Symmetric Criticality in Classical Field Theory

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Rotationally invariant solutions of the Laplace equation

\[ \nabla^2 \varphi = 0 \]

Laplace equation

\[ r = \sqrt{x^2 + y^2 + z^2} \]

Rotationally invariant function

\[ \varphi(x, y, z) = q(r) \]

\[ \nabla^2 \varphi = 0 \quad \iff \quad q'' + \frac{2}{r} q' = 0 \]

Symmetry reduction of Laplace equation
A variational approach

\[ S[\varphi] = -\frac{1}{2} \int_M dV \nabla \varphi \cdot \nabla \varphi \]

Variational principle for Laplace equation

\[ \frac{\delta S}{\delta \varphi} = 0 \iff \nabla^2 \varphi = 0 \]

Symmetry reduction of variational principle

\[ S[q] = -2\pi \int_a^b dr r^2 (q')^2 \]

Symmetry reduction of Euler-Lagrange equation

\[ \frac{\delta S}{\delta q} = 0 \iff q'' + \frac{2}{r} q' = 0 \]
Two procedures:

Action $\rightarrow$ Euler-Lagrange equations $\rightarrow$ Symmetry reduction

Action $\rightarrow$ Symmetry reduction $\rightarrow$ Euler-Lagrange equations

Are they equivalent?
• **Weyl (1917)** - Schwarzschild solution from reduction of Einstein-Hilbert action.

• **Pauli (1929)** - Same again.

• **Hawking (1969), MacCallum and Taub (1972)** - Procedure fails for certain symmetry groups.

• **Lovelock (1972)** - generalized Weyl’s treatment to more actions.

• **Palais (1979)** - The Principle of Symmetric Criticality (PSC)

• **Anderson, Fels, Torre (1996, 2000, 2002)** - Symmetric reduction of differential equations, Lagrangians and PSC.
Einstein equations for static, spherically symmetric spacetimes

\[
\begin{align*}
\text{ds}^2 &= -\mu(r)dt^2 + \lambda(r)dr^2 + r^2d\Omega^2 \\
S[\lambda, \mu] &= \int R \, dV = 8\pi \int dr \sqrt{\frac{\mu}{\lambda^3}} \left( r\lambda' - \lambda + \lambda^2 \right) + \text{t.d.} \\
\delta S = 0 &\iff \begin{cases}
  r\lambda' - \lambda + \lambda^2 = 0 \\
  r\mu' + \mu - \mu\lambda = 0
\end{cases} \iff \text{Einstein Field Equations}
\end{align*}
\]
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Hawking (1969)

\[ ds^2 = -e^\mu(r)dt^2 + e^\lambda(r)dr^2 + r^2 d\Omega^2 \]

\[ S[\lambda, \mu] = \int R dV = 8\pi \int dr \int_\mu \lambda^3 \left( r\lambda - \lambda + \lambda^2 \right) \delta S = 0 \]

\[ \Rightarrow \int r\lambda - \lambda + \lambda^2 \delta S = 0 \]

\[ r\mu + \mu - \mu\lambda = 0 \]

\[ \Rightarrow \text{Einstein Field Equations} \]

Spatially homogeneous cosmology

\[ M = \mathbb{R} \times G \]

\[ G = 3\text{-d Lie group} \]

Spacetime metric \( g \) is \( G \)-invariant \( \Rightarrow g = g(q), \) \( q = 10 \) functions of time.

Reduced Einstein action:

\[ S[q] = \int R(g(q)) dV(g(q)) \]

only yields correct equations of motion when \( G \) is \emph{unimodular}. 

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The Principle of Symmetric Criticality

“...critical symmetric points are symmetric critical points.”

For $G$-invariant $f$, $p$:

$$
\delta_1 f = 0 \Rightarrow \delta_2 f = 0
$$
The Principle of Symmetric Criticality

“...critical symmetric points are symmetric critical points.”

\[
\mu : G \times M \to M
\]

Group action

\[
\Sigma = \{p \in M | gp = p, \forall g \in G\}, \quad i : \Sigma \to M
\]

Symmetric points

\[
f : M \to \mathbb{R}, \quad \mu^*_g f = f, \quad \forall g \in G
\]

Invariant function

\[
df|_p = 0, \quad p \in M
\]

Critical point

\[
p \in \Sigma, \quad (df)_p = 0 \iff d(i^*f)|_p = 0
\]

PSC

Palais (1979)
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The Principle of Symmetric Criticality

“...critical symmetric points are symmetric critical points.”

