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Bangyu Shen
Xiaojing Wang
Chongyang Liu

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NONLINEAR STATE-DEPENDENT IMPULSIVE SYSTEM IN FED-BATCH CULTURE AND ITS OPTIMAL CONTROL

BANGYU SHEN, XIAOJING WANG
School of Mathematical Science, Huaiyin Normal University
No.111, Changjiang West Road, Huai’an 223300, China

CHONGYANG LIU
School of Mathematics and Information Science
Shandong Institute of Business and Technology
No.191, Binhai Road, Yantai 264005, China

ABSTRACT. In fed-batch culture, feeding substrates is to provide sufficient nutrition and reduce inhibitions simultaneously for cells growth. Hence, when and how much to feed substrates are important during the process. In this paper, a nonlinear impulsive controlled system, in which the volume of feeding is taken as the control function, is proposed to formulate the fed-batch fermentation process. In the system, both impulsive moments and jumps size of state are state-dependent. Some important properties of the system are investigated. To maximize the concentration of target product at the terminal time, an optimal control model involving the nonlinear state-dependent impulsive controlled system is presented. The optimal control problem is subject to the continuous state inequality constraint and the control constraint. The existence of optimal control is also obtained. In order to derive the optimality conditions, the optimal control model is transcribed into an equivalent one by treating the constraints. Finally, the optimality conditions of the optimal control model are obtained via calculus of variations.

1. Introduction. 1,3-Propanediol (1,3-PD) has numerous applications in polymers, medicines, lubricants, food and cosmetics [14]. Its microbial production has recently attracted much attention throughout the world due to its environmentally safe, high region specificity, cheaply available feedstock and relatively high theoretical molar yield [4]. Among various microbial productions of 1,3-PD, glycerol bioconversion to 1,3-PD by Klebsiella pneumoniae (K. pneumoniae) has been widely investigated due to its high productivity and yield since 1980s [18, 15]. However, in the process of practical production, it is difficult to obtain a high concentration in the microbial culture of glycerol to 1,3-PD by K. pneumoniae. Hence, it is an interest area to develop improved techniques to increase the productivity of 1,3-PD.

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This paper is dedicated to Professor Enmin Feng for his important contributions in control and optimization and on the occasion of his 75th birthday. The reviewing process of the paper was handled by Honglei Xu as Guest Editor.
During the bioconversion of glycerol to 1,3-PD by *K. pneumoniae*, the most efficient cultivation method appears to be a fed-batch culture which corrects pH by alkali addition with glycerol supply [20]. The fed-batch culture of glycerol to 1,3-PD by *K. pneumoniae* begins with a batch culture, then batch-fed glycerol and alkali is added into the bioreactor to provide sufficient nutrition and maintain a suitable environment for cells growth.

However, glycerol fermentation by *K. pneumoniae* is a complex bio-process since the microbial growth is subjected to multiple inhibitions of substrate and products, such as glycerol, 1,3-PD, ethanol and so on [21]. Hence, it is important to decide when and how much to feed substrate during the fed-batch process. In order to provide sufficient nutrition, maintain a suitable environment for cells growth and reduce the simultaneous inhibition by excessive substrate, the glycerol concentration in the bioreactor must be maintained in a given range. Namely, an appropriate amount of glycerol and alkali should be fed instantaneously into the bioreactor when the glycerol concentration drop to a certain value. And after the feeding, the glycerol concentration in the bioreactor does not exceed another certain value. Furthermore, since the feeding process is very short, it is taken as an impulsive form in the actual fermentation process.

Nonlinear impulsive systems are mathematical model to simulate an evolving process with short-term perturbations [9]. Since impulsive differential equations have wide applications in many fields, such as HIV treatment [8], population dynamics [1], and so forth, the theory of such equations has been given extensive attention. However, most of the research refers to the optimization, optimal control and necessary conditions for the impulsive systems with either state jumps or impulsive moments independence of the state [8, 3, 13, 2, 6, 7, 16], etc., and the research on optimal control and optimality conditions for the impulsive system, in which both jumps size of state and impulsive moments are state-dependent, is very few. Liu etc [10, 11, 12] discussed modelling, optimal control, sensitivity analysis and parameter identification for time-delayed switched system and nonlinear time-delay system in fed-batch fermentation.

