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Article info
Article history:
Received 6 January 2011
Received in revised form 17 August 2011
Accepted 3 February 2012
Available online 17 April 2012

Keywords:
Climate change
Energy markets
Environmental externalities
Nonrenewable resources
Technological change

Abstract
We study how environmental regulation in the form of a cap on aggregate emissions from a fossil fuel (e.g., coal) interacts with the arrival of a clean substitute (e.g., solar energy). The cost of the substitute is assumed to decrease with cumulative use because of learning-by-doing. We show that optimal energy prices may initially increase because of pollution regulation, but fall due to learning, and rise again because of scarcity of the resource, finally falling after transition to the clean substitute. Thus nonrenewable resource prices may exhibit cyclical behavior even in a purely deterministic setting.

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1. Introduction

More than 85% of commercial energy today is supplied by the three major fossil fuels, namely coal, oil and natural gas. Each of these resources, in varying degrees, is a major contributor to environmental problems such as global warming. These resources are also nonrenewable. However, there are many clean substitutes such as solar and wind energy which are currently more expensive in the cost of producing a unit of electricity or usable heat. Empirical evidence suggests that the cost of these clean substitutes fall as they begin to acquire more market share (McDonald and Schattenholzer, 2001).

In this paper, we examine the substitution of a clean energy source for a polluting one in energy production. For example, solar or wind energy are clean but expensive substitutes for coal in electricity generation. We posit a scenario in which an extension to the Kyoto Protocol or another international agreement imposes a binding target for atmospheric carbon. A forward-looking social planner internalizes the intertemporal knowledge spillovers from using the clean...
technology. We ask: if there is significant learning in the clean substitute, how will that affect optimal energy prices and the process of substitution?

Many studies have looked at the problem of nonrenewable resource extraction as well as learning-by-doing in new technologies. However, the focus of our paper is on the role played by environmental regulation in this substitution process. Specifically, we ask how a cap on the stock of pollution or, equivalently, a carbon concentration target may affect the switch to the clean substitute when the latter exhibits learning effects.

We characterize this problem by assuming a fairly general cost specification for the nonrenewable resource and the learning technology. Unit extraction costs for coal increase with cumulative depletion. The average cost of solar energy is assumed to decrease with cumulative use but increase with the quantity supplied each period. For example, the unit cost of a solar panel may decline over time, the higher the number of panels built in the past. But at any given time, the unit cost is increasing and convex with respect to the number of units supplied. This is quite realistic because as more and more solar units are brought into the market at any moment, they may have to be deployed in regions that are less favorable to solar energy such as those with lower incidence of solar radiation or in dense urban areas with higher installation costs.

There are several optimal solutions to this model, which we describe in the paper. But the key result arises when the clean technology is used before regulation becomes binding. We show that in the initial period, energy prices rise because of scarcity and impending regulation. As soon as regulation becomes binding and the stock of pollution is at its maximum level, energy prices start falling. Clean energy use increases during this period but emissions cannot increase because of regulatory constraints. However there comes a time when resources are scarce enough that regulation no longer binds, and prices rise again, driven by the scarcity of the fossil fuel until it is no longer economical to mine higher cost deposits. This rise in prices also leads to an increased adoption of clean energy. Finally the polluting fossil fuel becomes too expensive to mine and the clean alternative takes over as the sole supplier of energy and once again, energy prices fall because of learning.

In standard models of Hotelling, the price of a nonrenewable resource rises until a clean substitute is used. If the substitute is available in infinite supply and fixed cost, energy prices rise until this transition and then stay constant. When learning in the backstop is included, energy prices rise until the resource is economically exhausted then fall once substitution to the backstop has taken place (e.g., see Oren and Powell, 1985). We show that with both learning and regulation, optimal energy prices may rise and fall successively.

This long-run non-monotonic behavior of energy prices is counter-intuitive and occurs because of the interplay of regulation, scarcity of the fossil fuel and learning in the clean technology. It occurs when it is optimal to deploy the clean technology before regulation binds or during it, although in the latter case, the rise and fall of energy prices is not as pronounced, as shown later in the paper.

A recent review of the empirical significance of the Hotelling (1931) model suggests that “its most important empirical implication is that market price must rise over time in real terms, provided that costs are time-invariant” (Livernois, 2008). Livernois also points out that empirical tests of the model have been generally unsuccessful. Our results suggest that in the long run, resource prices may exhibit significant structural variations driven by regulatory policy and market forces, which may result in alternating phases with secular upward or downward price movements.

Although for convenience, the paper is motivated in terms of coal and a clean substitute such as solar energy, it is equally applicable to other settings, such as the monopoly production of oil by a cartel such as OPEC with a competitive clean technology (e.g., a hydrogen car). The solution predicts that oil prices may rise, followed by a decline when emissions constraints become binding. They rise again when regulation ceases to bind, followed by an eventual decline when there is a complete transition to the clean substitute. What is surprising is that energy prices may start decreasing upon attaining the regulated level of emissions.

(footnote continued)
of all fossil fuels) are much more expensive than the substitutes available in electricity generation such as hydro and nuclear power. That is, relative to coal, oil and natural gas have strong comparative advantage in their respective uses.

Footnote continued: 3 This is consistent with historical evidence which suggests that cheaper production units will be brought online over time, gradually replacing more expensive units. For instance, McDonald and Schattenholzer (2001) report average cost reductions of 5–35% from a doubling of cumulative production in solar and wind energy generation. Duke and Kammen (1999) also find significant reductions in average cost from a rise in the cumulative production of solar panels.

Footnote continued: 4 Empirically, this may be the most plausible case. Carbon regulation is not yet binding in most energy markets and stylized facts suggest that clean substitutes such as solar and wind energy already occupy a small but fast-growing share of the energy market. For example the global market for solar photovoltaics was worth more than 17 billion US dollars in 2007, exhibiting a 62% growth from the previous year (EETimes, 2008). The wind turbine market is even bigger, with revenues of $36 billion in 2007. It accounted for a significant 30% of new power generation in the United States in 2007, which along with Germany and Spain has the highest installed capacity (EWEA, 2008). However the global share of energy mix for these two energy sources is still less than 1%. This is consistent with most empirical modeling of climate stabilization scenarios considered by the IPCC and the IEA to meet long run goals such as limiting climate change to 2°C. In calculating the optimal energy supply portfolio to respect a 450 ppm atmospheric CO2 constraint, the IEA found that use of emissions-free sources would need to ramp up immediately, despite the fact that atmospheric CO2 concentrations do not peak in their scenario until 2035. Importantly, the plausibility of this scenario does not depend on the costs of renewables today absent subsidies—this scenario is the most plausible so long as it will be optimal to use renewables before the emissions ceiling binds rather than choosing to meet the emissions constraint without deploying any alternative energy sources.

