Pattern formation in two-frequency forced Faraday waves

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ABSTRACT
Pattern Formation in Two-Frequency Forced Faraday Waves
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This dissertation investigates pattern selection in two-frequency forced Faraday waves. In this system, a fluid layer is subjected to a periodic vertical acceleration 

\[ g_z [\cos(\chi) \cos(m\omega t) + \sin(\chi)(n\omega t + \phi)] \]

where \( m \) and \( n \) are co-prime integers. For sufficiently large \( g_z \), standing waves form on the free surface. Experiments have produced exotic patterns, including spatially-periodic superlattice (SL) patterns (Kudrolli, Pier and Gollub, Physica D, 1998) which contain two length scales. This dissertation determines the selection mechanism for the length scale ratio for the SL-I superlattice pattern.

A 12-dimensional \( D_6 + T^2 \) equivariant bifurcation theoretic framework (Dionne, Silber and Skeldon, Nonlinearity, 1997) is used to study the competition of stripes, hexagons, rhombs, and SL-I superlattice patterns. Symmetry considerations are used to demonstrate how weakly damped harmonic modes may affect the stability of SL-I patterns through spatiotemporally-resonant triad interactions, which produce resonant contributions to coefficients in the bifurcation equations. To demonstrate this effect explicitly, the coefficients are numerically calculated via a perturbation calculation on partial differential equations of Zhang and Viñals (J. Fluid Mech., 1997) which describe Faraday waves on a deep layer of weakly viscous fluid. A bifurcation analysis reveals that a weakly damped harmonic mode may help stabilize an SL-I superlattice pattern.

For weak damping and forcing, symmetry considerations also determine which
particular damped harmonic modes have the most significant effect. These are: (i) modes oscillating with twice the frequency of the pattern modes, (ii) “difference frequency” modes oscillating with dominant frequency $|m-n|\omega$ and (iii) “sum frequency” modes oscillating with dominant frequency $(m+n)\omega$. For weak damping and forcing and one dimensional waves, an analytical expression is derived for the cubic self-interaction term. For stronger damping and forcing and two-dimensional waves, the remaining coefficients are computed numerically. Both calculations yield results in good agreement with those obtained from symmetry arguments.

A bifurcation analysis demonstrates that the difference frequency modes help stabilize the SL-I pattern and determines the length scale ratio, which is well-predicted by the gravity-capillary wave dispersion relation. The SL-I stabilization effect may be enhanced by appropriate choice of periodic forcing functions with more than two frequency components.
For Jude.
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Chapter 1

Introduction

1.1 Motivation

Spontaneous pattern formation is a widespread natural phenomenon that has been observed in fluid convection [1], chemical reaction-diffusion experiments [2], visual hallucinations [3] and a host of other physical, chemical and biological systems. Partial reviews are available in [4, 5]. Such spatially-extended systems have been the focus of much investigation in the latter part of the 20th century, as the mathematical and experimental techniques necessary for their study have been developed and improved. The work presented in this dissertation is one attempt to achieve a synergy of these recent advancements. In particular, we use modern theoretical tools to gain insight into recent experiments on pattern formation in Faraday waves.

The observation of Faraday waves, which form on a fluid layer subjected to a periodic vertical acceleration of sufficient strength, was first reported by Michael Faraday in 1831 [6]. Many early investigations of Faraday wave pattern formation
focused on the simple patterns which can tile the plane, such as stripes, squares, and hexagons. These patterns may be regarded as a superposition of, respectively, one, two, or three standing waves oriented in appropriate directions. For instance, a square pattern consists of two standing waves oriented perpendicularly to each other. More recently, complex patterns, i.e. patterns composed of more than three waves, have become a subject of interest.

One example of a complex pattern is the “quasipattern”. Quasipatterns are the continuum analogues of quasicrystals. They are spatially nonperiodic, lacking the translational symmetries of the simpler planar patterns. However, their Fourier spectra possess discrete rotational symmetry. Quasipatterns have been observed in experimental systems including Faraday waves [9, 10, 11] and nonlinear optics [12]. One example, taken from [9], is shown in Figure 1.1. This 12-fold quasipattern consists of six standing waves oriented around a circle with equal spacing between them.

Another type of complex pattern is the “superlattice” pattern. This term refers to a periodic pattern that has spatial structure on more than one length scale – typically, a small scale structure and a large scale spatial periodicity. Any superlattice pattern, therefore, has an associated length scale ratio. A wealth of recent experimental work has produced superlattice patterns. They have been observed in Faraday wave experiments [10, 13, 14, 15, 16], in nonlinear optical systems [17], in vertically vibrated Rayleigh-Bénard convection [18, 19], in granular layers [20], and in ferrofluids driven by an a.c. magnetic field [21]. The superlattice patterns come in a variety of types, and may comprise different numbers of waves having different spatial arrangement and temporal dependence. One variety of superlattice pattern is the SL-I superlattice pattern observed in Faraday wave experiments in [10]. They used a periodic vertical
acceleration consisting of two rationally-related components with frequencies $m\omega$ and $n\omega$ and observed the pattern shown in Figure 1.2a. A key result of this dissertation is the identification of a mechanism responsible for stabilizing patterns like the SL-I shown here, and for selecting their associated length scale ratio.

Many spatially-extended pattern forming systems, including Faraday waves, may be described by nonlinear partial differential equations. In mathematical terms, the appearance of a pattern in these systems is due to a symmetry-breaking bifurcation of the trivial (fully symmetric) solution of the underlying nonlinear partial differential equations. We adopt this bifurcation theoretic framework. The mathematical investigation relies on a combination of tools from elementary group theory, bifurcation theory and asymptotic analysis.
Figure 1.2: (a) An SL-I superlattice pattern obtained in a Faraday wave experiment forced with two frequency components in ratio 6/7 (taken from work of Kudrolli, Pier and Gollub [10]). The picture is a blow-up of a small region near the center of the experimental cell. (b) A similar pattern obtained in a numerical simulation of chemical reaction-diffusion equations [22].
Near a bifurcation, and under certain mathematical assumptions, one may obtain a finite-dimensional system of ordinary differential equations describing the slow-time dynamics of the dominant waves in the pattern. The exact form of these equations (but not the coefficients of the terms) may be determined by considering the symmetry group of the problem and the type of bifurcation that is relevant. This symmetry-based approach, which does not depend on the details of the underlying partial differential equations, is developed in [23]. The key advantage is that results derived (such as the existence of solution branches) are applicable to any system with the same symmetry group and bifurcation type, suggesting why similar patterns occur in disparate systems; an example is given in Figure 1.2.

Alternatively, the bifurcation equations may be obtained from an explicit perturbation calculation on the governing partial differential equations. Such calculations are useful because they yield values of the bifurcation coefficients and thus enable one to make specific quantitative statements about solution branches and their stability, and to see how these depend on parameters in the underlying partial differential equations.

In this dissertation, we use both of the aforementioned approaches in our study of pattern formation in Faraday waves forced with two frequency components. We use symmetry arguments to elucidate the role that temporal symmetries play in the pattern selection problem. We also perform explicit calculations on Faraday wave equations in order to derive the bifurcation coefficients and thus obtain quantitative results. We pay special attention to the implications our results have for the formation of SL-I superlattice patterns like the one from [10] shown in Figure 1.2a. The particular partial differential equations we use for our calculations have been derived
by Zhang and Viñals [24] and apply to Faraday waves on deep layers of weakly viscous fluid. Although we do not expect the Zhang-Viñals equations to be quantitatively valid for the experimental conditions of [10], which involve a shallow viscous fluid layer, many of our qualitative results should carry over.

1.2 Summary of results

We now present the main results of this dissertation, which apply to Faraday waves forced according to the function $g_z[\cos(\chi) \cos(m\omega t) + \sin(\chi)(n\omega t + \phi)]$. Here $m$ and $n$ are co-prime integers, so that the forcing function has period $T = 2\pi/\omega$. The 12-dimensional bifurcation framework we adopt [25, 26] allows us to investigate the competition of the following patterns near the onset of the bifurcation: stripes, hexagons, three distinct rhombic patterns, and superhexagon and supertriangle SL-I patterns.

In the Faraday system, the parametrically exited modes are either harmonic (having temporal period $T$) or subharmonic (having temporal period $2T$). We use symmetry considerations to demonstrate how weakly damped harmonic modes may affect pattern stability by means of spatiotemporally-resonant triad interactions, which produce resonant contributions to coefficients in the bifurcation equations. Weakly damped subharmonic modes result in no such contribution, and thus are not expected to play an important role in pattern selection [27].

To demonstrate this effect explicitly, we numerically calculate the coefficients in the bifurcation equations via a perturbation calculation on partial differential equations of Zhang and Viñals [24] which describe Faraday waves on a deep layer of weakly viscous fluid. A bifurcation analysis reveals that a weakly damped harmonic mode
may help stabilize an SL-I pattern.

For sufficiently weak damping and forcing, symmetry considerations also determine which particular damped harmonic modes have the most significant effect in terms of their contributions to the coefficients in the bifurcation equations. These are: (i) modes oscillating with twice the frequency of the pattern modes, (ii) “difference frequency” modes oscillating with dominant frequency $|m - n|\omega$ and (iii) “sum frequency” modes oscillating with dominant frequency $(m + n)\omega$. While case (i) is possible in the classical single-frequency-forced Faraday problem, cases (ii) and (iii) are possible only for multiple frequency forcing.

