link at capacity}, \quad n_c = k_{\text{jam}} \cdot L \cdot N, \quad \text{where } k_{\text{jam}} = \text{jam density}, \quad k_{\text{jam}} = 140 \text{ veh/km/ln} \]

**Boundary Condition Analysis**

When the number of vehicles on the link ($n$) is very small, approximately 1, the average travel speed equals the free flow speed \( (v_n = v_f) \). In this case, \( P_b = \frac{1}{n_c} \approx 0 \). A blocking probability of zero indicates that there is no congestion on the road and drivers do not moderate their speed with respect to other vehicles. When the number of
vehicles on the link \((n)\) approaches capacity \((n_c)\), all traffic is at a standstill. From [4], the average travel speed is exactly zero \(v_n = 0\) when \(n = n_c + 1\). Although the link capacity by definition is the absolute limit for the road, there still may be some movement when \(n = n_c\) so \(v_n = 0\) for all situations where \(n \geq n_c + 1\), a theoretically improbable scenario [4]. In this case, \(P_b = 1\). A blocking probability of one indicates that there is absolutely no movement on the link since it is completely full.

**Expected Future Density** \(k_E\)

The expected future density \((k_E)\) is based on the Markov chain transition probabilities. Here, future means the subsequent time interval. In statistics, the expected value of an outcome is a weighted average of all possible results based on their likelihood or probability. From a certain traffic state, defined by its density, multiple traffic densities may immediately follow. The expected future density is given by

\[
k_E = \sum_{i=1}^{12} p_i \cdot k_i
\]

where

- \(p_i\) = transition probability
- \(k_i\) = possible density traffic state

With the transition probabilities, the expected future density can be generated. From the expected density, the expected number of vehicles on the road can be calculated \((n = k_E \cdot L \cdot N)\). The expected number of vehicles can be used directly in the blocking probability and used to define the average travel speed on the link for the expected number of vehicles \((v_n)\).

**Revised Linear and Exponential Models Used to Approximate Blocking Probability**

Linear and exponential models have been developed in [2]. Similar models of the average travel speed are used in this model however, the free flow speed to replaces the previous average travel speed of a lone occupant, which is defined as the posted speed limit [2]. The revised models are given by

Linear: \[
v_n = \frac{v_f}{n_c} (n_c + 1 - n)
\]

Exponential: \[
v_n = v_f \exp\left[\frac{n-1}{\beta}\right]
\]

where

- \(n = \text{expected number of vehicles on the link}\), \(n = k_E \cdot L \cdot N\), where \(k_E = \text{expected future density}\), \(L = \text{length of the link}\), \(L = 0.55 \text{ km}\) and \(N = \text{number of lanes on the link}\), \(N = 4\)
- \(v_n = \text{average travel speed on the link for } n\) vehicles
- \(v_f = \text{free flow speed}\), \(v_f = 115 \text{ km/h}\)
\( n_c = \text{number of vehicles on the link at capacity}, \ n_c = k_{\text{jam}} \cdot L \cdot N, \) where \( k_{\text{jam}} = \text{jam density}, \ k_{\text{jam}} = 140 \text{veh/km/ln} \)

\( \gamma = \text{shape parameter}, \ \gamma = \ln\left(\frac{\ln(v_a/v_f)}{\ln(v_b/v_f)}\right) / \ln\left(\frac{a-1}{b-1}\right), \ \gamma = 1.43 \)

\( \beta = \text{scale parameter}, \ \beta = \frac{a-1}{\ln(v_f/v_a)}^{\gamma/b} = \frac{b-1}{\ln(v_f/v_b)^\gamma} \), \ \beta = 104.92 \)

and

\( a = \text{number of vehicles on the link at an observed density}, \ a = k_a \cdot L \cdot N, \ k_a = 50 \text{veh/km/ln}, \ a = 27.5 \)

\( b = \text{number of vehicles on the link at an observed density}, \ b = k_b \cdot L \cdot N, \ k_b = 12.5 \text{veh/km/ln}, \ b = 110 \)

\( v_a = \text{average travel speed at an observed density}, \ \text{approximated from data}, \ v_a = 100 \text{km/hr} \)

\( v_b = \text{average travel speed at an observed density}, \ \text{approximated from data}, \ v_b = 40 \text{km/hr} \)