In this paper, a nonlinear impulsive controlled system in which the volume of feeding is the control function and both jumps size of state and impulsive moments are state-dependent, is proposed to formulate the fed-batch fermentation process. Subsequently, some important properties such as the existence, uniform boundedness, continuity and regularity of the solution are investigated. To maximize the concentration of 1,3-PD at the terminal time, an optimal control model involving the nonlinear state-dependent impulsive controlled system and subject to the continuous state inequality constraint and the control constraint is presented. The existence of optimal control is also obtained. In order to deduce the optimality conditions, the optimal control model is transcribed into an equivalent one by treating the constraints. Finally, the optimality conditions of the optimal control model are investigated by calculus of variations.

This paper is organized as follows. In Section 2, a nonlinear state-dependent impulsive controlled system of fed-batch culture is described and its optimal control model is developed. In Section 3, some important properties of the controlled system are investigated. In Section 4, the existence of optimal control is obtained, optimality conditions are also deduced by calculus of variations. Conclusions are provided in Section 5.
2. Controlled nonlinear state-dependent impulsive system. The fed-batch culture of glycerol to 1,3-PD by \textit{K. pneumoniae} begins with a batch culture, then glycerol and alkali are fed into the fermentor. In order to provide sufficient nutrition, maintain a suitable environment and simultaneously reduce the influence of the inhibition of glycerol to cells growth, when the glycerol concentration drop to the value \( \alpha \), glycerol and alkali are fed instantaneously into the bioreactor. Since the feeding process is very short, it is taken as impulsive form in the actual fermentation process. As the result, the concentrations of reactants in the bioreactor will jump when the feeding occurs. According to the actual process of fed-batch fermentation, we assume that

- (H1). The concentrations of reactants are uniform in the bioreactor, time delay and nonuniform space distribution are ignored.
- (H2). The feed rate of glycerol can be infinitely large. Moreover, the feeding velocity ratio \( r \) of alkali to glycerol is constant.
- (H3). In order to maintain a stable environment for cells growth, the volume of each feeding of glycerol and alkali is not less than a constant volume \( V_s \).

Under assumptions (H1) and (H2), the controlled nonlinear state-dependent impulsive system describing the process of fed-batch culture can be formulated as:

\[
\begin{aligned}
    \dot{x}(t) &= f(x(t)), & t &\notin \mathcal{I}, \\
    \Delta x(t) &= g(x(t^-), u(t)), & t &\in \mathcal{I}, & t &\in [0, t_f], \\
    x(0) &= \xi,
\end{aligned}
\]

where \( x_1(t), x_2(t), x_3(t), x_4(t), x_5(t) \) and \( x_6(t) \) are the concentrations of biomass, glycerol, 1,3-PD, acetate and ethanol and the broth volume at time \( t \) in bioreactor, respectively. \( x(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))^T \in \mathbb{R}^6 \) is the state variable, \( \xi \in \mathbb{R}^6 \) is the initial state of the culture. \( t_f \) is the terminal moment of the fermentation process. Let \( \mathcal{I} := \{ t_i \in (0, t_f) | x_2(t_i^-) = \alpha_s \} \) and \( t_i < t_{i+1} \).

\[
f(x(t)) = (\mu x_1(t), -q_2 x_1(t), q_1 x_1(t), q_4 x_1(t), q_5 x_1(t), x_6(t))^T \in \mathbb{R}^6.
\]

where the specific cellular growth rate \( \mu \), the specific consumption rate of glycerol \( q_2 \) and the specific product formation of 1,3-PD, acetic acid and ethanol \( q_3, q_4 \) and \( q_5 \) are expressed by the following equations [19].

\[
\mu = \mu_m \frac{x_2(t)}{x_2(t) + K_s} \prod_{i=2}^{5} (1 - \frac{x_i}{x_i^*})^{n_i},
\]

\[
q_2 = m_2 + \frac{\mu}{Y_2} + \Delta_2 \frac{x_2(t)}{x_2(t) + k_2},
\]

\[
q_3 = m_3 + \mu Y_3 + \Delta_3 \frac{x_2(t)}{x_2(t) + k_3},
\]

\[
q_4 = m_4 + \mu Y_4 + \Delta_4 \frac{x_2(t)}{x_2(t) + k_4},
\]

\[
q_5 = q_2 \left( \frac{b_1}{c_1 + \mu x_2(t)} + \frac{b_2}{c_2 + \mu x_2(t)} \right).
\]