• Setting: $N, \Sigma$ are Banach $G$-manifolds.

• Sufficient conditions for PSC:
  - $G$ compact
  - $G$ acts isometrically on Riemannian $M$
The Principle of Symmetric Criticality

“...critical symmetric points are symmetric critical points.”

Necessary and Sufficient:
(Linearizable G action)

\[ \Sigma^* \cap \Sigma^0 = 0 \]

- G-invariant linear functionals
- Annihilator of G-invariant vectors
The Principle of Symmetric Criticality

“...critical symmetric points are symmetric critical points.”

\[ p \in \Sigma \]

\[ df \bigg|_p \in \Sigma^* \]

\[ d(\iota^* f) = 0 \iff df(v) = 0 \quad \forall \ v \in T_p^* \Sigma \quad \implies \quad df \bigg|_p \in \Sigma^0 \]

\[ \Sigma^* \cap \Sigma^0 = 0 \quad \implies \quad df \bigg|_p = 0 \]
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• Palais (1979) - The Principle of Symmetric Criticality (PSC)

• Anderson, Fels, Torre (1996, 2000, 2002) - Symmetric reduction of differential equations, Lagrangians and a local version of PSC.
Why a local formulation of PSC is desirable...

• Focus on critical points forces one to consider:
  
  - function spaces
  - boundary conditions
  - asymptotic conditions
  - existence of the functionals
  - existence of G-actions
  - existence of (G-invariant) critical points
  - etc.

• Focus on field equations $\Rightarrow$ this is largely avoided.
Symmetry Reduction

• Fields
• Field Equations
• Lagrangians
Reduction: the fields

Rotation group action on a vector field:

\[ \varphi(\vec{x}) \implies R \cdot \varphi(R^{-1} \cdot \vec{x}) \]

Rotationally invariant vector field:

\[ \varphi(\vec{x}) = q(|\vec{x}|) \vec{x} \]
Reduction: the fields
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\[ \pi: E \to M, \quad \varphi: M \to E \] ---- Bundle of fields
Reduction: the fields

\[ \pi: E \to M, \quad \varphi: M \to E \]

\[ \varphi(x) \to g \cdot [\varphi(g^{-1} \cdot x)] \]

------ Bundle of fields

------ Group action
Reduction: the fields

\[ \pi: E \to M, \quad \varphi: M \to E \]

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\[ \varphi(x) = g \cdot [\varphi(g^{-1} \cdot x)] \]

------ Bundle of fields

------ Group action

------ G-invariance
Reduction: the fields

\[ \pi: E \to M, \quad \varphi: M \to E \]

\[ \varphi(x) \to g \cdot [\varphi(g^{-1} \cdot x)] \]

\[ \varphi(x) = g \cdot [\varphi(g^{-1} \cdot x)] \]

\[ \varphi(x) = gx \cdot [\varphi(x)] \]

------ Bundle of fields

------ Group action

------ G-invariance

------ Isotropy \( G_x \)
Reduction: the fields

\[ \pi: E \rightarrow M, \quad \varphi: M \rightarrow E \]  

------ Bundle of fields

\[ \varphi(x) \rightarrow g \cdot [\varphi(g^{-1} \cdot x)] \]  

------ Group action

\[ \varphi(x) = g \cdot [\varphi(g^{-1} \cdot x)] \]  

------ G-invariance

\[ \varphi(x) = g_x \cdot [\varphi(x)] \]  

------ Isotropy \( G_x \)

\[ \pi: \kappa(E) \rightarrow M, \quad \varphi: M \rightarrow \kappa(E) \]  