In this research, the density values chosen for \( k_a \) and \( k_b \) are 50 and 12.5 veh/km/ln, respectively, are where there seems to be an inflection in the empirical graph of the service rate, see Figure 1. Furthermore, the linear and exponential models of the blocking probability are given by

**Linear:**
\[
P_b = \frac{n^{(1-n_c/n_c)}}{n_c}
\]

**Exponential:**
\[
P_b = \frac{n\left\{1-\exp\left(-\frac{n-1}{\beta}\right)\right\}}{n_c}
\]

**EMPIRICAL STUDY**

An ideal means to validate this model is to regulate a section of road and compare its throughput with the model output. However, such an approach is not practical. Alternatively, simulation appears to be a reasonable choice. A common set of input is applied to both the simulation model and the mathematical model and comparison is made based on the outputs of both models. To facilitate the comparison, the transition probability matrix and the blocking probabilities are empirically obtained from GA 400 data.

**Study Site and Data Sampling**

The dataset used in this study was collected from GA 400 by Georgia NaviGAtor, the ITS of the State of Georgia, from 01/01/2003 to 12/31/2003. Traffic conditions at the site were monitored by video cameras deployed approximately every one third of a mile of the road in both directions. Each camera constitutes an observation station and samples all lanes at this location. Each sample contains a variety of information, among which speed, flow, and density are of major interest in this study. After considering data quality and site configuration, one section of the southbound GA 400 was selected as the study site. The study site has four lanes, contains Station 4001118, and is 550 m long. To provide a reasonable volume of data yet to reduce bias, thirty days out of about one year worth of data was randomly selected to generate the transition probability matrix. In order to deal with high variability in the original data as well as to facilitate the use of Markov chain theory, the data has been aggregated to a five minute time interval. In addition, the raw classified traffic counts has been converted to passenger car equivalents (PCE) [14].
Transition Matrix

The sampled thirty days data was used in the MATLAB® software to produce the transition probability matrix, see Table 3. In this research, the transition probability matrix represents the probability that a future state will follow the present state in the next five minutes. For example, if the present density is 37 vehicles per kilometer (i.e. State 4), the probability of having State 2 next is 0.033, i.e. Cell (4,2). It can be easily identified that there is a diagonal probability trend in the matrix. The diagonal trend represents the tendency of traffic to remain in the same or adjacent traffic state. A higher probability means that traffic is more likely to transition to that state if no disturbances occur. Around States 4-9, traffic becomes less stable and the transition probabilities are spread over many states. In States 10-12, the probabilities once again are high values. From the sampled thirty days data, there were less than 10 data points in States 10-12. However, for these states there is probability of zero that traffic will remain in its present state. Correspondingly, there is a high likelihood that traffic will transition to other states. The result is that as the traffic density increases, it is likely that traffic will transfer to other potential states.

Table 3 Transition Probability Matrix Approximated from Thirty Days Data

<table>
<thead>
<tr>
<th>Traffic States at time t+1</th>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
<th>State 4</th>
<th>State 5</th>
<th>State 6</th>
<th>State 7</th>
<th>State 8</th>
<th>State 9</th>
<th>State 10</th>
<th>State 11</th>
<th>State 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>State 1</td>
<td>0.846</td>
<td>0.152</td>
<td>0.001</td>
<td>0</td>
<td>0</td>
<td>0.001</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>State 2</td>
<td>0.107</td>
<td>0.882</td>
<td>0.009</td>
<td>0.001</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 3</td>
<td>0</td>
<td>0.212</td>
<td>0.433</td>
<td>0.269</td>
<td>0.077</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 4</td>
<td>0</td>
<td>0.033</td>
<td>0.17</td>
<td>0.49</td>
<td>0.157</td>
<td>0.072</td>
<td>0.046</td>
<td>0.013</td>
<td>0.013</td>
<td>0.007</td>
<td>0</td>
<td>0</td>
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<tr>
<td>State 5</td>
<td>0</td>
<td>0</td>
<td>0.231</td>
<td>0.481</td>
<td>0.173</td>
<td>0.058</td>
<td>0.038</td>
<td>0</td>
<td>0.019</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.371</td>
<td>0.229</td>
<td>0.143</td>
<td>0.029</td>
<td>0.2</td>
<td>0.029</td>
<td>0</td>
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<tr>
<td>State 7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.313</td>
<td>0.125</td>
<td>0.375</td>
<td>0.063</td>
<td>0.125</td>
<td>0</td>
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<tr>
<td>State 8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.143</td>
<td>0.071</td>
<td>0.286</td>
<td>0.214</td>
<td>0.143</td>
<td>0.071</td>
<td>0</td>
<td>0.071</td>
<td>0</td>
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<tr>
<td>State 9</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0.429</td>
<td>0.286</td>
<td>0</td>
<td>0.143</td>
<td>0</td>
<td>0.143</td>
<td>0</td>
<td>0</td>
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<tr>
<td>State 10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.333</td>
<td>0.333</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.333</td>
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<tr>
<td>State 11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.333</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.667</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>State 12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Linear and Exponential Blocking Probabilities