Under anaerobic conditions at 37°C and pH 7.0, the critical concentrations of biomass, glycerol, 1,3-PD, acetic acid and ethanol for cells growth are \( x_1^* = 10 \text{ gL}^{-1}, \)
\( x_2^* = 2039 \text{ mmolL}^{-1}, \)
\( x_3^* = 1036 \text{ mmolL}^{-1}, \)
\( x_4^* = 1026 \text{ mmolL}^{-1} \) and \( x_5^* = 360.9 \text{ mmolL}^{-1} \), respectively. The maximum specific growth rate of cells \( \mu_m \) and Monod constant \( K_s \) take the values of 0.9179 h\(^{-1}\) and 0.1203 mmolL\(^{-1}\), respectively. \( b_1 = \)
0.01, \( b_2 = 2.11742 \), \( c_1 = 0.096 \), \( c_2 = 79.279 \), \( n_5 = 3 \) and the values of other parameters in Eqs. (3)-(6) are listed in Table 1.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( m_i )</th>
<th>( k_i )</th>
<th>( Y_i )</th>
<th>( \Delta_i )</th>
<th>( n_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.882</td>
<td>18.288</td>
<td>0.01604</td>
<td>11.432</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-4.307</td>
<td>23.1489</td>
<td>54.0998</td>
<td>13.9538</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>-1.552</td>
<td>81.8317</td>
<td>50.1265</td>
<td>2.296</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ \Delta x(t) = x(t^+) - x(t^-) = x(t) - x(t^-) \] is the jump size of state variable, \( x(t^+) \) and \( x(t^-) \) denote the right-hand and left-hand limits of \( x(t) \) at time \( t \), respectively.

\[ g(x(t^-), u(t)) := D(x(t^-), u(t))(-x_1(t^-), -x_2(t^-), -x_3(t^-), -x_4(t^-), -x_5(t^-), x_6(t^-) + u(t))^T \in \mathbb{R}^6, \]

where \( C_{a0} \) is the glycerol concentration in feed. \( r \) is the velocity ratio of adding alkali to glycerol. \( u(t) \) and is the volume of feeding glycerol and alkali, and \( D(x(t^-), u(t)) \) is the dilution rate when \( t \in I \), respectively, and \( D(x(t^-), u(t)) \) is defined as

\[ D(x(t^-), u(t)) = \frac{u(t)}{x_6(t^-) + u(t)}. \] (9)

Based on the actual culture process,

\[ U(t) = 0, \quad \text{if} \ t \not\in I, \quad \text{or} \quad U(t) = [V_s, V^*], \quad \text{if} \ t \in I. \]

where \( V_s \) and \( V^* \) are positive constants which denote the minimal and maximal volumes of adding glycerol, respectively. Now, we define the class of admissible control functions as

\[ U_{ad} := \{ u(\cdot) | u(t) \in U(t), \ t \in [0, t_f] \}. \] (10)

Note that the concentrations of biomass, substrate, products and the volume of culture broth are restricted in a certain range. The minimal and maximal volumes of culture broth are \( x_{6s} \) and \( x_{6u} \), which are given. So we consider the properties of the system on

\[ W_{ad} := \{ x \in \mathbb{R}^6 | x_i \in [x_{is}, x_{iu}], i \in I_6 \}. \] (11)

3. Properties of the controlled state-dependent impulsive system. The purpose of this section is to explore some important properties of the system (1), such as the existence, uniqueness and regularity of solution of the system.

**Definition 3.1.** [9] For any \( u \in U_{ad} \), a function \( x : [0, t_f] \to \mathbb{R}^6 \) is said to be a solution to the system (1), denoted by \( x(\cdot; u) \), if

- \( x(0; u) = \xi \) and \( x(t; u) \in W_{ad} \) for \( t \in [0, t_f] \);
- \( x(\cdot; u) \) is continuously differentiable with respect to \( u \) and satisfies \( \dot{x}(t; u) = f(x(t; u)) \) for \( t \in [0, t_f] \cup I \);
- \( x(t^-; u) \) exists, \( x(t; u) = x(t^-; u) + g(x(t^-; u), u(t)) \) for \( t \in I \).