Footnote continued: 5 See Chakravorty et al. (2006) for a model of a scarce nonrenewable resource with regulation but no learning.
In reality, there may be many short-run factors (e.g., speculation in commodity markets) that are at play in the determination of energy prices, but these results may at least partly explain the fluctuations in the prices of fossil fuels such as crude oil, natural gas and coal in recent years at a time when there is a general expectation that environmental regulation will bind at some time in the near future. The model then predicts that if, say, an international treaty imposed a target of 450 ppm (parts per million) of carbon, we would expect prices to rise initially but start decreasing as soon as this constraint becomes binding. When the constraint no longer binds and we fall below the 450 ppm level, energy prices will rise again, and finally fall when we make a complete transition to the clean substitute. The textbook Hotelling model, with learning or pollution regulation, does not predict this cyclical behavior.

Our paper also contributes to a large literature on the role of environmental regulation in generating endogenous technological change, and the importance of considering these incentives in setting policy. Arrow (1962) introduced the notion of learning-by-doing, where cumulative experience rather than the passage of time or directed investment leads to lower marginal production costs. Newell et al. (2002) demonstrate how policies change the long-run cost structure for the firm and drive innovation. Popp (2006), Gerlagh and van der Zwaan (2003), Nordhaus (2002), and Gould and Mathai (2000) account for the potential for induced technological change in determining optimal climate change policy. Bramoullé and Olson (2005) show how new abatement technologies may be preferred to existing ones because of the dynamic incentives arising from learning-by-doing. Like these papers, endogenous technological change in response to emissions policy is important in our results, but we examine specifically how technological change interacts with resource scarcity.

Endogenous technological change has also been examined in the context of finite resource extraction, but not in conjunction with a binding constraint on emissions. Tahvonen and Salo (2001) characterize the optimal extraction of finite resources when physical capital becomes more productive with use, increasing the marginal productivity of alternative and traditional energy sources over time. Dasgupta et al. (1982) characterize optimal resource extraction rates when probability of the discovery of a substitute technology can be altered through investment. Tsur and Zemel (2005) have considered scarcity and research and development in a model of economic growth. None of these studies consider the impact of coincident environmental constraints, however.

Yet another set of papers have taken the traditional Hotelling framework and examined conditions under which resource taxes may be non-monotonic. For example, Sinclair (1994) shows the possibility of declining resource taxes when interest rates are allowed to vary exogenously. Single-peaked resource price paths have also been shown to occur when stock externalities are produced by use of the polluting resource, as in Ulph and Ulph (1994), Hoel and Kverndokk (1996) obtain an inverted U-shaped path of emissions taxes using an explicit pollution damage function. In a model with a polluting resource but a non-polluting backstop, Tahvonen (1997) shows the possibility of simultaneous use of the two resources and the potential for a single-peaked price path of the nonrenewable. In a Hotelling model with a resource-exporting cartel and a coalition of resource-importing governments, Rubio and ESCRiche (2001) show that the resource price and Pigouvian tax need not move in the same direction over time. While it is clear that many studies have shown the phenomena of non-monotonic resource prices, none have demonstrated the existence of multiple price peaks as we show in this paper.

Section 2 develops the dynamic model of a nonrenewable resource with learning in the clean technology. We discuss the necessary conditions and develop some basic insights. Section 3 characterizes optimal energy price paths and the sequence of resource use. Section 4 concludes the paper by highlighting policy implications and limitations of the framework considered in this paper.

2. The model

In this section we extend the textbook Hotelling model by adding an emissions-free alternative energy source whose unit cost is determined by past experience. Let the utility or gross surplus from energy consumption be given by \( U(q) \) where \( q(t) \) is the energy consumed at any time \( t \), which is the sum of the extraction rates for a polluting fossil fuel denoted by \( x(t) \) and the consumption of the clean backstop resource denoted by \( y(t) \) so that \( q = x + y \). For ease of exposition, we denote the fossil fuel as “coal” and the clean backstop as “solar.” The utility function is strictly increasing and concave with respect to \( q \), i.e., \( U(q) > 0, U'(q) < 0 \) and satisfies the Inada condition, \( \lim_{q \to 0} U(q) = \infty \). Denote the marginal surplus by \( p(q) = U'(q) \) and by \( d(p) \) the corresponding demand function.

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6 They are currently at about 390 ppm, not counting other greenhouse gases. Although the Kyoto Protocol mandates limits on carbon emissions, its ultimate goal is the stabilization of carbon in the atmosphere, as suggested by numerous studies of the Intergovernmental Panel on Climate Change (e.g., see IPCC, 2001; Gupta et al., 2007). We model regulation as a limit on the stock of carbon, not on flow as has been done by Smolders and van der Werf (2008). In reality it is some combination of the two, if one considers the short term limits imposed by the Kyoto Protocol on emissions within a 5-year period and the long term goal of climate stabilization. A limit on the stock allows for arbitrage over time, and implicitly assumes that damages are only a function of the stock. It may be important in future work to consider other more realistic but complex regulatory mechanisms, such as a hybrid flow and stock approach.

7 Technical change in resource extraction can lead to initially declining resource prices which eventually rise as scarcity rents increase. In such a situation, substitution to the backstop may also generate a peak, especially if the backstop resource is also subject to learning effects (Slade, 1982). An exogenous demand shock combined with the cost of changing levels of productive capital may give rise to a price cycle.

8 We sometimes avoid writing the time argument \( t \) to reduce notational clutter.

9 We also denote first and second derivatives by using notation \( U_q, U_{qq} \) as appropriate.
The initial stock of coal is defined by \( X_0 > 0 \) and the residual stock at any time \( t \) is given by \( X(t) = -x(t) \). The unit extraction cost of coal is assumed to be of the form \( c(X) \) with \( c(X) < 0, c'(X) > 0 \) and \( \lim_{X \to 0} c(X) = \infty \). Thus \( c(X(t))x(t) \) is the total cost of coal at any given time.