Figure 3 represents the blocking probabilities from the linear and exponential congestion models. The blue solid line represents the exponential blocking probability while the red dashed line denotes the linear blocking probability.
The difference between linear and exponential model is that service rate \( f(n) = \frac{V_a}{V_1} \), which was used to approximate the blocking probability, was developed separately with respect to each of the two models. However, the empirical curves for vehicular traffic flow strongly suggest that the exponential model may provide a more accurate approximation for the average travel speed with variations in vehicular density [2]. Therefore, this research has used the exponential blocking probability in the objective function.

From the linear and exponential blocking probability curves in figure 3, there is a drop in the blocking probability at traffic state 11. Theoretically, the trend should be increasing. By referring to the transition probability matrix, States 10-12 have a zero probability that they will remain in their present state. However, State 11 is spread between two traffic states, State 4 and State 10. This distribution of the empirical data results in the anomaly of the blocking probability curves.

![Figure 3 Linear and Exponential Model of the Blocking Probability](image)

**Empirical Performance of the Mathematical Model**

The empirical performance of the mathematical model is examined using two days other than those using in developing the transition probability matrix. Figure 4 illustrates the time-varying effect of the objective function’s components. This figure can be interpreted as follows. If one pumps traffic into a link at the rate specified by the red dashed curve, in the long run one can reasonably expect a throughput corresponding to the blue solid curve. In essence, the throughput is the input flow penalized by its associated probability of congestion (the green dash-dotted curve).
As stated above, a simulation model is developed to provide a ground against which the mathematical model is compared. The simulation model is developed using CORSIM and involves only two links, a long real link with multiple lanes and an upstream dummy link to hold queues, if any. A common set of input flow is created with an increasing demand. The simulation is run at each input flow level for multiple times in order to obtain the corresponding throughput in the long run. Meanwhile, the same set of input also applies to the mathematical model and the corresponding throughput is computed.

The comparison result is plotted in Figure 5, which shows how throughput varies with input/demand. Generally, the throughput increase linearly with demand up to about 1500veh/hr/ln. Under such a flow rate, traffic has enough room to digest the disturbances generated both environmentally and geometrically. However, as demand
increases close to capacity, congestion emerges along with a penalty. At this point, traffic cannot accommodate demand in a timely manner and smooth the disturbances generated by non-linear and stochastic natured traffic flow. The throughput will peak somewhere near the capacity because this is where the difference between the input flow and the penalty of congestion, represented by the blocking probability, is maximized.

Though slight discrepancies exist, the comparison of the output of the two models suggests a good fit in general. More specifically, the mathematical model tends to slightly underestimate congestion in the higher demand range and overestimate congestion in the mid-demand range. These discrepancies may be attributed to the transition probability matrix where the probabilities spread among many states in mid- to high-density range. In the high-density range, the probabilities suggest unexpected low densities than the present state.

CONCLUSIONS AND FUTURE WORK

Using the proposed blocking probability distribution, this paper presented a new mathematical model for the optimal throughput of a one-directional link. This model assumes the stochastic nature of traffic flow, Markovian traffic states and a non-linear decreasing service rate. The mathematical model was compared with a simulation model with limited success. However, the mathematical model obviously follows the trend of the empirical data. In this sense, the mathematical model may be a more realistic prediction method. Under a Vehicle infrastructure Integration (VII) scenario, this model may serve as the basis of link flow control in an effort to achieve the maximum link throughput in the long run.

Future work with this model may involve: finding an improved set of data including more data points as well as more instances of severe congestion and use of a different time and/or density interval as ITS and Vehicle Infrastructure Integration (VII) technology advances.

Figure 5 Comparison of the outputs of both models
REFERENCES