**Definition 3.2.** A solution \( x(\cdot; u) \) to the system (1) is said to be regular if for each \( \{t_i, t_j\} \subset I \ (i \neq j) \), there exists a positive constant \( M \) such that \( |t_i - t_j| \geq M \).
Similar to the proof of Properties in [7], we have similar conclusion as follows:

**Property 1.** The functions \( f, g \) defined in (2) and (8) satisfy that

(a). The function \( f \) and its derivative with respect to \( x \), are continuous on \( t \in [0, t_f] \)
for each \( x \in W_{ad} \);

(b). \( f \) is Lipschitz continuous on \( W_{ad} \), that is, for any \( x^1, x^2 \in W_{ad} \), there exists a positive constant \( L_1 \) such that

\[
||f(x^1) - f(x^2)|| \leq L_1 ||x^1 - x^2||;
\]

(c). \( f \) is of linear growth, that is, there exists a positive constant \( K_1 \) such that

\[
||f(x)|| \leq K_1(||x|| + 1), \quad \forall x \in \mathbb{R}^n;
\]

(d). \( g \) is Lipschitz continuous in both the variables and bounded on \( W_{ad} \times U_{ad} \),
where \( ||\cdot|| \) is the Euclidean norm.

**Property 2.** For any \( u \in U_{ad} \), the system (1) undergoes an impulsive effect a finite number of times, i.e., the system (1) is non-Zeno, that is, for each \( \{t_i, t_j\} \subset \mathcal{I} \ (i \neq j) \), there exists a positive constant \( M \) such that \( |t_i - t_j| \geq M \).

Proof. For any \( u \in U_{ad} \) and each \( \{t_i, t_j\} \subset \mathcal{I} \ (i \neq j) \), since \( f \) is continuous on \( W_{ad} \) for each \( t \in [t_i, t_i+1] \subset [0, t_f] \), there exists a positive constant \( M' \) such that

\[
||f(x(t; u))|| \leq M', \quad \text{for each } t \in [t_i, t_i+1].
\]

It is obtained that

\[
|t_i - t_j| \geq |t_i - t_{i+1}| \geq \frac{1}{M'} \left| D(x(t_i^-; u), u(t_i)) \left( \frac{C_0}{1 + r} - \alpha_s \right) \right|
\]

\[
\geq \frac{1}{M'} \left| D(x(t_i^-; V_*), V_*) \left( \frac{C_0}{1 + r} - \alpha_s \right) \right|.
\]

Take \( M = \frac{1}{M'} \left| D(x(t_i^-; V_*), V_*) \left( \frac{C_0}{1 + r} - \alpha_s \right) \right| \), the proof is completed.

In view of Property 2, for each \( u \in U_{ad} \), the corresponding set of impulsive moments can be denoted by \( \mathcal{I}_n := \{t_1, \cdots, t_m\} \) such that

\[
0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = t_f.
\]

**Theorem 3.3.** For any \( u \in U_{ad} \), the nonlinear state-dependent impulsive system (1) has a unique regular solution \( x(\cdot; u) \), which can be written as

\[
x(t; u) = \begin{cases}
\xi + \int_{t_0}^t f(x(s))ds, & t \in [t_0, t_1),
\xi + \sum_{j=1}^i g(x(t_j^-), u(t_j)) + \sum_{j=1}^{i-1} \int_{t_j}^{t_j^-} f(x(s))ds + \int_{t_i}^t f(x(s))ds, & t \in [t_i, t_{i+1}], \ i = 1, \cdots, m.
\end{cases}
\]
Proof. The proof can be easily obtained from Property 1 and Property 2.