For weak damping and forcing and one-dimensional waves, we derive analytical expressions for the critical forcing and wave number, and for the cubic self-interaction coefficient in the bifurcation equations which determines the amplitude of the waves as a function of the control parameter near onset. We quantify the predicted resonance effects and determine how their existence depends on the forcing frequency ratio $m/n$. For two-dimensional waves and stronger damping and forcing, we compute onset parameters and all of the coefficients numerically, as before. The analytical and numerical results are in good agreement with the predictions based on the symmetry arguments.

A bifurcation analysis reveals that the difference frequency mode helps stabilize the SL-I pattern and determines the length scale ratio. This ratio is well-predicted by a simple dispersion relation which depends only on $m$, $n$, and the physical properties of the fluid. Based on the understanding of this tunable stabilization mechanism, we demonstrate how forcing functions composed of more than two frequency components may be used to further enhance the stability of SL-I patterns.
1.3 Outline of dissertation

We begin in Chapter 2 by reviewing the mathematical techniques which we use in our study of Faraday wave pattern formation. We review key concepts from bifurcation theory and discuss how linear stability analysis may be used to determine how the trivial solution of a system of partial differential equations loses stability. We then discuss how a center manifold reduction may be used, under suitable conditions, to reduce the governing partial differential equations to a finite-dimensional bifurcation problem. We review how the form of these bifurcation equations may be deduced from symmetry arguments, and how the coefficients in the equations may be calculated using a multiple scales perturbation method.

In Chapter 3 we review background on Faraday waves, discussing many of the previous experimental and theoretical results. We also present two mathematical formulations of the Faraday problem—the Navier-Stokes equations with a free boundary, and the Zhang-Viñals equations [24]—and briefly compare linear stability results for the two.

In Chapter 4 we present results for Faraday waves close to onset. We use symmetry arguments to show the importance of weakly damped harmonic modes for the pattern selection problem. Then, we calculate the coefficients in the bifurcation equations from the Zhang-Viñals equations in order to demonstrate the symmetry results. Implications for SL-I pattern stabilization are discussed in a simple bifurcation example. The results of this chapter were published in [28].

In Chapter 5 we present results for Faraday waves close to onset with weak damping. Symmetry arguments are used to identify the particular weakly damped har-
monic modes that are most important for SL-I pattern selection. We use a perturbation method to obtain analytical expressions for onset parameters and for the cubic coefficient in the bifurcation equation for one-dimensional waves. We analyze these expressions, paying special attention to the effect of the second forcing component. We also verify the symmetry results for stronger damping and forcing, and for two-dimensional waves, using the numerical calculation from Chapter 4. We discuss implications for SL-I pattern selection. The results of this chapter were published in [29].

In Chapter 6, we apply our understanding of the mechanism identified in Chapters 4 and 5 to suggest how multiple frequency forcing functions may be used to further enhance the stability of SL-I patterns.

Finally, in Chapter 7, we conclude by reviewing our main results and discussing directions for further research.
Chapter 2

Mathematical background

2.1 Introduction

In this chapter we review the mathematical background necessary for our investigation of pattern formation. We use a combination of tools from dynamical systems, elementary group theory, and perturbation theory. Our approach is relevant for the study of pattern formation due to local bifurcations. That is to say, it gives information about the qualitative changes in the dynamics of a system in a neighborhood of a particular point in phase space. In this dissertation, the focus is on applying these techniques to Faraday wave pattern formation; nonetheless, they may in fact be applied to a wide class of pattern formation problems.

This chapter is organized as follows. In Section 2.2, we review key concepts from bifurcation theory. We begin by focusing on finite systems of ordinary differential equations. For such systems, the Hartman-Grobman Theorem provides a criterion for determining when a local bifurcation occurs. We then turn to the case of partial
differential equations, and review how linear analysis may be used to determine when
the trivial solution loses stability. In Section 2.3, we review the Center Manifold
Theorem for systems of ordinary and partial differential equations. For the latter
case, when suitable mathematical conditions are satisfied, we may invoke the theorem
to reduce the governing partial differential equations to a finite number of bifurcation
equations, which are ordinary differential equations describing the slow evolution of
the critical modes. In Section 2.4, we discuss how symmetry considerations may
be used to deduce the form of the bifurcation equations. For our study of Faraday
waves, we follow [25, 26, 30] and restrict attention to solutions which are spatially
doubly-periodic; this restriction is also discussed in this section. Finally, in Section
2.5, we explain how a multiple scales perturbation method may be used to calculate
the values of the coefficients in the bifurcation equations.

We apply these techniques to the Zhang-Viñals partial differential equations [24]
describing Faraday waves later in this dissertation. In the present chapter, we demon-
strate some of the main ideas for a simpler example, namely the Brusselator reaction-
diffusion equations [31, 32] describing a toy chemical reaction that may undergo both
Turing and Hopf bifurcations.

2.2 Bifurcation theory

In this section, we review key ideas from bifurcation theory. We first state the relevant
concepts as they apply to a finite set of ordinary differential equations, and then
extend some of the ideas to the case of partial differential equations.
We begin by considering the finite system of ordinary differential equations

\[ \dot{U} = G(U; R). \]  

(2.1)

Here \( U \) represents the time-dependent functions \( U_1(t), \ldots, U_n(t) \). The functions \( G \) are nonlinear in their arguments. Equations (2.1) also depend on control parameters \( R = (R_1, \ldots, R_p) \in \mathbb{R}^p \). We assume that the system has a fixed point \( U = U^* \). We perturb the fixed point solution to determine its linear stability, writing \( U = U^* + u \) and neglecting nonlinear terms to obtain

\[ \dot{u} = J(U^*)u. \]  

(2.2)

Here, \( J(U^*) \) is the \( n \times n \) Jacobian matrix evaluated at the fixed point.

We refer to all of the eigenvalues of \( J(U^*) \) collectively as the eigenvalue spectrum. If the entire spectrum is contained in the left half of the complex plane then the perturbations \( u \) decay in time and the equilibrium solution \( U^* \) is linearly stable. If any part of the spectrum lies in the right half-plane then some perturbations grow in time and \( U^* \) is linearly unstable. If any eigenvalues lie on the imaginary axis, and the rest lie in the left half-plane, then \( U^* \) is neutrally stable.

The linear stability of a fixed point is related to the structural stability of the nearby phase portrait in an important and useful way. This relationship is described by the Hartman-Grobman theorem, which is discussed in many dynamics texts, including [33, 34, 35, 36, 37]. Before we can discuss this relationship, we must introduce several concepts.
A phase portrait for a system is structurally stable if its topology is unchanged by perturbing the equations, \textit{i.e.} by adding small terms to the right side of (2.1). The topological properties of the phase portrait include, for instance, the number of fixed points and their stability. Two phase portraits are said to be topologically equivalent if there is a homeomorphism (a continuous deformation with a continuous inverse) that maps one onto the other. Qualitatively, this means that one phase portrait may be bent and warped (but not ripped) to obtain the other. Finally, we define the notion of hyperbolicity of an equilibrium solution. The fixed point \( U^* \) is said to be hyperbolic if none of the eigenvalues of (2.2) have real part equal to zero. If any eigenvalue has a zero real part, then \( U^* \) is said to be nonhyperbolic.

We now state the Hartman-Grobman theorem, which relates information about the solutions of (2.1) to the solutions of (2.2). The Hartmann-Grobman theorem states that if \( U^* \) is hyperbolic, then the phase portraits near \( U^* \) for the systems (2.1) and (2.2) are topologically equivalent. For example, this means that if we determine that a hyperbolic fixed point \( U^* \) is linearly unstable then it is also nonlinearly unstable, \textit{i.e.} small perturbations from \( U^* \) will grow in time according to the dynamics prescribed by (2.1). This is not necessarily the case for a nonhyperbolic fixed point. If \( U^* \) is nonhyperbolic for some value of the control parameters \( R \), then the stability properties of \( U^* \) for the dynamics of (2.1) are not necessarily the same as those for (2.2). Furthermore, the Implicit Function Theorem (see, for example, [37]) tells us that when the loss of hyperbolicity occurs such that \( \det J(U^*) = 0 \), the number of solution branches may change. We refer to these situations, in which the number of equilibria and their stability properties may change, as “bifurcations,” or changes in the qualitative dynamics of the system.
We now briefly review the canonical or "normal forms" for bifurcations in simple systems of ordinary differential equations with one control parameter, i.e. $R = R \in \mathbb{R}$; see, for example, [33, 34, 35, 37]. By "normal form" we mean the simplest nonlinear system which retains the essential characteristics of a particular bifurcation. Bifurcation diagrams provide a convenient way to visualize the bifurcations described by these normal forms. A bifurcation diagram is a plot of some variable representing a solution to the system versus a control parameter. Stable equilibria are indicated by solid lines, and unstable equilibria by dotted lines. Examples are shown in Figure 2.1.

The saddle-node bifurcation is the generic mechanism for the creation of two equilibria. The normal form is

$$\dot{u} = R + au^2. \quad (2.3)$$

We assume first that $a < 0$. For $R < 0$, there is no steady state solution. For $R > 0$, there are two steady state solutions $u_+ = \pm \sqrt{-R/a}$. The $u_+$ solution is stable and the $u_-$ solution is unstable. The corresponding bifurcation diagram, which plots the steady solutions $u$ versus $R$ and indicates their stability, is shown in Figure 2.1a. For $a > 0$, the nontrivial solutions $u_{\pm}$ exist only for $R < 0$, and their stability assignments are the reverse of those for the $a < 0$ case.