Coal use leads to emissions. By appropriate choice of units, we can specify a 1:1 relationship between coal use and the pollution emitted.\(^{10}\) The accumulated stock of the pollutant (carbon) in the atmosphere is given by \( Z(t) \) with initial stock \( Z(0) = Z_0 > 0 \) and the law of motion for \( Z \) is given by

\[
\dot{Z}(t) = x(t) - f(Z(t)),
\]

where \( f(Z) \) is the natural rate of dilution of pollution in the atmosphere. A higher stock of pollution implies a higher rate of dilution, i.e., \( f(Z) > 0 \).\(^{11}\) Similar to the stabilization scenarios envisaged by the IPCC or Hansen et al. (2008), we assume that the planner seeks to maximize utility under a constraint \( Z > Z_0 \) which imposes that at any time \( t \), \( Z(t) \leq Z \).

The unit cost of the clean substitute, solar energy decreases with cumulative production denoted by \( Y(t) \), i.e., \( Y(t) = Y_0 + \int_0^t y(t) \, dt \), where \( Y(0) = Y_0 \) is the cumulative production of solar energy at the initial period. We specify this unit cost as \( g(Y,Y) \), shown in Fig. 1.

The total cost of solar energy then becomes \( g(Y,Y) \). We assume that for any \( y > 0 \) and \( Y > 0 \), the unit cost \( g(y,Y) \) is strictly positive. We further assume that \( g_y(Y,Y) > 0 \) and \( g_{yy}(Y,Y) > 0 \).\(^{13}\) That is, increased use of solar energy raises its unit cost at an increasing rate. The marginal cost of solar energy is positive, i.e., \( (\partial / \partial y) [g(Y,Y)] = g + g_y > 0 \). It is also increasing since \( (\partial / \partial y) [g_y(Y,y)] = 2g_y + g_{yy} > 0 \). We further assume that \( g_y < 0 \) and \( g_{yy} > 0 \). The average cost declines with experience at a decreasing rate.\(^{14}\)

The unit cost of solar declines because as new capital stock (solar plants) are added, their average cost is expected to be lower because of learning. As old plants are retired or become obsolete, they are replaced by new ones, thus bringing down the unit cost of producing renewable energy. The marginal cost in any period is increasing in quantity supplied because of lower because of learning. As old plants are retired or become obsolete, they are replaced by new ones, thus bringing down the unit cost of producing renewable energy. The unit cost of producing renewable energy is increasing in quantity supplied because of lower because of learning. As old plants are retired or become obsolete, they are replaced by new ones, thus bringing down the unit cost of producing renewable energy.

Let the social rate of discount be given by

\[
\delta = \frac{1}{\theta}.
\]

We can easily re-work the model by assuming \( \theta \) units of pollution per unit of coal use, but that will not change any of our results below.

We assume the function \( f(Z) \) because it is more general than assuming a constant dilution rate and the results are developed without loss of generality. However, if a part of the stock does not decay, that could imply different phases even at the ceiling. There is some ongoing work on this topic which identifies conditions under which these various phases will occur (see Amigues and Moreaux, 2011).

Most Hotelling models assume a constant cost \( g_{yy} = 0 \) in the backstop resource. Constant costs over the quantity supplied will change many of our results. This point is discussed again below.

A unit cost function that satisfies all the above conditions is \( g(Y,Y) = AY^{-\eta} + By^2 \), where \( A, B \) and \( \eta \) are strictly positive constants. It is easy to check that \( g_y > 0, g_{yy} > 0, g_y < 0 \) and \( g_{yy} > 0 \).

In the case of wind power, the same effect occurs. Over time, wind turbines have become more efficient, but older ones remain in use leading to a merit order of installed technologies. The merit order schedule will fall with time as more efficient plants replace more expensive ones. The extensive margin effect in any period, holding technology constant, implies that as more wind turbines are installed, high-gradeing of sites leads to sites with larger or more reliable wind resources being exploited first.

Although we discuss solar energy as a likely candidate for the clean technology, we avoid being very specific about the nature of the technology in this formulation. For example, solar panels may have a long life (30–40 years), in which case, at any given time, there would be an inherited installed capacity of solar panels, net of obsolescence. This will introduce complications in the model. However, the qualitative results will not change because all the solutions to the model imply increasing solar use over time, so the question of unusable capacity does not arise. Any technology that exhibits a unit cost that increases with the quantity supplied works, provided there are cost reductions due to experience.

Let the social rate of discount be given by \( \delta = \frac{1}{\theta} \). The average cost declines with experience at a decreasing rate.\(^{14}\)

The marginal cost of solar declines because as new capital stock (solar plants) are added, their average cost is expected to be lower because of learning. As old plants are retired or become obsolete, they are replaced by new ones, thus bringing down the unit cost of producing renewable energy. The marginal cost in any period is increasing in quantity supplied because of lower because of learning. As old plants are retired or become obsolete, they are replaced by new ones, thus bringing down the unit cost of producing renewable energy.
The current value Lagrangian is given by

\[ L = U(x+y) - c(X)x - g(y,Y)y - b\dot{y} + \mu[x-f(Z)] + \gamma_x(Z-Z) + \gamma_y y + \gamma_Z Z, \]

where \( \lambda(t), \beta(t) \) and \( \mu(t) \) respectively, are the co-state variables attached to the three equations of motion (2)–(4) and \( \gamma_x, \gamma_y \) and \( \gamma_Z \) are Lagrange multipliers associated with the Kuhn–Tucker conditions in (5). We obtain the following\(^\text{18}\) first order conditions:

\begin{align}
\dot{U}_q &= c + \lambda - \mu - \gamma_x \\
\dot{U}_q &= g + g_y y - \beta - \gamma_y \\
\dot{\lambda}(t) &= r\lambda + c'(X)x \\
\dot{\beta}(t) &= r\beta + g_Y y \\
\dot{\mu}(t) &= (r + f'(Z))\mu + \gamma_Z.
\end{align}

The complementary slackness conditions are

\begin{align}
\gamma_x x &= 0, \quad \gamma_x \geq 0, \quad x \geq 0 \\
\gamma_y y &= 0, \quad \gamma_y \geq 0, \quad y \geq 0 \\
\gamma_Z (Z-Z) &= 0, \quad \gamma_Z \geq 0, \quad Z-Z \geq 0
\end{align}

and the transversality conditions are given by

\begin{align}
limit_{t \to \infty} e^{-rt}\lambda(t)x(t) &= 0, \\
limit_{t \to \infty} e^{-rt}\beta(t)y(t) &= 0 \text{ and} \\
limit_{t \to \infty} e^{-rt}\mu(t)(Z-Z) &= 0.
\end{align}

The three costate variables capture the dynamic incentives created by resource scarcity, the environmental constraint, and learning-by-doing. Condition (8) is the usual Hotelling rule for the scarcity rent of the nonrenewable resource \( \lambda(t) \) when the unit cost is declining with the residual stock. The variable \( \beta(t) \) in (9) represents the shadow value of solar energy production: the additional benefit from producing one more unit of solar energy which comes from the learning benefit from future reductions in cost. Finally \( \mu(t) \) is the shadow price of the pollution stock and is negative. Its rate of increase over time is given by (10).