**Theorem 3.4.** The solution $x(\cdot; u)$ of the system (1) with given initial value $\xi$ is uniformly bounded.

Proof. In view of Property 1, Property 2 and Theorem 3.3, for each $u \in U_{ad}$, the corresponding set of impulsive moments $I_u := \{t_1, \ldots, t_m\}$ such that

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \tau,$$

we obtain that:

If $t \in [0, t_1)$, then, in the light of Gronwall inequality, it follows that

$$||x(t; u)|| \leq ||\xi|| + \int_0^t ||f(x(s))||ds$$

$$\leq ||\xi|| + \int_0^t K_1(||x(s)|| + 1)ds$$

$$\leq ||\xi|| + \int_0^t K_1(||x(s)|| + 1)ds$$

$$\leq (||\xi|| + K_1t_1)\exp(K_1t_1) := N_1.$$

If $t \in [t_1, t_2)$, then there exists a positive constant $K_2$ such that

$$||x(t; u)|| \leq ||\xi|| + ||g(x(t_1^{-}), u(t_1))|| + \int_{t_1}^t ||f(x(s))||ds$$

$$\leq ||\xi|| + K_2 + \int_{t_1}^t K_1(||x(s)|| + 1)ds$$

$$\leq (||\xi|| + K_2 + K_1t_2)\exp(K_1t_2) := N_2.$$

Similar as the above discussion, if $t \in [t_m^+, t_f)$, then there exists a positive constant $K_{m+1}$ such that

$$||x(t; u)|| \leq ||\xi|| + \sum_{j=1}^m ||g(x(t_j^{-}), u(t_j))|| + \int_{t_m}^t ||f(x(s))||ds$$

$$\leq ||\xi|| + K_{m+1} + \int_{t_m}^t K_1(||x(s)|| + 1)ds$$

$$\leq (||\xi|| + K_{m+1} + K_1t_f)\exp(K_1t_f) := N_{m+1}.$$

Take $N = \max\{N_1, N_2, \cdots, N_{m+1}\}$, then we have $||x(t; u)|| \leq N$, $\forall t \in [0, t_f]$. The proof is completed.

Since the continuity of solution of impulsive differential system with respect to the parameter is different from that of ordinary differential system, similar to Definition 2.3.1 in [9], we give the following definition.

**Definition 3.5.** A solution $x(\cdot; u)$ to the system (1) is said to have continuous dependence relative to the control function $u$ if given any $\epsilon > 0$, there is a $\delta > 0$ and a closed set $T_{\epsilon} \subset [0, t_f]$ such that $m([0, t_f]\setminus T_{\epsilon}) < \epsilon$ and

$$||x(t; u_1) - x(t; u)|| < \epsilon, \quad t \in T_{\epsilon}$$

provided $||u_1 - u|| < \delta$, where $m$ denote the Lebesgue measure.

The impulsive moments of the system (1) are state-dependent, which are not preassigned. So, in order to prove the continuous dependence on control function $u$, the general Euclidean norm cannot give us the desired result, we need the Skorohod topology defined in [5].
In [17], the author constructs a time scaling function $\lambda(\cdot)$ to prove the continuous dependence of the solution in nonlinear impulsive switching system on different parameters. Similarly, in our paper, we also adopt the method.

Let $\Lambda$ denote the class of strictly increasing, continuous mappings of $[t_0, t_f]$ onto itself, $\mathcal{D} = \mathcal{D}[t_0, t_f]$ be the space of functions $x$ on $[t_0, t_f]$ that are right-continuous and have left limits.

If $\lambda \in \Lambda$, then $\lambda(t_0) = t_0$ and $\lambda(t_f) = t_f$. For $x, y \in \mathcal{D}$, we define the Skorohod topology as:

$$\inf \{ \epsilon > 0 \mid \exists \lambda \in \Lambda \text{ such that } \sup_t |\lambda(t) - t| \leq \epsilon \text{ and } \sup_t |x(t) - y(\lambda(t))| \leq \epsilon \}. $$

Here, $\lambda$ is kind of time scaling, it can be used to compare the different solution with respect to the different control function.

![Figure 1. Comparison of two solutions with same initial data and different parameters.](image)

**Theorem 3.6.** If $x(\cdot; u)$ is a solution of the system (1) with given initial value $\xi$, then it is continuous in $u$.