The transcritical bifurcation is the generic mechanism by which two equilibria exchange their stabilities. The normal form is

$$\dot{u} = Ru + au^2 \quad (2.4)$$

There are two steady state solutions $u_0 = 0$ and $u_1 = -R/a$. For $R < 0$, $u_0$ is stable
Figure 2.1: Bifurcation diagrams for some of the simple normal forms. Stable (unstable) branches are indicated by a solid (dotted) line. For all of the following, we set the parameter $a$ to be negative. (a) Saddle-node bifurcation specified by (2.3). (b) Transcritical bifurcation specified by (2.4). (c) Supercritical pitchfork bifurcation specified by (2.5). (d) Supercritical Hopf bifurcation specified by (2.6).
and $u_1$ is unstable. For $R > 0$, $u_0$ is unstable and $u_1$ is stable. The bifurcation
diagram is given in Figure 2.1b.

The pitchfork bifurcation is the characteristic bifurcation for systems with reflection
symmetry. The normal form is

$$\dot{u} = Ru + au^3.$$ \hfill (2.5)

We assume first that $a < 0$. For $R < 0$ the only steady state solution is the trivial
solution $u_0 = 0$, which is stable. For $R > 0$, $u_0$ is unstable and there are two additional
stable solutions $u_{\pm} = \pm \sqrt{-R/a}$. Since the nontrivial solutions exist only above the
$R$ value where the trivial solution becomes unstable, this bifurcation is said to be
“supercritical.” The bifurcation diagram is shown in Figure 2.1c. It is also possible
to have a subcritical pitchfork bifurcation (not shown) when $a > 0$. In this case,
the nontrivial solutions are unstable, and exist only in the regime where the trivial
solution is stable.

The three bifurcations previously discussed are of steady state type. That is
to say, they are due to a single real eigenvalue crossing the imaginary axis. When
two complex conjugate eigenvalues cross the imaginary axis (and if the genericity
conditions of the Hopf Bifurcation Theorem are satisfied, as described, for example,
in [36]) then a Hopf bifurcation occurs and the system undergoes a transition to an
oscillatory solution. The normal form is two dimensional and takes the form

$$\dot{\rho} = R\rho + a\rho^3$$ \hfill (2.6a)  
$$\dot{\phi} = \omega + b\rho^2$$ \hfill (2.6b)
where $\rho e^{i\phi} = u_1 + iu_2$ and all coefficients are real. The bifurcation diagram for the supercritical Hopf bifurcation ($a < 0$) is shown in Figure 2.1d (the subcritical case, for $a > 0$, is not shown). The bifurcation diagram is given in $R - \rho$ space. The broken ellipse indicates a periodic solution, or limit cycle, due to the $\phi$ dynamics.

We now turn away from the case of finite systems of ordinary differential equations and focus instead on partial differential equations, which are infinite dimensional dynamical systems (in the sense that the phase space is infinite dimensional). We assume that the governing partial differential equations have been linearized around the trivial solution to obtain a problem of the form

$$\partial_t u = Lu$$

(2.7)

with appropriate boundary conditions. Here, $L$ is a linear operator. If the entire eigenvalue spectrum is contained in the left half of the complex plane and is bounded away from the imaginary axis then the equilibrium solution $U^*$ is linearly stable. When the spectrum crosses the imaginary axis, the trivial solution becomes unstable.

The linear analysis identifies a bifurcation point of the system at which the trivial solution loses stability. Such bifurcations may lead to patterned states of the type studied later in this dissertation. Thus, the linear analysis is necessarily the starting point for our investigation, and for many investigations of pattern formation. A wealth of sources on linear stability analysis is available; introductions are provided in [4, 38]. We now clarify the concept of linear stability analysis of equilibrium solutions of partial differential equations by means of a concrete example.

We examine a system of two coupled equations called the “Brusselator” [31, 32]
which models a toy chemical reaction-diffusion system. The Brusselator equations are

\begin{align}
\partial_t U &= A - (B + 1)U + U^2V + D_U \nabla^2 U \\
\partial_t V &= BU - U^2V + D_V \nabla^2 V
\end{align}

(2.8)

and we consider an infinite domain of two spatial dimensions. In (2.8), \(U(x, y, t)\) and \(V(x, y, t)\) are concentrations of the two chemical species under consideration, and \(D_U\) and \(D_V\) are their respective diffusion coefficients. The constant parameters \(A\) and \(B\) are concentrations of species being continuously fed into the system. A general introduction to the chemistry and mathematics of reaction diffusion systems may be found in [39]. A thorough linear stability analysis of (2.8) is performed in [40].

The spatially homogeneous steady state solution is \(U^* = (U^*, V^*)^T = (A, B/A)^T\). The linearized system for the perturbations \(u = (u, v)^T\) is given by (2.7) with linear operator

\[
L = \begin{pmatrix}
B - 1 + D_U \nabla^2 & A^2 \\
-B & -A^2 + D_V \nabla^2
\end{pmatrix}
\]

(2.9)

For this problem, the eigenvalue spectrum is continuous and the eigenfunctions are simple Fourier modes. (Other problems involving other boundary conditions may yield different results. For instance, if equations 2.8 are posed on a finite square domain with no flux boundary conditions, the spectrum is discrete and the eigenfunctions are products of cosines.) Thus, to determine the linear stability of the
steady state solution, we consider perturbations of the form

\[
\mathbf{u} = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} e^{\sigma t + i k \cdot \mathbf{x}}.
\]

(2.10)

By substituting (2.10) into (2.7) and writing the characteristic polynomial for the resulting eigenvalue problem, we obtain the dispersion relation, which relates the eigenvalues \(\sigma\) to the wave number \(|k| \equiv k\) of the perturbation. The quadratic dispersion relation for our example is

\[
\sigma^2 + \{A^2 - B + 1 + [D_U + D_V] k^2\} \sigma \\
+ \{D_U D_V k^4 + [D_U A^2 + D_V (1 - B)] k^2 + A^2\} = 0.
\]

(2.11)

The neutral curve corresponding to a steady state transition is determined by the condition \(\det \mathbf{L} = 0\). Furthermore, if we insist that the steady state curve lie below the Hopf curve, we also obtain the condition \(\text{Tr} \mathbf{L} < 0\). These two conditions yield

\[B = D_U k^2 + 1 + \frac{D_U}{D_V} A^2 + \frac{A^2}{D_V} \frac{1}{k^2}\] (2.12a)

\[B < (D_U + D_V) k^2 + A^2 + 1\] (2.12b)

The neutral curve corresponding to a Hopf transition is determined by the condi-
tions $\text{Tr} L = 0$ and $\det L > 0$. These conditions lead to

$$B = (D_U + D_V)k^2 + A^2 + 1$$

(2.13a)

$$B < D_Uk^2 + 1 + \frac{D_U}{D_V}A^2 + \frac{A^2}{D_V} \frac{1}{k^2}.$$  

(2.13b)

We now examine a sample case where $A = 3$, $D_U = 0.1$ and $D_V = 0.4$ and consider bifurcations that are due to increasing the parameter $B$. The neutral stability curve, showing the values of $B$ at which the equilibrium solution $U^*$ loses stability to a particular perturbation wave number $k$, is shown in Figure 2.3a.

For sufficiently small values of $B$, the equilibrium solution $U^*$ is stable to perturbations of all Fourier wave numbers $k$. As the parameter $B$ is increased, the system undergoes its primary bifurcation at the minimum of the neutral stability curve. The location of this point in $k - B$ space can be determined by solving $\partial B/\partial k = 0$ as well as (2.12a) and (2.13a). For the present example, the primary bifurcation point, labeled $P_1$ in Figure 2.3a, is at a critical parameter value $B_c = 6.25$ and corresponds to a critical wave number $k_c = \sqrt{15}$. This is an example of a Turing bifurcation that leads to a Turing pattern [41].

Another neutral stability curve is shown in Figure 2.3b, for $A = 1.8$, $D_U = 0.1$ and $D_V = 0.2$. In this case, the primary bifurcation, indicated by point $P_2$, is of Hopf type and occurs at critical wave number $k_c = 0$ for a critical parameter value $B_c = 4.24$. This is an oscillatory instability leading to a state of homogeneous oscillations since $k_c = 0$.

In the next section we will explain how, when appropriate mathematical conditions are satisfied, the study of bifurcations in partial differential equations may be reduced
Figure 2.2: Right-most portion of eigenvalue spectra in the complex plane for two types of bifurcations. Critical eigenvalues are marked with an “x”. The data are computed from the linearization of the Brusselator equations (2.8). (a) A steady state bifurcation corresponding to $P_1$ in Figure 2.3a. The parameters are $A = 3, B = 6.25, D_U = 0.1$ and $D_V = 0.4$. (b) A Hopf bifurcation corresponding to $P_2$ in Figure 2.3b. The parameters are $A = 1.8, B = 4.24, D_U = 0.1$ and $D_V = 0.2$. 
Figure 2.3: Neutral stability curves for the Brusselator equations (2.8). The spatially homogeneous steady state is stable below the curves and unstable above the curves. The solid curve corresponds to the steady state bifurcation specified by (2.12); the dotted curve corresponds to the Hopf bifurcation specified by (2.13). Only the bottom of the curve is shown, i.e. for each wave number $k$, only the first instability to occur with increasing $B$ is indicated. (a) $A = 3$, $D_U = 0.1$ and $D_V = 0.4$. The primary bifurcation is of steady state type and occurs at $P_1 = (k_c, B_c) = (\sqrt{15}, 6.25)$. The eigenvalue spectrum is shown in Figure 2.2a. (b) $A = 1.8$, $D_U = 0.1$ and $D_V = 0.2$. The primary bifurcation is of Hopf type and occurs at $P_2 = (k_c, B_c) = (0, 4.24)$. The eigenvalue spectrum is shown in Figure 2.2b.
to the study of a normal form bifurcation problem consisting of a finite number of ordinary differential equations. For the research presented in this dissertation, the relevant finite-dimensional system describes the evolution of the critical modes which make up various spatially-periodic Faraday wave patterns near the onset of the primary bifurcation. The normal form is more complicated than the simple ones which we reviewed at the beginning of this section because of the presence of symmetries, and will be presented in Section 2.4.