Note from (7) that when solar energy is being used \( (y(t) > 0) \), then \( \gamma_y = 0 \) so that we must have \( p(t) = g + g_y y - \beta \). Recall that the average cost of solar energy is \( g \) and the marginal cost is \( g + g_y y \). Since \( \beta > 0 \), this means that price is less than marginal cost. The planner internalizes the value of learning via beta, and chooses the optimal quantity. In a decentralized implementation, this would be achieved via a feed-in tariff or subsidy.\(^\text{19}\) Moreover if the average cost of solar energy is more than \( \beta \), then the firm would be making a loss in the absence of the compensatory payment.

\(^\text{18}\) According to Seierstad and Sydsaeter (1987, Theorem 1, p. 317), since our problem is a pure state variable constraint problem, a sufficient optimality condition is that the optimized Hamiltonian of the problem be concave jointly in the state variables \( X,Y \) which is the case when the Hessian of the Hamiltonian is a negative semi-definite matrix. Note that \( Z \) appears in the Lagrangian but not in the Hamiltonian. All the cross second derivatives of the Hamiltonian are zero. Sufficient conditions for negative semi-definiteness are: \( c'(X) \) is positive, \( g_{yy} \) is positive; under these assumptions all the principal minors of order 1 are negative, since they are simply \( -c'(X)x \) and \( -g_{yy} \), and the determinant of the Hessian of order 2 is positive.

\(^\text{19}\) See Kverndokk and Rosendahl (2007) for a discussion of subsidy and feed-in tariffs when learning occurs outside the firm.
From (9), it is clear that if over some time interval \([t_0, t_1]\), there is no solar energy use, i.e., \(y(t) = 0\), then the shadow price of the stock of cumulative experience \(\beta\) exhibits exponential growth, i.e., \(\beta = r\beta\) and \(\beta(t) = \beta(t_0) e^{(r-\frac{1}{2}+\gamma) t}, t \in (t_0, t_1)\). Suppose solar energy were employed only after some delay, i.e., no solar was used over the interval \([0, t_1]\). Then the right hand side of (7) becomes \(g(0, Y_0) - \beta_0 e^{\gamma t}, t \in [0, t_1]\), where \(\beta_0 \equiv \beta(0)\) is the initial value of the shadow price for solar industry experience. When solar is being used however, (9) and (15) yield

\[
\beta(t) = \int_t^\infty e^{-(r-\frac{1}{2})t} [-g_y(y(t), Y(t))y(t)] dt
\]

which will decline towards zero for large \(t\) if \(g_y\) goes to zero.

Cumulative experience \(Y(t)\) is an asset, the rate of return for which must be the same as for any other asset in the economy. That is, even if the renewable energy industry is idle, i.e., no solar energy is being consumed, from (9), \(\beta\) must grow at the rate of discount. We ignore any depreciation of experience and assume that accumulated experience is always preserved.\(^{20}\)

If the stabilization target does not bind initially but will in the future, then \(\gamma_2 = 0\) so that (10) yields \(\mu(t) = (r + f'(Z))\mu\), which gives \(\mu(t) = \mu_0 e^{\int_0^t [r + f'(Z)] dt}\), where \(\mu_0\) is its initial value of the shadow price of the pollution stock at time \(t=0\), which is negative. Thus \(\mu(t)\) is strictly decreasing (since \(f(Z) > 0\)), i.e., its absolute value is strictly increasing. If the target is not binding and will never bind in the future, then \(\mu(t) = 0\).

A binding target on the concentration of carbon in the atmosphere implies three phases of resource extraction—before, during and after the period when the cap is binding. We show below that coal use must be decreasing everywhere except when the cap binds. We also show that solar energy consumption must increase if solar is being used. If the ceiling never binds in time, then the analysis is similar to the post-ceiling phase. In both cases, the environmental constraint is no longer relevant, hence \(\mu(t) = 0\). Coal use will be declining due to scarcity and solar use will increase because of learning.\(^{21}\)

Before characterizing the solution, let us develop intuition by discussing what happens before, during and after the period when the stock constraint binds.

### 2.1. Before the ceiling is reached

If the ceiling is not yet binding but will be, then by (13) and (10), \(\bar{Z} > Z(t)\) hence \(\gamma_2 = 0\) so that \(\mu = \mu(r + f'(Z))\). Suppose there is an initial period when solar energy is not used, and only coal is used. Then \(\gamma_1(t) = 0\). Hence from (6), \(p = c + \lambda - \mu\). Differentiating with respect to time, using (8) and canceling terms gives \(p = r\lambda - \mu\). Substituting from (10) yields \(U'(x)\tilde{x} = r\lambda - \mu - f'(Z)\mu\). That is, the use of coal must decrease as in the textbook Hotelling case. This is because of increasing scarcity and the marginal shadow cost of the future pollution constraint given by \(\mu\) which too increases over time.

Now consider the situation when both energy sources are being used simultaneously, i.e., \(x(t) > 0\) and \(y(t) > 0\). Differentiating (6), we get \(p = U'(q)\tilde{q} = r\lambda - r + f'(Z)\mu > 0\) which also implies that \(\tilde{q} < 0\). The price of energy increases and aggregate energy use decreases. To see what happens to the energy mix over time, we differentiate (7), use (9) and cancel terms to get \(p = 2g_y\tilde{y} + g_{yy}\tilde{y} + g_{yy}\tilde{y}^2 - r\beta\). Substituting for \(p\) from above, we have

\[
y = \frac{r(\lambda + \beta) - f'(Z) + r\mu - g_{yy}\tilde{y}^2}{2g_y + g_{yy}\tilde{y}} > 0
\]

since the numerator and denominator are both positive. That is solar energy use must increase with time under joint use. The use of the nonrenewable resource is given by \(\tilde{x} = \tilde{q} - \tilde{y} < 0\). Since aggregate energy use is decreasing and solar use is increasing, coal use must decline over time. The increase in the value of coal because of its scarcity drives the adoption of solar energy. This effect is then compounded by cost reductions achieved by learning, although learning also reduces the importance of resource scarcity in the long run.