**Proof.** In view of Property (2), without loss of generality, we prove the theorem for a special case showed in Figure 1, that is, the number of impulsive moments of the solution $x(t, u)$ is one more time than that of the solution $x(t, u')$; furthermore, $x(t, u)$ has an impulsive moment at the final time $t_f$. We construct the time scaling function $\lambda(\cdot)$ as follows:

$$\lambda(t) = \begin{cases} 
  t_0(u') + \frac{t_1(u') - t_0(u')}{t_1(u) - t_0(u)} (t - t_0(u)), & t \in [t_0(u), t_1(u)], \\
  \vdots \\
  t_{i-1}(u') + \frac{t_i(u') - t_{i-1}(u')}{t_i(u) - t_{i-1}(u)} (t - t_{i-1}(u)), & t \in [t_{i-1}(u), t_i(u)], \\
  \vdots \\
  t_{m-1}(u') + \frac{t_m(u') - t_{m-1}(u')}{t_m(u) - t_{m-1}(u)} (t - t_{m-1}(u)), & t \in [t_{m-1}(u), t_m(u)], \\
  t_m(u') + \frac{t_f - t_m(u')}{t_f - t_m(u)} (t - t_m(u)), & t \in [t_m(u), t_f]. 
\end{cases}$$

(13)
Based on the above definition of \(\lambda(t)\), the time scale of different impulsive moments is transformed to be consistent. So, it is only needed that proving the desired result on the time interval \([t_m, t_f]\). For different control functions \(u\) and \(u'\), we have

\[
x(t_f^+, u) = x(t_f^-, u) + \Delta x(t_f),
\]

\[
x(\lambda(t), u') = x(t_m(u'), u') + \Delta x(t_m(u')) + \int_{t_m(u')}^{t} \dot{x}(s) ds,
\]

while \(u' \rightarrow u\), \(t_m(u') \rightarrow t_f\). On the basis of implicit function theorem, it can be easily obtained that \(\Delta x(t_m(u')) \rightarrow \Delta x(t_f)\).

Summing up the above, the desired result can be obtained. \(\square\)

4. Optimal control model and optimality conditions. Now, we define the set of the solutions to the system (1) as follows:

\[
S_0 := \{x(\cdot; u) \in \mathbb{R}^6 \mid x(t; u) \text{ is the solution to (1) with initial state } \xi \text{ for any } u \in U_{ad} \text{ and } t \in [0, t_f]\}. 
\]

Since the concentrations of biomass, substrate, products and the volume of culture broth are restricted in \(W_{ad}\), the set of admissible solutions is denoted by

\[
S := \{x(\cdot; u) \in S_0 \mid x(t; u) \in W_{ad}, \; t \in [0, t_f]\}. 
\]

Furthermore, we denote the set of feasible control functions as

\[
U := \{u \in U_{ad} \mid x(\cdot; u) \in S\}. 
\]

To improve 1,3-PD production, feeding appropriate amounts of glycerol and alkali is required in the fed-batch culture. The optimal control problem of the whole bioprocess can be described as follows:

\[
\min J(u) := -x_3(t_f; u) \quad \text{s.t.} \quad u \in U,
\]

where \(x_3(\cdot; u)\) is the third component of the solution to the system (1) and we call it Problem (OSICP).

Since the Problem(OSICP) is an optimal control problem with the inequality constraints of continuous state, and the impulsive moments are not determined a priori, this follows from the degree of freedom allowed on the state jumps, i.e., variations of jump sizes of state induces variations of the impulsive moments. Note also that the jump sizes of state are dependent on the state and the volumes of feeding glycerol and alkali at the impulsive moments. Therefore, the existing methods cannot be directly used to solve the problem. In order to handle with the constraints, we introduce a new state variable \(x_7\) satisfying

\[
\begin{align*}
\dot{x}_7(t) &= h(x(t), u(t)), \quad t \in [0, t_f], \\
x_7(0) &= 0,
\end{align*}
\]

where

\[
h(x(t), u(t)) = \sum_{i=1}^{6} \left[(x_{i*} - x_i)^2 \eta(x_{i*} - x_i) + (x_i - x_i^*)^2 \eta(x_i - x_i^*)\right] \\
&\quad + (a_*(t) - u(t))^2 \eta(a_*(t) - u(t)) + (u(t) - a^*(t))^2 \eta(u(t) - a^*(t)),
\]

and

\[
\eta(\zeta) = \begin{cases} 0, & \zeta \leq 0, \\
1, & \zeta > 0. \end{cases}
\]
Obviously, $x(t)$ and $u(t)$ satisfy the inequality constraints of continuous state in the system iff