2.3 Center manifold reduction

In this section we explain how, under suitable mathematical conditions, the technique of center manifold reduction may be used to reduce the governing partial differential equations to a finite dimensional set of ordinary differential equations. This technique will be used in Chapters 4 and 5 in order to derive bifurcation equations describing the evolution of Faraday wave patterns near the onset of the primary bifurcation.

For simplicity, we begin by discussing center manifold reduction in the context of a finite system of ordinary differential equations. The discussion is similar to that found in many sources; see, for example, [37]. We consider the system (2.1) and assume as before that the system has a fixed point $U = U^*$. The eigenvalue spectrum of $J(U^*)$
may be divided into three sets depending on the real parts of the eigenvalues \( \sigma_i \):

\[
\sigma_s = \{ \sigma_i | \text{Re} \sigma_i < 0 \} \tag{2.14a}
\]
\[
\sigma_u = \{ \sigma_i | \text{Re} \sigma_i > 0 \} \tag{2.14b}
\]
\[
\sigma_c = \{ \sigma_i | \text{Re} \sigma_i = 0 \}. \tag{2.14c}
\]

The respective generalized eigenspaces are the stable eigenspace \( E^s \), the unstable eigenspace \( E^u \), and the center eigenspace \( E^c \). A schematic diagram of these eigenspaces is shown in Figure 2.4b. This example corresponds to a steady state bifurcation, as shown by the eigenvalue spectrum in Figure 2.4a. For this example, \( \dim E^s = 4 \), \( \dim E^u = 0 \), and \( \dim E^c = 1 \). A general and useful property of \( E^s \), \( E^u \), and \( E^c \) is that they are dynamically invariant for the linearized problem, i.e. an initial condition in one of these subspaces will remain in the subspace for all time according to dynamics prescribed by the linearization (2.2).

In order to study systems of nonlinear equations, we would like to have a nonlinear analogue to the linear eigenspaces discussed above. This is provided by the Center Manifold Theorem; see, for example, [36]. The Center Manifold Theorem guarantees the existence of the stable manifold \( W^s \), the unstable manifold \( W^u \), and the center manifold \( W^c \). These are tangent, respectively, to \( E^s \), \( E^u \), and \( E^c \). Examples are shown in Figure 2.4b. Furthermore, they are invariant under the dynamics prescribed by the nonlinear system (2.1). The center manifold \( W^u \) is not guaranteed to be unique. Intuitively, this non-uniqueness is related to the fact that different trajectories tangent to \( E^c \) can differ by exponentially small amounts but still have the same asymptotic behavior.
In a neighborhood of a bifurcation, there is a separation of time scales between the dynamics in $W^c$ and the dynamics in $W^s$. An initial condition beginning in a neighborhood of the fixed point $U^*$ will evolve very quickly in $W^s$ and approach $W^c$. Then, the system will undergo a slow time evolution along $W^c$. In other words, the asymptotic state of the system will be determined by the dynamics in $W^c$. Thus, we adopt the viewpoint that the “interesting” dynamics occur on $W^c$, and focus our efforts on reducing the full governing system down to a simpler system which describes the slow time evolution of the critical mode(s) on the center manifold. This so-called center manifold reduction results in an enormous mathematical simplification of the problem.

The ideas discussed above have been presented in the context of ordinary differ-
ential equations. A similar set of ideas holds for partial differential equations. Here, we give the statement of the Center Manifold Theorem as it appears in [42]:

(i) Let $Z$ be a Banach space admitting a $C^\infty$ norm away from $0$ and let $F_t$ be a $C^0$ semiflow defined in a neighborhood of zero for $0 \leq t \leq \tau$. Assume $F_t(0) = 0$, and that for $t > 0$, $F_t(x)$ is $C^{k+1}$ jointly in $t$ and $x$.

(ii) Assume that the spectrum of the linear semigroup $DF_t(0) : Z \to Z$ is of the form $e^{t(\sigma_1 \cup \sigma_2)}$ where $e^{t\sigma_1}$ lies on the unit circle (i.e. $\sigma_1$ lies on the imaginary axis) and $e^{t\sigma_2}$ lies inside the unit circle a nonzero distance from it for $t > 0$, i.e. $\sigma_2$ is in the left half-plane.

(iii) Let $Y$ be the generalized eigenspace corresponding to the part of the spectrum on the unit circle. Assume $\dim Y = d < \infty$.

Then there exists a neighborhood $V$ of $0$ in $Z$ and a $C^k$ submanifold $M \subset V$ of dimension $d$ passing through $0$ and tangent to $Y$ at $0$ such that

(a) If $x \in M$, $t > 0$ and $F_t(x) \in V$, then $F_t(x) \in M$.

(b) If $t > 0$ and $F^n_t(x)$ remains defined and in $V$ for all $n = 0, 1, 2, \ldots$ then $F^n_t(x) \to M$ as $n \to \infty$.

Condition (i) above, namely the existence of a suitable Banach-space formulation, is satisfied for the unforced Navier-Stokes equations (as discussed in [42]). As the starting point for the mathematical investigation of Faraday waves, we use these equations. However, we add a periodic forcing term, and we also formulate our center manifold dynamics in terms of a map, rather than a flow. Nonetheless, we will not
consider condition (i) further in this dissertation. Rather, when invoking the Center Manifold Theorem, we will focus on conditions (ii) and (iii), which we now discuss in more detail.

Condition (ii) requires that the stable portion of the eigenvalue spectrum be bounded a finite distance from the imaginary axis. For the case of ordinary differential equations discussed above, the eigenvalue spectrum is discrete and thus the condition is automatically satisfied. For instance, in Figure 2.4 there is a gap between the eigenvalue at the origin and the eigenvalue just to its left. In contrast, for the case of partial differential equations, this condition may be violated; see, for example, Figure 2.2. In such a situation, there are modes in the system that are arbitrarily close to criticality, and thus cannot be “slaved” in a center manifold reduction.

Condition (iii) requires that the dimension of the generalized eigenspace corresponding to the neutral eigenvalues be of finite dimension. This condition may be violated due to the symmetries present in many physical systems of interest. For instance, when posed on an infinite plane, the Brusselator equations (2.8) and the Zhang-Viñals Faraday wave equations both have a continuous rotational symmetry. Linear analysis on these equations tells us the critical wave number $k_c$, but because of the rotational symmetry, perturbations having any Fourier wave vector with magnitude $k_c$ will have zero growth rate at the primary bifurcation. This results in a so-called “critical circle” of wave vectors in Fourier space which contains an infinite number of modes, as shown in Figure 2.5. Thus, even if there were a gap in the eigenvalue spectrum, a center manifold reduction to a finite number of modes would be impossible.

In practice, we may bypass both of the problems discussed above and thus satisfy
Figure 2.5: Critical circle of wave vectors in Fourier space. The circle consists of all wave vectors $\mathbf{k}$ having magnitude equal to the critical wave number $k_c$. This situation arises when the system under consideration has a continuous rotational symmetry.

the requirements necessary to invoke the Center Manifold Theorem by restricting our attention to a class of spatially-periodic solutions. This is the standard approach used, for example, in [43]. This restriction is arbitrary, but is still useful since many of the Faraday wave patterns observed in experiment are periodic (see Section 3.2 for a discussion of experimentally observed patterns). Under this restriction, the eigenvalue spectrum is discrete and bounded away from the imaginary axis (except for the critical eigenvalue or eigenvalues), and the center eigenspace is finite dimensional at the primary bifurcation. The particular class of periodic solutions we consider in this dissertation and the form of the resulting bifurcation equations describing the slow-time dynamics on the center manifold are introduced at the end of the next section.
2.4 Symmetries and normal forms

Equivariant bifurcation theory provides us with tools for studying bifurcations in the presence of symmetry. We may use this machinery to deduce the terms in the normal form bifurcation equations describing the slow dynamics on the center manifold, and to determine the existence of certain solution branches of the resulting bifurcation problem. The methods of equivariant bifurcation theory are given a full treatment in [23]. In this section, we mention a few of the main ideas used in deducing the form of the bifurcation equations from the symmetries of the problem. Throughout, we illustrate the ideas by means of a simple example using the Brusselator equations (2.8).

We begin by commenting that some systems of partial differential equations, considered together with their boundary conditions, have a non-trivial symmetry group $\Gamma$ corresponding to geometric operations which leave the equations unchanged. For instance, when posed on an unbounded horizontal domain, the Brusselator equations (2.8) (and the Zhang-Viñals Faraday wave equations which we introduce in Section 3.4) both possess the symmetry of the Euclidean group $E(2)$.

The bifurcation equations describing slow evolution along the center manifold, such as (2.32), may be written in the form

$$\dot{z} = f(z; \lambda).$$

(2.15)

Here, $\lambda$ represents one or more bifurcation parameters. These bifurcation equations inherit the symmetry of the governing equations and boundary conditions. For the present moment, let us consider a simple example, namely the restriction of the Brus-
Selator equations (2.8) to one spatial dimension. We assume that the one dimensional

domain has periodic boundary conditions, and that its size may be selected to fit the
critical wavelength $2\pi/k_c$. For the case of a steady-state bifurcation, there is only one
mode (and its complex conjugate) on the center manifold, which we write as

$$z(t) = u e^{ik_c x} + c.c. .$$  \hspace{1cm} (2.16)

Here, $u$ is the critical eigenvector determined from linear stability analysis and $z(t)$
is its slowly varying amplitude.