\(^{20}\) A more realistic formulation, especially from an empirical point of view, could include a constant rate of attrition \(\theta \geq 0\) of the asset when the industry is idle, i.e.,

\[
\tilde{Y}(t) = \begin{cases} -\theta Y(t) & \text{if } Y(t) = 0, \\ Y(t) & \text{if } Y(t) > 0. \end{cases}
\]

We implicitly assume \(\theta = 0\) in this paper.

\(^{21}\) When knowledge spillovers are not taken into account, the model reverts to an optimal carbon pricing model with spillovers and \(\beta(t) = 0\). Qualitatively, this will mean less alternative energy adoption at a given level of \(Y\), all else equal. This lower level of adoption will, in turn, slow the rate of increase in \(Y\), which means that alternative energy costs will be higher than they otherwise would be, which exacerbates the difference in deployment. A countervailing effect occurs as the higher alternative energy costs due to lower deployment will increase the shadow cost of the constraint, and therefore increase the cost of carbon \(\mu(t)\). The particular quantitative implications for a planner who only accounts for the carbon constraint, and not for learning potential, are not generalizable. The qualitative implications are lower deployment of alternatives, and higher than optimal costs of meeting the emissions constraint.
2.2. When the ceiling is binding

When the concentration of carbon is at the ceiling \( Z \) for an interval of time, coal must be in use, since emissions need to be positive and equal to natural dilution as shown in (1). To remain at the constraint, \( \dot{x}(t) = 0 \), so define \( x = f(Z) \). This is the constant amount of coal use each period that will keep the stock of pollution at the ceiling. There are two possibilities at the ceiling depending upon whether solar energy is also being used or not. If there is no solar in use at the ceiling, then \( q(t) = x(t) = \pi \) and \( p(t) = \pi = U'(x) \). Coal use is constant hence the price of energy is constant as well.

The more interesting case is when solar energy is being used at the ceiling. Note that the same amount of coal must be emitted each time period at the ceiling, otherwise the stock of pollution will not remain at \( Z \). So \( q(t) = \pi + y(t) \) where \( y(t) > 0 \). This implies that \( \gamma_y = 0 \). Then from (7) and following the same procedure as before, we get \( y = ((g_{yy}y^2 - r\beta)/(U''(q) - 2g_{yy} - g_{yp}y)) > 0 \) since both numerator and denominator are negative. That is, solar use must be increasing at the ceiling while coal use will remain constant, by definition. Aggregate energy use will increase. Then there is increasing solar energy use, we get \( \dot{p} = U_{\pi}q = U_{\pi}(\dot{x} + \dot{y}) = U_{\pi}q \dot{y} < 0 \). In other words, the price of energy when the pollution ceiling is binding must be decreasing.

2.3. Once the ceiling is no longer binding

There are three possibilities in the post-ceiling phase—only coal is used, both resources are used, or only solar is used. Since the ceiling is no longer binding and will not be again, the shadow price of the stock of carbon \( \mu \) is zero.

If only coal is used over some period in the post ceiling phase, then from (6) we have \( p = c(X) + \lambda \). which upon differentiation yields \( \dot{\lambda} = c'(X)\dot{X} + \lambda = r\lambda \) using (8) and canceling terms. Thus the price of energy is increasing and energy use is decreasing, i.e., \( \dot{q} = \dot{x} < 0 \). This phase is essentially a pure Hotelling path with a rising extraction cost function. Of course, prices cannot rise forever because at some point, \( c(X) + \lambda \) will catch up with the price of solar, given by \( g(0,Y_0) - \beta \) (see (7)).

If both coal and solar are used simultaneously, then (6) continues to hold as \( p = c(X) + \lambda \). Time differentiating gives \( \dot{p} = r\lambda > 0 \). Energy prices must rise and aggregate consumption must decline, \( \dot{q} < 0 \). As before, we can use the necessary conditions to determine the rate of change of solar energy use with respect to time as

\[
\dot{y} = \frac{r(\lambda + \beta) - g_{yy}y^2}{2g_{yy} + g_{yp}y} > 0,
\]

which is a special case of the pre-ceiling result with \( \mu(t) = 0 \). Since solar use is increasing and aggregate energy use is decreasing, then coal use must decline, \( \dot{x} = q - y < 0 \).

When only solar energy is being used, \( \gamma_y = 0 \) and using (7), we have

\[
\dot{y} = \frac{g_{yy}y^2 - r\beta}{U''(q) - 2g_{yy} - g_{yp}y} > 0.
\]

This is the same result as in the ceiling with joint use, except that instead of \( x = \pi \), now \( x = 0 \) since no coal is being used. Then \( \dot{p} = U'(y)Y < 0 \). The price of energy must fall over time.

As solar energy continues to be used, cumulative experience \( Y(t) \) increases with time. What is the limiting value of \( Y \)? We have \( \lim_{t \to \infty} Y(t) = \infty \) and the average cost of solar energy approaching the limit cost \( g(y) \). In the limit the supply of solar energy is given by \( p = U'(Y) = g(y) + g'(y)Y \). Since \( U'(Y) \) is downward sloping and the marginal cost of solar energy is upward sloping, the solution is unique. Let us define this limit quantity as \( y \).

3. Price paths and the energy mix

We now proceed to discuss all possible solutions to the planner’s problem. They can be classified as follows:

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\(^{23}\) It can be shown that the ceiling can be hit at most over one time interval, since coal which causes the pollution gets increasingly scarce over time.