$$x_7(t_f) = 0.$$ 

Let $\hat{x} := (x_1, \ldots, x_6, x_7)^T$, $\tilde{f}(\hat{x}(t), u(t)) := (f^T(x(t)), h(x(t), u(t)))^T$ and $\tilde{g}(\hat{x}(t^-), u(t)) := (g^T(x(t^-), u(t)), 0)^T$. Given $u \in U_{ad}$ and the corresponding set of impulsive moments $T_u := \{t_1, \ldots, t_m\}$ ($0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = t_f$), let

$$\hat{x}(t) = \begin{cases} 
\tilde{f}(\hat{x}(t), u(t)) := F^0(\hat{x}(t), u(t)), & t \in [t_0, t_1], \\
\tilde{f}(\hat{x}(t), u(t)) + \tilde{g}(\hat{x}(t^-), u(t_i)) := F^i(\hat{x}(t), u(t)), & t \in [t_i, t_{i+1}], \; i = 1, \ldots, m.
\end{cases}$$

where $\hat{x}(\cdot)(i = 0, \ldots, m)$ is considered as a weak derivative of $\tilde{x}(\cdot)$ defined on the interval $[t_i, t_{i+1}]$. It is obvious that $\tilde{x}(\cdot)$ consists of continuous part defined at $t \in (t_i, t_{i+1})$ and jumps at $t_i$. Then the Problem(OSICP) can be approximated by the Problem OSICP($\gamma$)

$$\text{OSICP}(\gamma) \quad \min J_\gamma(u) := -\tilde{x}_3(t_f; u) + \gamma \tilde{x}_7(t_f; u)$$

s.t. $\tilde{x}(t) = F^i(\tilde{x}(t), u(t))$, $i = 0, \ldots, m$,

$$\tilde{x}_2(t^-_i) = \alpha_+, \; i = 1, \ldots, m,$$

$$u(t) \in \mathbb{R}^1, \; t \in [0, t_f],$$

where $\gamma > 0$ is a penalty parameter.

**Theorem 4.1.** If $u \in U_{ad}$ is the optimal control function of (OSICP($\gamma$)), $T_u = \{t_i\}_{i=1}^m$ is the corresponding sequence of impulsive moment, then there exist vector functions $\lambda^i(t) = (\lambda^i_1(t), \ldots, \lambda^i_7(t))^T$, $i = 0, 1, \ldots, m$, and a vector $b = (b_1, \ldots, b_m)^T$ such that the following conditions hold

1. $\dot{\lambda}^i(t) = -H^i_\lambda(\hat{x}(t), u(t), \lambda^i(t))$ a.e. on $[t_i, t_{i+1}]$, $i = 0, \ldots, m$. 
2. $\lambda^m(t_f) = -e_3 + \gamma e_7$. 
3. $\lambda^{i-1}(t_i) = \lambda^i(t_i) + b_i e_2$, $i = 1, \ldots, m$. 
4. $H^{i-1}(\tilde{x}(t_i), u(t_i), \lambda^{i-1}(t_i)) = H^i(\tilde{x}(t_i), u(t_i), \lambda^i(t_i))$, $i = 1, \ldots, m$. 
5. $H^i_\lambda(\hat{x}(t), u(t)) = 0$, a.e. on $[t_i, t_{i+1}]$, $i = 0, \ldots, m$. 

Here $e_2, e_3$ and $e_7$ are unit vector of $\mathbb{R}^7$, and the $i$th component of $e_i$ is 1.

$$H^i(\hat{x}(t), u(t), \lambda^i(t)) := (\lambda^i(t), F^i(\hat{x}(t), u(t)))$$

is a “partial” Hamiltonian on $[t_i, t_{i+1}]$, $i = 0, \ldots, m$, and

$$H^i_\lambda(\hat{x}(t), u(t), \lambda^i(t)) = \begin{cases} 
(\lambda^i(t), \tilde{f}(\hat{x}(t), u(t))), & t \in [t_0, t_1], \\
(\lambda^i(t), \tilde{f}(\hat{x}(t), u(t)) + \tilde{g}(\hat{x}(t^-), u(t_i))), & t \in [t_i, t_{i+1}], \; i = 1, \ldots, m.
\end{cases}$$

**Proof.** The augmented Lagrangian can be written as

$$\mathcal{L} = -\tilde{x}_3(t_f) + \gamma \tilde{x}_7(t_f) + \sum_{i=1}^m b_i (\tilde{x}_2(t^-_i) - \alpha_+)$$

$$+ \sum_{i=0}^m \int_{t_i}^{t_{i+1}} (H^i(\dot{x}(t)) - \lambda^i_7(t)\dot{x}(t))dt,$$

(23)
where $H^i(\ddot{x}(t)) = H^i(\ddot{x}(t), u(t), \lambda^i(t))$.