------ Fiber reduction
Reduction: the fields

\[ \pi: E \to M, \quad \varphi: M \to E \]

\[ \varphi(x) \to g \cdot [\varphi(g^{-1} \cdot x)] \]

\[ \varphi(x) = g \cdot [\varphi(g^{-1} \cdot x)] \]

\[ \varphi(x) = g_x \cdot [\varphi(x)] \]

\[ \pi: \kappa(E) \to M, \quad \varphi: M \to \kappa(E) \]

\[ \hat{\pi}: \kappa(E)/G \to M/G \]

------ Bundle of fields

------ Group action

------ G-invariance

------ Isotropy \( G_x \)

------ Fiber reduction

------ Bundle of invariant fields
Reduction: the fields

\[ \pi : E \to M, \quad \varphi : M \to E \]  
\[ \varphi(x) \to g \cdot [\varphi(g^{-1} \cdot x)] \]  
\[ \varphi(x) = g \cdot [\varphi(g^{-1} \cdot x)] \]  
\[ \varphi(x) = g_x \cdot [\varphi(x)] \]  

\[ \pi : \kappa(E) \to M, \quad \varphi : M \to \kappa(E) \]  
\[ \hat{\pi} : \kappa(E)/G \to M/G \]  
\[ q : M/G \to \kappa(E)/G \]  
\[ \varphi = \varphi(q) \]  

------ Bundle of fields
------ Group action
------ G-invariance
------ Isotropy \( G_x \)
------ Fiber reduction
------ Bundle of invariant fields
------ G-invariant fields
Reduction: the equations

Field Equations:

\[ \nabla^2 \phi = 0 \]

\[ \nabla^2 (R \cdot \phi) = R \cdot (\nabla^2 \phi) \]

Reduced Equations:

\[ \phi(x) = q(|x|) x \]

\[ \nabla^2 \phi = \left( q'' + \frac{4}{|x|} q' \right) x \]
Reduction: the equations

G-invariant field equations

\[ \Delta[\varphi] = 0, \quad g^{-1} \cdot \Delta[g \cdot \varphi] = \Delta[\varphi] \]

restricted to G-invariant fields become G-invariant fields and, by the same kind of reduction, can be viewed as equations for the reduced fields.

\[ q: M/G \to \kappa(E)/G \]

\[ \hat{\Delta}[q] = 0 \]

Reduction:
- number of independent variables
- number of dependent variables
- number of field equations
Reduction: the Lagrangian

\[ \pi: M \rightarrow M/G \]

Dimension: \( n \) \( n - q \)
Reduction: the Lagrangian

\[ \pi: M \rightarrow M/G \]

Dimension: \( n \quad n - q \)

\[ \lambda = \lambda[\varphi] \quad \text{G-invariant Lagrangian n-form} \]
Reduction: the Lagrangian

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Dimension: \( n \) \( n - q \)

\[ \lambda = \lambda[\phi] \quad \text{G-invariant Lagrangian n-form} \]

\[ \xi \quad \text{G-invariant skew tensor of type (q,0) tangent to orbits} \]
Reduction: the Lagrangian

$$\pi: M \rightarrow M/G$$

Dimension: \( n \), \( n - q \)

$$\lambda = \lambda[\varphi]$$ \quad G-invariant Lagrangian n-form

$$\xi$$ \quad G-invariant skew tensor of type \((q,0)\) tangent to orbits

$$\hat{\lambda}[q]$$ \quad Reduced Lagrangian \((n-q)\) form on \( M/G \)

$$\pi^* \hat{\lambda} = \xi \cdot \lambda$$
Reduction: the Lagrangian

**Rotationally invariant Lagrangian:**

\[ \lambda = -\frac{1}{2} \| \nabla \phi \|^2 dx \wedge dy \wedge dz \]

**That skew tensor:**

\[ \xi = \frac{r}{2z} (y \partial_z - z \partial_y) \wedge (z \partial_x - x \partial_z) \]

**Reduced Lagrangian:**

\[ \hat{\lambda} = -\frac{1}{2} r^2 \left( r^2 q'^2 + 2rqq' + 3q^2 \right) dr \]
Principle of Symmetric Criticality - Local version
(Anderson, Fels, Torre)
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\[ \delta \lambda = E(\lambda) \cdot \delta \varphi + d\eta \]

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First variation.
Euler-Lagrange operator \( E \).
Boundary term \( d\eta \).