\(^{22}\) There exists a finite time \( t \) at which the use of the nonrenewable resource is terminated provided that the cost of the clean substitute declines sufficiently due to cumulative experience. Then since \( \lim_{t \to \infty} c(X) = \infty \), the terminal condition that determines when no more of the coal is used is given by \( c(X(t)) = p(t) \). As pointed out by Salant et al. (1983), the extraction period of coal may be infinite even if the inverse demand function is bounded from above, i.e., \( p(0) < \infty \). In our case, the inverse demand is not bounded because of the Inada condition. But the residual demand of coal is bounded and decreasing over time. Therefore coal extraction over an infinite period cannot be excluded \( a \ priori \). However, suppose the initial extraction cost of coal is very low relative to \( p \), the price at the ceiling and this extraction cost does not increase significantly over a large part of the coal stock. Suppose also that the initial cost of solar is very high, but the limit price \( g(y) \) is lower than the initial extraction cost of coal and that this limit price is low over a large range of values of the solar extraction rate. If with given demand parameters, it takes a long time for solar energy to come "close" to the limit price, then the price of energy cannot increase forever. The necessary conditions for an infinite period of coal extraction will not be satisfied. This is the setting we consider in this paper.
3.1. The ceiling never binds

To obtain some intuition, we first discuss the case when the ceiling on atmospheric concentration never binds. Here the solution is a pure Hotelling model with learning-by-doing in the backstop resource. If the unconstrained solution to the planner’s problem implies a path for the pollution stock \( Z(t) \) that never equals the stock constraint \( \bar{Z} \) for a non-degenerate interval of time, then the ceiling is redundant. In that case, we may have two possible outcomes depending on whether solar energy is used right from the beginning of the planning horizon or not.

When solar energy is used from the beginning, the resource and price use profile is shown in Fig. 2.

During the initial period of joint use, \( p(t) > 0, \dot{q}(t) < 0, \dot{x}(t) < 0 \) and \( \dot{y}(t) > 0 \). Finally at time \( t_\alpha \), coal is no longer used, and all energy is supplied by solar. Then the price of energy decreases and solar use increases because of learning. Note that the price of energy rises at first then declines. In the figure, \( x(0) \) may be more or less than \( y \). Similarly, there is no a priori ordering of \( q(0) \) and \( \bar{Y} \). When the ceiling never binds but solar energy is not deployed from the beginning, we may have three phases, as shown in Fig. 3.24

In the first phase \([0,t_\alpha]\), only coal is exploited. The price of energy keeps increasing while the true cost of solar energy given by (7), i.e., \( g(0,Y_0) - \beta \) keeps decreasing as the value of future cost reductions in solar energy increases even through no learning is occurring. The gap between them shrinks. Finally at time \( t_\alpha \), in the second phase solar energy becomes economical and both resources are exploited at the same time. The price of energy keeps increasing and peaks at time \( t_\alpha \) when coal is economically exhausted. Beyond this time, the price of energy falls and solar energy use increases and approaches an upper bound in the limit.

There must always be an intermediate phase in which both resources are used, as a result of our assumption that the marginal cost of solar energy increases with the volume supplied at any instant, i.e., \( \frac{\partial c(y)}{\partial y} [g + g_y y] = 2g_y + g_{yy} y > 0 \). Suppose that there is no joint use, which would imply that before time \( t_\alpha \), there is no solar energy use and beginning at \( t_\alpha \), solar supplies the whole industry. Then at \( t_\alpha \), there is a jump in solar energy consumption from zero to \( y = q(t_\alpha) = \lim_{t \to t_\alpha} x(t) > 0 \) which would contradict a cost-minimizing solution to the planner’s energy supply problem. That is at time \( t_\alpha \), by (7), we must have \( p(t_\alpha) = g(q(t_\alpha),Y_0) + g(q(t_\alpha),Y_0)q(t_\alpha) - \beta(t_\alpha) \). Now consider the fact that \( p(t) \) is continuous and increasing for \( t < t_\alpha \), \( \beta = r(\beta) \) by (9) hence \( \beta(t) = \beta_0 e^{rt} \) for \( t < t_\alpha \) and \( g(y,Y_0) + g(y,Y_0)y \) is strictly increasing in \( y \). Then for \( t=t_\alpha - \epsilon \) where \( \epsilon > 0 \) is sufficiently small, there is a level of solar energy use \( y_\alpha \), \( 0 < y_\alpha < q(t_\alpha) \), which satisfies condition (2) with \( \gamma_y = 0 \) and \( p(t) = g(y,Y_0) + g(y,Y_0)y - \beta(t) \). But this violates the fact that since solar energy use is zero before time \( t_\alpha \), we must have \( p(t) \leq g(0,Y_0) - \beta(t) - \gamma_y \) at time \( t = t_\alpha - \epsilon \) with \( \gamma_y \geq 0 \). Similar arguments hold in each of the solutions described below.

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24 See Appendix for the system of differential equations that must be solved for this case.
25 In all these figures, it is generally hard to predict the curvature of the price and quantity profiles, without making further assumptions on the functional forms for the demand for energy and the cost function for coal. In general, the price paths are not differentiable at the peaks. The curvature properties of the figures must not be interpreted as having been derived analytically. However, some additional results on this topic can be obtained by writing to the authors.
3.2. A binding ceiling

When the ceiling binds over a non-zero interval, then solar energy may become economical before, during or after the ceiling. We consider each of these below.

Fig. 3. Ceiling does not bind: solar is initially too expensive (the dashed line represents the price of solar energy which is lower than its unit cost).

Fig. 4. Solar energy is first used after the ceiling is no longer binding.
### 3.2.1. Solar is first used after the ceiling is no longer binding

The price path and resource extraction is shown in Fig. 4. Initially emissions are higher than natural dilution, i.e., \( x(t) > f(Z) \) so the stock of pollution \( Z(t) \) increases as given by (1). However, because of rising resource prices, coal extraction and emissions decline and the higher stock of pollution means increased dilution since \( f(Z) > 0 \). Finally at time \( t = t_c \), emissions equal exactly equal natural dilution. In the following phase, the stock of pollution stays at the constrained level, \( Z(t) \), emissions equal \( x(t) \), and energy consumption is constant. At the end of this phase, \( \mu(t) = 0 \) and is always zero beyond.