Let $L_H = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} (H^i(\dot{x}(t)) - \lambda^iT(t)\dot{x}(t))dt$, then the increment of $L_H$ with respect to $\ddot{x}$ can be written as

$$\triangle_{\ddot{x}} L_H = \mathcal{L}_H(\ddot{x} + p) - \mathcal{L}_H(\ddot{x})$$

$$= \sum_{i=0}^{m} \left[ \int_{t_i}^{t_{i+1}+dt_i} (H^i(\dot{x}(t) + p(t)) - \lambda^iT(t)(\dot{x}(t) + p(t)))dt - \int_{t_i}^{t_{i+1}} (H^i(\dot{x}(t)) - \lambda^iT(t)\dot{x}(t))dt \right],$$

where $p(t)$ is a continuously differentiable function, $dt_i$ is a small time increment. After integration by parts and rearrangement, the above equation can be written as

$$\triangle_{\ddot{x}} L_H = \sum_{i=0}^{m} \left[ \int_{t_i}^{t_{i+1}} (H^i(\dot{x}(t) + p(t)) - H^i(\ddot{x}(t)) + \dot{x}^T(t)p(t))dt - (H^i(\ddot{x}(t)) - \lambda^iT(t)\dot{x}(t))|_{t_i}^{t_{i+1}} \right].$$

Using Taylor’s theorem, we obtain the following expression up to first order

$$\triangle_{\ddot{x}} L_H = \sum_{i=0}^{m} \left[ \int_{t_i}^{t_{i+1}} \left( H^i(\dot{x}(t)) + \dot{x}^T(t)p(t) \right) dt - (H^i(\ddot{x}(t)) - \lambda^iT(t)\dot{x}(t))|_{t_i}^{t_{i+1}} \right].$$

In view of Eq. (25), making use of the relation $d\ddot{x}(t_i) = p(t_i) + \dot{x}(t_i)dt_i$, the first order variation of $L_H$ with respect to $\ddot{x}$ can be written as

$$\delta_{\ddot{x}} L_H = \sum_{i=0}^{m} \left[ \int_{t_i}^{t_{i+1}} \left( H^i(\dot{x}(t)) + \dot{x}^T(t)p(t) \right) dt - (H^i(\ddot{x}(t)) - \lambda^iT(t)\dot{x}(t))|_{t_i}^{t_{i+1}} \right].$$

Hence, the first order variation of $L$ with respect to $\ddot{x}$ can be written as

$$\delta_{\ddot{x}} L = -e_3d\ddot{x}(t_f) + \gamma e_7d\ddot{x}(t_f) + \sum_{i=0}^{m} \left[ \int_{t_i}^{t_{i+1}} \left( H^i(\dot{x}(t)) + \dot{x}^T(t)p(t) \right) dt - (H^i(\ddot{x}(t)) - \lambda^iT(t)\dot{x}(t))|_{t_i}^{t_{i+1}} \right].$$

According to the calculus of variations, a necessary condition for a solution to be optimal is $\delta_{\ddot{x}} L = 0$. After rearrangement and setting to zero the coefficients of the independent increments yields the necessary conditions (18)-(21). The necessity of (22) can be shown analogously.
5. Conclusion. The paper proposed a state-dependent impulsive controlled system, contrasting with existing model to describe the fed-batch process. We prove some properties, i.e., the existence, uniform boundedness, continuity and regularity of the solution. The aim of feeding glycerol is to obtain as high concentration of 1,3-PD as possible at terminal time. Hence, the optimal state-dependent impulsive control model (OSICP) is proposed to maximize the production of 1,3-PD. Finally, we deduced an optimality condition which plays an important role in judging whether a control is the optimal control. In the future, some algorithm research will be done on solving (OSICP(γ)).

REFERENCES


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E-mail address: bangysen@163.com
E-mail address: wwxjj@163.com
E-mail address: liu_chongyang@yahoo.com