\[ E(\lambda)[\varphi] = 0 \iff \Delta[\varphi] = 0 \]
Principle of Symmetric Criticality - Local version
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\[ \delta \lambda = E(\lambda) \cdot \delta \varphi + d\eta \]

First variation. Euler-Lagrange operator \( E \).
Boundary term \( d\eta \).

\[ E(\lambda)[\varphi] = 0 \iff \Delta[\varphi] = 0 \]

PSC: There exists \( \xi \) such that for any \( G \)-invariant Lagrangian

\[ E(\hat{\lambda})[q] = 0 \iff \hat{\Delta}[q] = 0 \]
The crux of the matter...

\[ \delta \hat{\lambda}[q] = \]

\[ E(\hat{\lambda})[q] \cdot \delta q + d\hat{\eta}[q, \delta q] = \xi - E(\lambda)[\varphi(q)] \cdot \delta \varphi(q) + \xi - d\eta[\varphi, \delta \varphi] \]
The crux of the matter...

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\[ E(\hat{\lambda})[q] \cdot \delta q + d\hat{\eta}[q, \delta q] = \xi - E(\lambda)[\varphi(q)] \cdot \delta \varphi(q) + \xi - d\eta[\varphi, \delta \varphi] \]

For PSC we need:

\[ (I) \quad \xi - d\eta[\varphi(q), \delta \varphi(q)] = d\hat{\eta}[q, \delta q] \]

\[ (II) \quad \xi - E(\lambda)[\varphi(q)] \cdot \delta \varphi(q) = 0 \quad \implies \quad E(\lambda)[\varphi(q)] = 0 \]
Theorem

Necessary and sufficient condition for

\[ \xi - d\eta [\varphi(q), \delta \varphi(q)] = d\hat{\eta}[q, \delta q] \]

Assuming \( \eta \) is \( G \)-invariant, Lie algebra cohomology of \( G \) relative to \( G_x \), \( \forall x \in M \), is non-vanishing at degree \( q = \) dimension of orbits.

\[ H^q(G, G_x) \neq 0, \quad \forall x \in M \]
Theorem

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Assuming \( \eta \) is \( G \)-invariant, Lie algebra cohomology of \( G \) relative to \( G_x \), \( \forall x \in M \), is non-vanishing at degree \( q \) = dimension of orbits.

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\( G \)-invariant closed, not exact \( q \)-forms on \( G/G_x \).
Theorem

Necessary and sufficient conditions such that, for all $\lambda$,

$$\xi - E(\lambda)[\varphi(q)] \cdot \delta\varphi(q) = 0 \implies E(\lambda)[\varphi(q)] = 0$$

- $V_x$ Vector space of $G_x$ - invariant field variations at $x$
- $V_x^*$ Dual space
- $V_x^0$ Annihilator of $V_x$
Theorem

Necessary and sufficient conditions such that, for all $\lambda$,

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$V_x$ Vector space of $G_x$ - invariant field variations at $x$

$V_x^*$ Dual space

$V_x^0$ Annihilator of $V_x$

$$V_x^* \cap V_x^0 = 0$$
Necessary and Sufficient Conditions for PSC*
(Anderson, Fels, Torre)

(1) \( H^q(G, G_x) \neq 0 \)

(2) \( V_x^* \cap V_x^0 = 0 \)

*Assuming G-invariant boundary form.
Special Cases
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- PSC is valid for compact $G$. \textit{(e.g., rotational symmetry)}
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Special Cases

• PSC is valid for compact $G$. (e.g., rotational symmetry)

• If $G$ acts freely on $M$, PSC $\iff$ unimodular $G.$
  (e.g., Hawking)

• If $G$ acts transversely on $E$, then
  PSC $\iff$ Lie algebra cohomology condition.
  (e.g., Anderson & Fels)
Special Cases

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• If $G$ acts transversely on $E$, then
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• If there is a $G$-invariant Riemannian metric on $M$, then
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- If there is a $G$-invariant Riemannian metric on $M$, then
  PSC $\iff$ Lie algebra cohomology condition.

- For tensor fields in spacetime, intersection condition can fail only for null rotation subgroups of the Lorentz group.