A one-dimensional spatial translation symmetry acts on $z$ as

$$\Theta(x \to x + \Delta x) : z \to z e^{i\theta}$$  \hspace{1cm} (2.17)

where $\theta = k_c \Delta x$. A reflection symmetry acts on $z$ as

$$\kappa(x \to -x) : z \to \bar{z}.$$  \hspace{1cm} (2.18)

These actions generate the group $O(2)$. The general differential equations (2.15) are
$\Gamma$-equivariant, which means that

$$f(z) = \gamma^{-1}f(\gamma z)$$  \hspace{1cm} (2.19)

or equivalently, if $z_0$ is a solution to (2.15), then so is $\gamma z_0$. From the condition (2.19)
we may deduce the form of the equations (2.15) as we now describe.

We begin the procedure by writing the functions $f$ in (2.15) as Taylor series in
terms of their arguments $z$. For our simple example, we may write

$$f(z) = \sum_{\alpha, \beta} a_{\alpha, \beta} z^\alpha \bar{z}^\beta. \quad (2.20)$$

Here, $\alpha, \beta \in \mathbb{Z}$ and $\alpha, \beta \geq 0$. We then enforce the condition (2.19) to deduce restrictions on $f$.

For our example, using $\gamma = \Theta$ in (2.19) yields the condition

$$\sum_{\alpha, \beta} a_{\alpha, \beta} z^\alpha \bar{z}^\beta = \sum_{\alpha, \beta} a_{\alpha, \beta} z^\alpha \bar{z}^\beta e^{i\theta(\alpha-\beta-1)} \quad (2.21)$$

from which it follows (for arbitrary $\theta$) that $\alpha = 1 + \beta$. Thus, we may now write

$$f(z) = \sum_{\beta} a_{\beta} z^{1+\beta} \bar{z}^\beta. \quad (2.22)$$

Now, using $\gamma = \kappa$ in (2.19) with (2.22) as a starting point yields the condition

$$\sum_{\beta} a_{\beta} z^{1+\beta} \bar{z}^\beta = \sum_{\beta} a_{\beta} z^{1+\beta} \bar{z}^\beta \quad (2.23)$$

from which it follows that $a_{\beta} \in \mathbb{R}$. Thus, the normal form is $\dot{z} = g(|z|^2)z$, where $g$ is real-valued. The cubic truncation of the bifurcation equation is

$$\dot{z} = a_0 z + a_1 |z|^2 z \quad (2.24)$$

which we mention now for comparison with a result in the following section.

Here we have shown how symmetry arguments may be used to determine the form
of bifurcation equations. While the our example is an application to the Brusselator equations, the result holds for any system with the same symmetries and bifurcation type.

The procedure is similar in principle, but more complicated in practice, for the bifurcation equations which are relevant for our study of Faraday waves later in this dissertation. We now turn to a discussion of this normal form. In order to reduce the governing partial differential equations to a finite dimensional system describing the slow center manifold dynamics, we will restrict our attention to solutions which are doubly periodic on a hexagonal lattice and consider the 12-dimensional $D_6+T^2$ equivariant bifurcation problems that result. With this restriction, we will be able to determine the relative stability of simple stripe patterns, simple hexagonal patterns, three distinct rhombic patterns, and superhexagon and supertriangle SL-I patterns. (For the reader unfamiliar with these patterns, some examples are shown in Section 3.2.) This approach is developed in [25, 26, 30]. The form of the bifurcation equations is determined by symmetry arguments, and the stability of various bifurcation branches is calculated. We now review the main points.

We consider solutions which are doubly periodic on a hexagonal lattice. For instance, the surface height of the fluid $h(x, t)$ is written in the form

$$h(x, t) = \sum_{m_1, m_2 \in \mathbb{Z}} z_{m_1, m_2}(t)e^{i(m_1k_1 + m_2k_2) \cdot x} + c.c. \quad (2.25)$$

Here, $k_1$ and $k_2$ generate a hexagonal dual lattice. That is,

$$k_1 = \frac{k_c}{r} (0, 1), \quad k_2 = \frac{k_c}{r} \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \quad (2.26)$$
where $r$ is a length scale parameter which determines the lattice spacing.

The twelve-dimensional bifurcation problem we consider applies when there are twelve integer pairs $(m_1, m_2)$ in (2.25) such that

$$|m_1 k_1 + m_2 k_2| = k_c,$$

(2.27)

where $k_c$ is the critical wave number associated with the primary bifurcation. Following [30] we associate with each lattice an integer pair $(n_1, n_2)$. Without loss of generality, we assume that $n_1$ and $n_2$ are co-prime, $n_1 > n_2 > n_1/2 > 0$, and $n_1 + n_2$ is not a multiple of 3. The twelve neutral modes that span the center eigenspace at the bifurcation point take the form

$$\sum_{j=1}^{6} z_j e^{iK_j \cdot \mathbf{x} + c.c.}$$

(2.28)

where $z_j \in \mathbb{C}$ are the complex mode amplitudes. Here,

$$K_1 = n_1 k_1 + n_2 k_2$$
$$K_2 = (n_1 - n_2) k_1 - n_2 k_2$$
$$K_3 = -n_2 k_1 + (n_1 - n_2) k_2$$
$$K_4 = n_1 k_1 + (n_1 - n_2) k_2$$
$$K_5 = -n_2 k_1 - n_1 k_2$$
$$K_6 = (n_2 - n_1) k_1 + n_2 k_2.$$

(2.29)

See Figure 2.6 for an example of two lattices and the corresponding twelve critical
wave vectors. Note that $\pm \mathbf{K}_1, \pm \mathbf{K}_2, \pm \mathbf{K}_3$ point to the vertices of a hexagon, as do $\pm \mathbf{K}_4, \pm \mathbf{K}_5, \pm \mathbf{K}_6$, and that the two hexagons are rotated relative to each other by an angle $\theta_h \in (0, \pi/3)$ indicated by the shaded sector in Figure 2.6. This angle is related to $(n_1, n_2)$ by

$$\cos(\theta_h) = \frac{n_1^2 + 2n_1n_2 - 2n_2^2}{2(n_1^2 - n_1n_2 + n_2^2)}. \quad (2.30)$$

Also note that each lattice determines a length scale ratio, namely $r$ in (2.26), which depends on $(n_1, n_2)$ as

$$r = |\mathbf{K}_j|/|\mathbf{k}_1| = \sqrt{n_1^2 - n_1n_2 + n_2^2} \geq \sqrt{7}. \quad (2.31)$$

The example in Figure 2.6a corresponds to $(n_1, n_2) = (3, 2)$, for which $\theta_h \approx 22^\circ$ in (2.30) and the lengthscale ratio $r$ in (2.31) is the smallest associated with a hexagonal lattice, namely $\sqrt{7}$. The finer lattice in Figure 2.6b corresponds to $(n_1, n_2) = (5, 3)$, for which $\theta_h \approx 13^\circ$ and the lengthscale ratio is $r = \sqrt{19}$.

Through cubic order in the amplitudes $z_j$, the resulting twelve-dimensional bifurcation problem takes the form

$\dot{z}_1 = \lambda z_1 + \epsilon \overline{z}_2 \overline{z}_3 + (b_1|z_1|^2 + b_2|z_2|^2 + b_2|z_3|^2 + b_4|z_4|^2 + b_5|z_5|^2 + b_6|z_6|^2)z_1 \quad (2.32a)$

$\dot{z}_2 = \lambda z_2 + \epsilon \overline{z}_1 \overline{z}_3 + (b_1|z_2|^2 + b_2|z_1|^2 + b_2|z_3|^2 + b_4|z_5|^2 + b_5|z_6|^2 + b_6|z_4|^2)z_2 \quad (2.32b)$

$\dot{z}_3 = \lambda z_3 + \epsilon \overline{z}_1 \overline{z}_2 + (b_1|z_3|^2 + b_2|z_1|^2 + b_2|z_2|^2 + b_4|z_6|^2 + b_5|z_4|^2 + b_6|z_5|^2)z_3 \quad (2.32c)$

$\dot{z}_4 = \lambda z_4 + \epsilon \overline{z}_5 \overline{z}_6 + (b_1|z_4|^2 + b_2|z_5|^2 + b_2|z_6|^2 + b_4|z_1|^2 + b_5|z_3|^2 + b_6|z_2|^2)z_4 \quad (2.32d)$

$\dot{z}_5 = \lambda z_5 + \epsilon \overline{z}_4 \overline{z}_6 + (b_1|z_5|^2 + b_2|z_4|^2 + b_2|z_6|^2 + b_4|z_2|^2 + b_5|z_1|^2 + b_6|z_3|^2)z_5 \quad (2.32e)$

$\dot{z}_6 = \lambda z_6 + \epsilon \overline{z}_4 \overline{z}_5 + (b_1|z_6|^2 + b_2|z_4|^2 + b_2|z_5|^2 + b_4|z_3|^2 + b_5|z_2|^2 + b_6|z_1|^2)z_6 \quad (2.32f)$
Figure 2.6: Fourier space diagram of the wave vectors (2.29) corresponding to the critical modes in the 12-dimensional bifurcation problems (2.32) and (4.11). The hexagonal lattice is specified by the integer pair \((n_1, n_2)\). Short arrows indicate the sum \(K_1 = n_1 k_1 + n_2 k_2\) in (2.29a). The shaded sector indicates the characteristic angle \(\theta_h\) determined by (2.30). (a) \((n_1, n_2) = (3, 2)\), length scale ratio \(r = \sqrt{7}\) in (2.31), and \(\theta_h \approx 22^\circ\). (b) \((n_1, n_2) = (5, 3)\), \(r = \sqrt{19}\), and \(\theta_h \approx 13^\circ\).
where \( \lambda \) measures the distance from the bifurcation point.