From time \( t_c \) only the nonrenewable resource is used still, and the solution is pure Hotelling since the environmental constraint is no longer binding. Emissions fall below \( x(t) \) and the stock of pollution begins to fall as resource prices rise due to the traditional Hotelling rule. All this time, as shown in Fig. 4, the real price of solar energy continues to decline but not enough to be economical. Finally at time \( t_x \) we have \( p(t_x) = g(0,Y_0) - \beta x^e \). Both resources are exploited, although coal use declines while solar use increases, driven by both increasing resource opportunity costs and by learning effects. At time \( t_x \), coal extraction is complete with \( p(t_x) = c(X(t_x)) \) implying that the scarcity rent of coal \( \lambda(t_x) \) is zero. Solar energy becomes the sole supplier of energy and its use increases over time with learning, leading to falling energy prices.\(^{26}\)

#### 3.2.2. Solar is first used while the ceiling is binding

When the equilibrium price of energy is high relative to the cost of solar energy, it is possible that solar energy may become economical when the economy is still at the constrained level of pollution. In this case, over the time interval \( [t_y, t_c] \) the use of solar energy must rise because of learning-induced decreases in cost (see Fig. 5). Since coal use must be constant, the aggregate consumption of energy rises, which in turn implies that the price of energy declines during this period of joint use at the ceiling. That is, \( q(t) = x(t) + y(t) \), so that \( \dot{q}(t) = \dot{y}(t) > 0 \) and \( \dot{p}(t) = U'(q) \dot{q}(t) < 0 \). As before the shadow price of the pollution stock is zero once the ceiling is no longer binding. The remaining path is similar to that shown in Fig. 4. Here the price of energy is initially increasing before the ceiling binds, then may be constant for some time while the economy faces the binding constraint but before solar energy is used, then declining as solar energy is used and the constraint slackens and finally increasing with resource scarcity and finally decreasing with learning once fossil fuels are economically exhausted.

#### 3.2.3. Solar is first used before the ceiling is reached

Perhaps the most interesting cases are those in which solar energy is first used before the constraint on pollution concentration binds.\(^{27}\) This may happen if the cost of solar is low relative to the price of energy or ceiling is less stringent.

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\(^{26}\) A degenerate case may occur if solar energy is first used exactly when the ceiling ceases to be binding. In this case we will not get a period with only coal use after the ceiling. This may happen if coal resources are limited or learning effects are higher in the solar technology.

\(^{27}\) This case is consistent with most empirical modeling of climate stabilization scenarios considered by the IPCC and the IEA. For example, in the IEA (2009) 450 ppm scenario, while atmospheric CO\(_2\) concentrations are not stabilized for 25–40 years, renewable energy use is higher than the reference case immediately, and in every period up to the point at which the concentration stabilizes.
Here we obtain two “peaks” in energy prices—one when the ceiling first binds and another when coal gets exhausted. There are two corresponding troughs in energy consumption, shown in Fig. 6. Coal is used initially until solar becomes economical at a price below the ceiling, i.e., \( p(t) = g(0, Y_0) - \beta(t_c) < \beta \). Both resources are exploited and the price of energy continues to rise. However, solar energy use expands while coal use declines although the emissions generated are higher than the dilution rate, so that the stock of pollution continues to build towards \( Z \). Finally the ceiling becomes effective at time \( t_c \). The remaining path is similar to the previous case when solar is first used at the ceiling. We can summarize the main results as follows:

**Proposition.** When a polluting nonrenewable resource has a clean substitute which exhibits learning-by-doing, the price path is single-peaked if the pollution ceiling does not bind. Under a binding stock pollution target, the price path is again single-peaked if the substitute is first used only after the ceiling on pollution is no longer binding. Double-peaked price paths are observed when the substitute is first used during or before the period when the ceiling binds.

The double-peaked paths in energy prices are counter-intuitive and so more discussion is warranted. These price paths occur due to the timing of the interaction of economic and imposed constraints on the extraction of fossil fuels and the potential for technological progress in the alternative energy source. The first peak occurs when the use of coal is restricted by an environmental target. Here, energy prices first increase due to both physical scarcity and the future stock pollution constraint, and so optimal solar energy may increase simply due to the scarcity imposed by the ceiling. These effects are exacerbated by learning, which implies that more of the alternative energy source will be used over time. When fossil fuel use is capped by the binding constraint, aggregate energy use increases (because of rising solar consumption) which means that the energy price must decline for some time. However, fossil fuel reserves eventually get sufficiently depleted so that the stock pollution constraint no longer binds, and extraction decreases as the textbook Hotelling model takes over, but with a twist. In our model, a positive-feedback loop occurs whereby fossil fuel scarcity leads to increasing adoption of the alternative, which leads to increased learning which in and of itself leads to increased adoption and accelerates the economic exhaustion of the resource. This transition eventually leads to a second peak in the price path, at which point coal is economically exhausted. Beyond this point, the economy is entirely supplied with emissions-free energy and so neither scarcity nor pollution constraints have any further effect, and energy prices must again fall due to learning in the clean technology.

The assumption of increasing unit cost of solar energy each period is important for our results. If the unit cost were constant \( g_{yy} = 0 \) then there may be jumps in the supply of solar energy along the optimal path, and periods of joint use of the two resources will not occur, except when the ceiling is binding and the “double peaks” result may not hold. This is because the marginal cost of solar will be constant in that case, and when solar energy kicks in, it will supply all the energy. Rising instantaneous unit costs as in our model allow for the coal price and solar price to increase over time, resulting in joint use.

![Fig. 6. Solar energy is first used before the constraint binds.](image-url)
Double-peaked paths occur only under selected circumstances. It must be the case that the alternative energy source is used before or during the period in which the constraint on the stock of pollution is binding. This is most likely to occur when alternative energy is relatively cheap, learning effects are significant, and/or when the shadow cost of the environmental cap $\bar{Z}$ is relatively high. For example, if alternative energy sources are prohibitively expensive and/or learning effects are small, it will take a very stringent cap on the accumulation of atmospheric pollutants to make their use economical. The fact that we observe many renewable resources being used even without subsidies (e.g., hydropower and wind energy) may be considered as supportive evidence for this case.

4. Concluding remarks

We consider the effect of an aggregate pollution target on substitution of a fossil fuel by a clean backstop technology, when the latter exhibits learning-by-doing. We show that the price of energy under such a pollution constraint may exhibit cyclical trends driven by scarcity of the nonrenewable resource, the effect of environmental constraints, and potential cost reductions in the clean technology. Such price cycles do not emerge in traditional models of energy use.\(^{28}\)

Although our results were developed in the context of environmental regulation which has not been binding except in some isolated instances (as in the regional climate change initiatives in the US), it is clear from our analyses that economic forces that can act as a temporary shock on the price of a nonrenewable resource (such as adjustment costs) may have the same effect as discussed in the paper. If these shocks cease to be important after a period of time, resource prices may fall because of learning, and the same price cycles may emerge.