Symmetry methods may also be used to determine which solution branches are guaranteed to exist as a result of the Equivariant Branching Lemma (detailed treatments of this method are given in [23, 44]). The branching equations and stability assignments for these branches for the system (2.32) (and for one additional primary branch) are given in Table 2.1.

The generic presence of a quadratic term in (2.32) renders all of the solutions in Table 2.1 unstable at bifurcation. Hence the transition from the trivial state to a nontrivial branch is expected to be hysteretic. In order to capture stable weakly nonlinear solutions, we must focus our analysis on the unfolding of the degenerate bifurcation problem \( \epsilon = 0 \). Note that when \( \epsilon = 0 \) the stability of simple and super hexagons/triangles is not determined by the cubic order since the phases \( \phi_j \) of solutions \( z_j = r_j e^{i\phi_j} \) to (2.32) are then arbitrary. Even in the case of \( 0 < |\epsilon| \ll 1 \) the relative stability of super hexagons and super triangles depends on terms that are at least fifth order. However, we may use the cubic truncation to determine that one (and only one) of these two solutions is stable. The higher order terms are only needed to determine whether it is the hexagonal or triangular superpattern [25].

For some of our theoretical investigation of Faraday waves, it will be useful to formulate the bifurcation equations in terms of a stroboscopic map, which describes the complex amplitudes of the critical modes once per period of the forcing function. This formulation is discussed in Section 4.3.

While symmetry methods yield the form of the bifurcation equations, they cannot yield the values of the coefficients since these depend on the specifics of the partial differential equations under consideration. In the next section, we show how the
<table>
<thead>
<tr>
<th>Planform and branching equation</th>
<th>Stability</th>
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</table>
| **Stripes**  
\( z = (x, 0, 0, 0, 0, 0) \)  
\( 0 = \lambda x + b_1 x^3 + \mathcal{O}(x^5) \) | \( \text{sgn}(b_1), \text{sgn}(\epsilon x + (b_2 - b_1) x^2), \text{sgn}(-\epsilon x + (b_2 - b_1) x^2), \text{sgn}(b_4 - b_1), \text{sgn}(b_5 - b_1), \text{sgn}(b_6 - b_1) \) |
| **Simple Hexagons**  
\( z = (x, x, x, 0, 0, 0) \)  
\( 0 = \lambda x + \epsilon x^2 + (b_1 + b_2) x^3 + \mathcal{O}(x^4) \) | \( \text{sgn}(\epsilon x + 2(b_1 + 2b_2) x^2), \text{sgn}(-\epsilon x + (b_1 - b_2) x^2) \)  
\( \text{sgn}(-\epsilon x + (b_1 + b_5 + b_6 - b_2 - b_6) x^2) \)  
\( \text{sgn}(-\epsilon x + \mathcal{O}(x^3)) \) |
| **Rhombs (Rh\(_4\))**  
\( z = (x, 0, 0, x, 0, 0) \)  
\( 0 = \lambda x + (b_1 + b_4) x^3 + \mathcal{O}(x^5) \) | \( \text{sgn}(b_1 + b_4), \text{sgn}(b_1 - b_4), \text{sgn}(\zeta_1), \text{sgn}(\zeta_2), \)  
\( \zeta_1 + \zeta_2 = (-2b_1 - 2b_4 + 2b_2 + b_5 + b_6)x^2, \)  
\( \zeta_1 \zeta_2 = -\epsilon^2 x^2 + (b_1 + b_4 - b_2 - b_5)(b_1 + b_4 - b_2 - b_6)x^4 \) |
| **Rhombs (Rh\(_5\))**  
\( z = (x, 0, 0, 0, x) \) | \( \text{same as Rh}_4 \text{ with } b_4 \leftrightarrow b_5 \) |
| **Rhombs (Rh\(_6\))**  
\( z = (x, 0, 0, 0, 0, x) \) | \( \text{same as Rh}_4 \text{ with } b_4 \leftrightarrow b_6 \) |
| **Super Hexagons**  
\( z = (x, x, x, x, x, x) \)  
\( 0 = \lambda x + \epsilon x^2 + (b_1 + 2b_2)x^3 \)  
\( + (b_4 + b_5 + b_6)x^3 + \mathcal{O}(x^4) \) | \( \text{sgn}(\epsilon x + 2(b_1 + 2b_2 + b_4 + b_5 + b_6) x^2) \)  
\( \text{sgn}(\epsilon x + 2(b_1 + 2b_2 - b_4 - b_5 - b_6) x^2) \)  
\( \text{sgn}(-\epsilon x + \mathcal{O}(x^3)), \text{sgn}(\zeta_1), \text{sgn}(\zeta_2), \)  
\( \zeta_1 + \zeta_2 = -4\epsilon x + 4(b_1 - b_2) x^2, \)  
\( \zeta_1 \zeta_2 = 4(\epsilon x - (b_1 - b_2) x^2)^2 \)  
\( -2((b_4 - b_5)^2 + (b_4 - b_6)^2 + (b_5 - b_6)^2) x^4 \)  
\( \text{sgn}(\zeta_3), \text{where } \zeta_3 = \mathcal{O}(x^{2n_1 - 1}) \) |
| **Super Triangles**  
\( z = (z, z, z, z, z), \)  
\( z = xe^{i\psi}, \psi \neq 0, \pi, \ldots \) | \( \text{Same as super hexagons except } \zeta_3 \rightarrow -\zeta_3 \) |

Table 2.1: Branching equations and stability assignments for the bifurcation equations (2.32), and for (4.11) with \( \sigma = 1 \). A solution is stable if all quantities in the right column are negative. See [25, 26, 30] for more details.
coefficients can be calculated using a multiple scales perturbation method.

2.5 Normal form coefficients and multiple scales perturbation theory

In the previous section we stated that the form of the bifurcation equations may be deduced from symmetry arguments. However, for our studies of Faraday wave equations, we would like to have not only the form of the equations, but also the values of the coefficients, so that we may make quantitative predictions. In practice, these coefficients are obtained by an explicit calculation from the governing partial differential equations. In this dissertation, we accomplish this center manifold reduction via a multiple scales perturbation method. Such methods are discussed in many standard perturbation theory texts including [45].

In this section, we introduce the method with a simple example, namely by performing a simple multiple scales calculation on the Brusselator equations (2.8). We focus on the steady-state bifurcation example from Section 2.2 and also restrict our attention to one spatial dimension. We assume the one dimensional domain has periodic boundary conditions, and that its size may be selected to fit the critical wavelength $2\pi/k_c$. The goal is to obtain an equation describing the slowly-varying amplitude of the critical mode for values of the parameter $B$ near onset. As before, we choose the parameters $A = 3$, $D_U = 0.1$ and $D_V = 0.4$. The critical wave number was found to be $k_c = \sqrt{15}$ which corresponds to the critical parameter value $B_c = 6.25$. (We choose specific parameter values here for simplicity. A general treatment, in which
\( k_c, B_c, \) and the coefficients in the bifurcation equations are given in terms of \( A, D_U, \)
and \( D_V, \) is given in [46]).

We begin by writing the full nonlinear equations for the perturbations \( u = (u, v)^T \)
of the spatially homogeneous steady state of (2.8). We find

\[
\partial_t u = Lu + N
\]

where the linear operator \( L \) is

\[
L = \begin{pmatrix} B - 1 + 0.1 \nabla^2 & 9 \\ -B & -9 + 0.4 \nabla^2 \end{pmatrix}
\]

and the nonlinear terms are

\[
N = \begin{pmatrix} 6uv + B/3u^2 + u^2v \\ -6uv - B/3u^2 - u^2v \end{pmatrix}.
\]

We now invoke multiple scales by assuming that the fields \( u \) and \( v \) depend on a
"slow-time". This slow time scale on the center manifold results from the gap in the
eigenvalue spectrum which is necessary to invoke the Center Manifold Theorem, as
described in Section 2.3. We expand the fields, the control parameter, and the time
derivative as follows:

\[
\begin{align*}
\mathbf{u}(x,T_2) & = \epsilon \mathbf{u}_1(x,T_2) + \epsilon^2 \mathbf{u}_2(x,T_2) + \epsilon^3 \mathbf{u}_3(x,T_2) + \ldots \\
B & = 6.25 + \epsilon^2 B_2 + \ldots \\
\partial_t & = \epsilon^2 \partial_{T_2} + \ldots
\end{align*}
\]

(2.36a)  

(2.36b)  

(2.36c)

Here, \( \epsilon \) is a small bookkeeping parameter. We have also used the notation \( \mathbf{u}_j = (u_j, v_j)^T \). Terms of \( \mathcal{O}(\epsilon) \) in the time derivative and in \( B \) are not necessary. (Other problems may admit other scalings in which case these terms must be included.) The governing equation (2.33) is now expanded in powers of \( \epsilon \) and solved at each order.

At \( \mathcal{O}(\epsilon) \) we have

\[
\mathbf{L}_0 \mathbf{u}_1 = 0.
\]

(2.37)

Here, \( \mathbf{L}_0 \) is \( \mathbf{L} \) in (2.34) with \( B = B_c = 6.25 \). This is simply the linear problem we solved in Section 2.2. Since (2.37) is an eigenvalue problem, the solution \( \mathbf{u}_1 \) is defined only to within a multiplicative constant. We take the solution to be

\[
\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -12 \\ 5 \end{pmatrix} z(T_2) e^{ik_c x} + c.c.
\]

(2.38)

Here, \( z(T_2) \) is the slowly varying amplitude of the critical mode, and we will determine an equation governing its dynamics. From now on, for brevity, we drop the \( T_2 \) dependence of \( z \).