A major assumption in the model is the goal of climate stabilization which implicitly means an inverted L-shaped damage function—which suggests that marginal damages are zero until we hit a threshold and then infinite beyond. This makes the model analytically tractable. Alternatively, one could specify a more realistic smooth damage function as has been assumed in the literature (e.g., Hoel and Kverndokk, 1996). However, it is not possible to predict the profile of resource use ex-ante—both when only coal is used, and when both resources are used jointly. There may be multiple peaks in resource use. There will be no ceiling, hence the comparison with the solution developed in this paper is not immediate. Tahvonen (1997) has examined a model with a damage function but with a fixed unit cost for the clean substitute and no learning.

Although not considered here, the emergence of new energy sources and substitution possibilities over time (Rhodes, 2007) may lead to repeated cycles in energy prices in the same manner that we have demonstrated in this paper. Further, multiple environmental constraints such as those on water use or toxic emissions may add to the complexity of optimal price paths. Finally, a model with many nonrenewable resources in conjunction with substitutes and environmental constraints would be expected to yield multiple peaks and troughs in optimal energy prices. This has important implications for empirical tests of the Hotelling model, and warrants very careful attention to the real opportunity costs of resource extraction at play before concluding that non-increasing or non-monotonic prices paths refute the Hotelling rule.

There are many extensions of the proposed framework which deserve consideration in future work. First, it may be interesting to see how substitution of the clean energy may occur under exogenous demand growth or when demand declines and how this differs from the constant energy demand case we have considered here. The second issue is uncertainty over the form and stringency of environmental regulation, the stocks of the fossil fuel and the cost reductions that could be achieved through learning-by-doing. Ex-ante, it is not clear how these uncertainties will affect the substitution process. A third issue ignored here is technological progress through research and development, rather than through learning-by-doing. Here, the industrial organization of the fossil fuel industry and of the industry producing clean alternatives will affect the process of technology adoption.

Acknowledgment

Chakravorty and Leach would like to thank the Social Sciences and Humanities Research Council (SSHRC) for generous research support. The authors also thank three anonymous referees for reports that significantly improved the quality of the paper and seminar participants at ETH Zurich, California State University at San Luis Obispo, and University of British Columbia, Heidelberg, Copenhagen, Rice, Alabama and Tilburg for valuable comments.

Appendix. Determination of the optimal sequence when the cap does not bind

Here we only consider the case illustrated in Fig. 2 in which solar energy is first used at the beginning of the planning horizon and the cap is not binding. The other cases are similar and therefore are not examined separately. We show how the optimal paths for $x(t)$, $X(t)$, $y(t)$, $Y(t)$, $\mu(t)$ and $\beta(t)$ are determined in the first phase when both resources are used,
together with the time $t_x$ at which coal use is terminated. The equations determining the above six variables are:

$$
\dot{X}(t) = -x(t) \\
\dot{Y}(t) = y(t) \\
U_q = c + \dot{\lambda} \\
U_q = g + g_y y - \beta \\
\dot{\lambda}(t) = r\dot{\lambda} + c'Xx, \text{ and} \\
\dot{\beta}(t) = r\dot{\beta} + g_y y
$$

We need to have six conditions that determine the six dimensional vector $(x,X,y,Y,\lambda,\beta)$ and another condition that determines the time $t_x$ when the first phase ends. These are

$$
X(0) = X_0 \\
Y(0) = Y_0 \\
U'(x(0) + y(0)) = c(X_0) + \dot{\lambda}_0 \\
U'(x(0) + y(0)) = g(y(0),Y_0) + g_y(y(0),Y_0)y(0) - \beta_0; \quad Y_0 \text{ given} \\
U'(x(t_x) + y(t_x)) = c\left(X_0 + \int_0^{t_x} X(t)\,dt\right) \text{ which implies } \dot{\lambda}(t_x) = 0. \\
U'(x(t_x) + y(t_x)) = g\left(y(t),Y_0 + \int_0^{t_x} y(t)\,dt\right) + g_y\left(y(t_x),Y_0 + \int_0^{t_x} y(t)\,dt\right)y(t_x) - \beta(t_x); \\
x(t_x) = 0
$$

Equivalently, the system of four first order differential equations in $X, Y, \lambda$ and $\beta$ can be solved. They are as follows:

$$
U'(\dot{X}(t) + \dot{Y}(t)) = c(X(t)) + \dot{\lambda}(t) \\
U'(\dot{X}(t) + \dot{Y}(t)) = g(Y(t),Y(t)) + g_y(\dot{Y}(t),Y(t)\dot{Y}(t)) - \beta(t) \\
\dot{\lambda}(t) = r\dot{\lambda}(t) - c'(X(t))\dot{X}(t) \\
\dot{\beta}(t) = r\dot{\beta}(t) + g_y(\dot{Y}(t),Y(t)\dot{Y}(t))
$$

We need four points in the $(X,Y,\lambda,\beta)$ space to determine the solution. These are

$$
X(0) = X_0 \text{ given} \\
Y(0) = Y_0 \text{ given} \\
U'(\dot{X}(t_x) + \dot{Y}(t_x)) = c\left(X_0 + \int_0^{t_x} X(t)\,dx\right) \text{ implies } \dot{\lambda}(t_x) = 0 \\
U'(\dot{X}(t_x) + \dot{Y}(t_x)) = g\left(Y(t_x),Y_0 + \int_0^{t_x} Y(t)\,dx\right) + g_y\left(Y(t_x),Y_0 + \int_0^{t_x} Y(t)\,dx\right)\dot{Y}(t_x) - \beta(t_x)
$$

Furthermore, at $t = t_x$, we must have $X(t_x) = 0$, which determines $t_x$.

In the second phase, only solar is used. The corresponding differential equations that need to be solved for $Y(t)$ and $\beta(t)$ are

$$
U'(\dot{Y}(t)) = g(Y(t),Y(t)) + g_y(\dot{Y}(t),Y(t)\dot{Y}(t)) - \beta(t) \quad \text{and} \\
\dot{\beta}(t) = r\dot{\beta}(t) + g_y(\dot{Y}(t),Y(t)\dot{Y}(t))
$$

together with the initial conditions $Y(t_x)$ and $\beta(t_x)$ given by the first phase.

References