At \( \mathcal{O}(\epsilon^2) \) we have

\[
\mathbf{L}_0 \mathbf{u}_2 = -\mathbf{N}_2
\]

(2.39)
where
\[
N_2 = \begin{pmatrix}
6u_1v_1 + B_c/3u_1^2 \\
-6u_1v_1 - B_c/3u_1^2
\end{pmatrix}.
\tag{2.40}
\]

The particular solution is
\[
u_2 = -\frac{160}{9}z^2e^{2ik_c x} + c.c.
\tag{2.41}
\]
\[
v_2 = \frac{140}{27}z^2e^{2ik_c x} + c.c. + \frac{40}{3}|z|^2.
\tag{2.42}
\]

At \(O(\epsilon^3)\) we have
\[
L_0 u_3 = H \tag{2.43}
\]
where
\[
H \equiv \partial_{\tau_2} u_1 - L_2 u_1 - N_3.
\tag{2.44}
\]

Here,
\[
L_2 = \begin{pmatrix}
B_2 & 0 \\
-B_2 & 0
\end{pmatrix}
\tag{2.45}
\]
and
\[
N_3 = \begin{pmatrix}
6u_1v_1 + 6u_2v_1 + 2B_c/3u_1v_1 + u_1^2v_1 \\
-6u_1v_2 - 6u_2v_1 - 2B_c/3u_1v_1 - u_1^2v_1
\end{pmatrix}.
\tag{2.46}
\]

At this stage, we encounter a complication. The right hand side \(H\) of (2.43) contains terms with spatial dependence \(e^{ik_c x}\) for which the operator \(L_0\) is not necessarily invertible. To guarantee that these terms lie in the range of \(L_0\), we must use the Fredholm Alternative Theorem (see, for example, [38]) which yields a so-called solvability
condition. The solvability condition is

\[ \langle \tilde{u}_1, H \rangle = 0 \quad (2.47) \]

where the appropriate inner product for this problem is defined as

\[ \langle \tilde{u}_1, H \rangle \equiv \frac{k_c}{2\pi} \int_0^{2\pi/k_c} \tilde{u}_1^\dagger H dx. \quad (2.48) \]

Here, the $\dagger$ denotes complex conjugate transpose, and $\tilde{u}_1$ is the solution to the adjoint linear problem

\[ L_0^T \tilde{u}_1 = 0. \quad (2.49) \]

We take the eigenvector $\tilde{u}_1$ to be

\[ \tilde{u}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} e^{ik_cx}. \quad (2.50) \]

Applying (2.47) we obtain the bifurcation equation

\[ \partial_{r_2} z = \frac{8}{15} B_2 z - \frac{4256}{81} |z|^2 z. \quad (2.51) \]

This equation has the same form as (2.24) which we derived using symmetry methods in Section 2.4.

We use similar multiple-scales perturbation techniques in Chapters 4 and 5 to compute the coefficients in the bifurcation equations (2.32) from the Zhang-Viñals Faraday wave equations.
Chapter 3

Faraday wave background

3.1 Introduction

In this chapter we discuss the experimental and theoretical background important to our understanding of Faraday waves. In the Faraday system, an initially quiescent fluid is subjected to a time-periodic vertical acceleration. For small accelerations, the fluid surface remains flat. For sufficiently strong accelerations, the fluid undergoes an instability to standing waves on the free surface. This phenomenon was first reported in 1831 by Michael Faraday [6], who drew a violin bow down the side of a square plate covered with various fluids (water, alcohol, oil, egg white, ink, milk) and observed “crispations” on the fluid surface. These standing wave crispations later became known as Faraday waves. Another of Faraday’s key observations was that the standing waves oscillated with one-half the frequency of the applied acceleration—this is the so-called subharmonic response. Over a century later, this subharmonic response was explained by the linear stability analysis in [47].
In the last 25 years or so, investigators have performed numerous theoretical and experimental studies of the Faraday system. Partial reviews are available in [7, 8]. Theoretical interest in the Faraday problem has been fueled in part by copious amounts of recent experimental data. Faraday waves are an attractive and convenient experimental system because of the numerous control parameters (the properties of the fluid and of the periodic forcing) and because the time scale for pattern formation is typically much faster than for other canonical pattern forming systems such as Rayleigh-Bénard convection. See [48] for a discussion of the advantages and disadvantages of Faraday waves from an experimental point of view.

The Faraday system provides the canonical example of how spatiotemporal patterns form through a parametric instability. Most of the experimental work on the Faraday system has utilized Newtonian fluids and a sinusoidal or two-frequency acceleration function, though some studies, such as [49], have used viscoelastic fluids. Faraday or Faraday-like instabilities have also been observed in a number of other experimental systems, and we mention a few of these now.

One variation on the Faraday experiment involves excitation of a ferrofluid, which is a colloidal suspension of magnetic powder. Experimentalists have excited standing waves on the ferrofluid surface by applying a.c. and/or d.c magnetic fields to the fluid surface [21, 50, 51, 52, 53, 54]. Standing waves may also be excited by the combination of a d.c. magnetic field applied and the usual periodic vertical acceleration [55]. In another variation of the Faraday problem, one applies a vertical temperature gradient, as in Rayleigh-Bénard convection, simultaneously with the vertical vibration [18, 19]. Parametric instability and pattern formation also occur in nonfluid systems such as vertically vibrated granular layers forced with one [56, 57, 58] or two [20] frequency
components.

Many theorists have used model partial differential equations to study parametric instability in a general framework. These studies include investigations of dynamics in a nonlinear Mathieu equation with spatial dependence [59, 60], of pattern formation in a Swift-Hohenberg-like equation [61], and of a Faraday-like instability in a Gross-Pitaevskii equation modeling a Bose-Einstein condensate subjected to a time-periodic electromagnetic field [62], to name a few.

For the remainder of this chapter, we focus on Faraday waves in Newtonian fluids forced with one or two frequency components. In Section 3.2 we summarize some of the relevant previous experimental and theoretical work on the problem. In Sections 3.3 and 3.4 we present two mathematical formulations of the Faraday problem, namely the Navier-Stokes equations with a free boundary and the Zhang-Viñals equations. In Section 3.5 we briefly compare their linear stability properties.

### 3.2 Review of previous work

Faraday waves have proven to be an especially versatile hydrodynamic pattern forming system, exhibiting an impressively wide range of patterns. Reviews are provided in [7, 8]. Many experimental investigations of Faraday wave pattern formation, such as [9, 63, 48], have utilized a simple sinusoidal forcing function to accelerate the fluid. In this case, one may observe familiar simple patterns such as stripes, squares and hexagons, targets, and spirals. Examples are shown in Figure 3.1. Other experimental studies with sinusoidal forcing have investigated the effects of container depth [64, 65] and of the presence of sidewalls [66].
Figure 3.1: Faraday wave patterns observed in experiments with sinusoidal forcing. (a) Stripe pattern from [9]. (b) Square pattern from [63]. (c) Hexagonal pattern from [48]. (d) Target pattern from [9]. (e) Spiral pattern from [9]. (f) Region of coexisting squares and hexagons from [48].
More recently, experimentalists have used a two-frequency forcing function consisting of rationally-related frequency components $m\omega$ and $n\omega$ to accelerate the fluid. They have observed the simple patterns seen in the single frequency case as well as a wealth of quasipatterns [9, 10, 67] and superlattice patterns [10, 68, 13, 14, 15]. Quasipatterns are the continuum analogues of quasicrystals. They are spatially non-periodic, and thus lack translational symmetry, though their Fourier spectra do have discrete rotational symmetries (see [69] for a thorough discussion of the definition of a quasicrystal). An example of a quasipattern is shown in Figure 1.1. In contrast to quasipatterns, superlattice patterns are spatially periodic, and they contain structure on more than one length scale. Though these two types of complex patterns were first observed in experiments utilizing two-frequency forcing, they were subsequently observed in experiments with sinusoidal forcing, as reported in [63, 11, 64] for quasipatterns and in [16] for superlattice patterns. Finally, experimentalists using two-frequency forcing have also observed exotic states such as triangles [70], time-dependent rhombic patterns [15] and localized wave structures [68]. Some examples are shown in Figure 3.2.

Theorists have performed numerous investigations of the linear (in)stability of the flat surface in the Faraday problem. The nature of the parametric instability was first identified in [47], which demonstrated that the linear stability of an ideal fluid subjected to single frequency forcing obeys a Mathieu equation (see [22] for an introduction to this equation). Since that seminal work, other analytical investigations have determined the linear stability of the free surface for fluids of finite viscosity subjected to single frequency forcing [71, 72, 73, 74] and impulsive forcing [75]. Other studies have used symmetry arguments to demonstrate how hidden symmetries may
Figure 3.2: Faraday wave patterns observed in experiments which used a two-frequency forcing composed of rationally-related frequency components $m\omega$ and $n\omega$. (a) Localized hexagonal oscillon structure from [68] with $m/n = 2/3$. (b) Triangular pattern from [70] with $m/n = 1/2$. (c) SL-II superlattice pattern from [10] with $m/n = 4/5$. See also Figures 1.1 and 1.2a for examples of a Faraday wave quasipattern and superlattice pattern which were obtained with two-frequency forcing.
lead to a degeneracy in the linear stability problem [76, 77]. A thorough numerical study of the linear stability was performed for single frequency forcing in [78] and extended to the case of two-frequency forcing in [79]. These results will play a key role in the work presented in Chapters 4 – 6 and will be discussed in detail in Section 3.5.

Many theoretical studies of the nonlinear aspects of Faraday waves have made use of symmetry methods, for instance to investigate primary and secondary pattern forming instabilities in a bifurcation theoretic setting [25, 80, 81]. Other studies have used numerical simulations to investigate Faraday wave patterns [82] and to assess the importance of dimensionality (i.e. three versus two dimensions) in the formation of localized wave structures [83].

Finally, we mention other theoretical studies of Faraday wave pattern formation, such as [84, 85], which were accomplished largely by means of weakly nonlinear analysis. Many such investigations have been carried by Víñals and collaborators. For instance, Zhang and Víñals [24, 86] derived equations for the Faraday waves on deep layers of fluid in the limit of weak viscosity and small amplitude waves. They emphasized resonant triad interactions and their contribution to nonlinear terms in the relevant amplitude equations; this subject is discussed in depth in Chapters 4 – 6. For the two-frequency case, their nonlinear investigation is limited to waves with forcing frequencies in ratio \( m/n = 1/2 \); the results in this dissertation will apply to higher values of \( m \) and \( n \). The weakly nonlinear analysis [24], namely the calculation of bifurcation coefficients from the Zhang-Víñals Faraday wave equations, was extended to the Navier-Stokes equations in [87]. In the following sections, we review both the Navier-Stokes and the Zhang-Víñals formulations of the Faraday problem and briefly compare their linear stability properties.
Figure 3.3: Schematic diagram of the Navier-Stokes formulation of the Faraday problem. Two fluids of densities $\rho_1, \rho_2$, dynamic viscosities $\eta_1, \eta_2$, and filling depths $h_1, h_2$ (either of which may be infinity) are superposed with the heavier fluid on the bottom. The surface tension at the interface is $\sigma$. A periodic vertical acceleration $g(t)$ is applied. For sufficiently strong $g(t)$, Faraday waves will form around the fluid interface at $z = 0$.

3.3 Navier-Stokes description of Faraday waves

In this section we review the mathematical formulation of the Faraday problem provided by the Navier-Stokes equations. Detailed treatments are given in [8, 75, 78, 79, 87] and we summarize the main points here.

We begin by considering two superposed layers of immiscible, incompressible fluids. The heavier, lower fluid and the lighter, upper fluid have respective densities $\rho_1, \rho_2$ and dynamic viscosities $\eta_1, \eta_2$. The fluids are filled to depths $h_1, h_2$, either or both of which may be infinity. The surface tension at the interface is $\sigma$. The fluids are enclosed between horizontal plates and a periodic vertical acceleration $g(t)$ is applied. For sufficiently strong $g(t)$, Faraday waves will form around the fluid interface at $z = 0$. A schematic diagram is given in Figure 3.3.

The well-known Navier-Stokes equations follow from conservation of momentum
and from the incompressibility condition. They are:

\[
\begin{align*}
\rho_j \left[ \partial_t + (\mathbf{u}_j \cdot \nabla) \right] \mathbf{u}_j &= -\nabla P_j + \eta_j \nabla^2 \mathbf{u}_j - \rho_j G(t) \mathbf{e}_z \\
\nabla \cdot \mathbf{u}_j &= 0.
\end{align*}
\] (3.1a, 3.1b)

Here, the vector \( \mathbf{u} = (u, v, w) \) contains the components of the fluid velocity field, \( P \) is the fluid pressure, and \( \mathbf{e}_z \) is the unit vector in the positive \( z \) direction. The index \( j = 1, 2 \) indicates the lower or upper layer of fluid. In a frame of reference moving with the oscillating container, the effect of the periodic forcing \( g(t) \) is a modulation of gravity. This enters (3.1a) as the gravity term

\[
G(t) = g_0 - g(t)
\] (3.2)

where \( g_0 = 980.665 \text{ cm/s}^2 \) is the usual acceleration due to gravity. For the bulk of this dissertation, we focus on the case of two-frequency forcing, which we write in the following forms:

\[
\begin{align*}
g(t) &= g_z [\cos(\chi) \cos(m\omega t) + \sin(\chi) (n\omega t + \phi)] \\
&= g_m \cos(m\omega t) + g_n (n\omega t + \phi) \\
&= G_m e^{im\omega t} + G_n e^{in\omega t} + c.c.
\end{align*}
\] (3.3)

Here, \( m \) and \( n \) are co-prime integers.

We compensate for the gravity term with a pressure field \( P_j^* = -\rho_j G(t)z \), substi-
tuting $P_j = P_j^* + p_j$ in (3.1) to obtain

$$
\rho_j [\partial_t + (\mathbf{u}_j \cdot \nabla)] \mathbf{u}_j = -\nabla p_j + \eta_j \nabla^2 \mathbf{u}_j \tag{3.4a}
$$

$$
\nabla \cdot \mathbf{u}_j = 0. \tag{3.4b}
$$

The boundary conditions for (3.4) are:

(i) No slip conditions at the lower and upper boundaries:

$$
\mathbf{u}_1(z = -h_1) = \mathbf{u}_2(z = h_2) = \mathbf{0}. \tag{3.5}
$$

(ii) A kinematic condition stating that the interface is advected by the fluid at the (deformed) interface, whose position we denote by $z = h(x, y, t)$:

$$
\partial_t + [\mathbf{u}(z = h) \cdot \nabla] h = w(z = h). \tag{3.6}
$$

(iii) A continuity condition stating that the velocity field is continuous across the interface $z = h$:

$$
\mathbf{u}_1(z = h) = \mathbf{u}_2(z = h). \tag{3.7}
$$

(iv) A continuity condition stating that the tangential components of the stress tensor are continuous across the interface. The elements of the stress tensor $\pi_j$ are

$$
\pi_{j,\ell m}(z) = -(p_j + P_j^*) \delta_{\ell m} + \eta_j (\partial_{\ell} u_{j,m} + \partial_{m} u_{j,\ell}). \tag{3.8}
$$

Here, the indices $\ell, m$ refer to the components $x, y, z$. The continuity condition is
Here, $a(x, y, t)$ and $b(x, y, t)$ are two tangential unit vectors at the fluid interface. The unit normal vector at the interface, pointing from the bottom layer to the top layer, is
\[ n(x, y, t) = \left(-\partial_x h, -\partial_y h, 1\right). \] (3.10)

(v) A jump condition in the normal component of the stress tensor across the interface $z = h$ due to the curvature induced by the surface tension:
\[ n \cdot \pi_2(z = h) \cdot n - n \cdot \pi_1(z = h) \cdot n = \sigma \kappa. \] (3.11)

Here, the mean curvature of the surface is
\[ \kappa = \nabla \cdot n. \] (3.12)

The quiescent steady state of the fluid is $u_j = 0$, $p_j = 0$, $h = 0$. We now linearize the problem, following the treatment in [78]. Equations (3.4) become
\[ \partial_t u_j = -\frac{1}{\rho_j} \nabla p_j + \nu_j \nabla^2 u_j \] (3.13a)
\[ \nabla \cdot u_j = 0 \] (3.13b)
where \( \nu_j = \eta_j/\rho_j \) is the kinematic viscosity. Notice that gravity does not appear in these equations. The effect of gravity (and its periodic modulation) is captured in the \( P_j^* \) term in (3.8) in boundary condition \((v)\) above. Applying the operator \( e_z \cdot \nabla \times \nabla \times \) to (3.13a) and making use of (3.13b) we obtain the linearized equation of motion for \( w_j \), the vertical component of the velocity \( u_j \):

\[
(\partial_t - \nu_j \nabla^2) \nabla^2 w_j = 0. \tag{3.14}
\]

We now assume that the problem may be decomposed into normal Fourier modes \( e^{i\mathbf{k} \cdot \mathbf{x}} \) (for instance, this is the case if we consider a problem of infinite horizontal extent). Here, the wave vector \( \mathbf{k} = (k_x, k_y) \) has magnitude \( k \) and the horizontal coordinate is \( \mathbf{x} = (x, y) \). We replace the vertical velocity \( w_j(x, z, t) \) by \( e^{i\mathbf{k} \cdot \mathbf{x}} w(z, t) \) and the free surface \( h(x, t) \) by \( e^{i\mathbf{k} \cdot \mathbf{x}} h(t) \).

After some mathematical manipulation (including linearization of the boundary conditions) the details of which may be found in [78], we arrive at the complete mathematical statement of the linear stability problem. We summarize the results here, which are equations (2.17) – (2.27) in [78]. The equations of motion, which follow from (3.13a), are

\[
[\partial_t - \nu_1(\partial_{zz} - k^2)](\partial_{zz} - k^2)w_1 = 0 \quad \text{for } -h_1 \leq z < 0 \tag{3.15a}
\]

\[
[\partial_t - \nu_1(\partial_{zz} - k^2)](\partial_{zz} - k^2)w_2 = 0 \quad \text{for } 0 < z \leq h_2. \tag{3.15b}
\]
Boundary condition (i) implies

\[ w_1 = 0 \quad \text{at} \quad z = -h_1 \]  \hspace{1cm} (3.16a)
\[ w_2 = 0 \quad \text{at} \quad z = h_2 \]  \hspace{1cm} (3.16b)
\[ \partial_z w_1 = 0 \quad \text{at} \quad z = -h_1 \]  \hspace{1cm} (3.16c)
\[ \partial_z w_2 = 0 \quad \text{at} \quad z = h_2. \]  \hspace{1cm} (3.16d)

Boundary condition (ii) implies

\[ \partial_t h = w\big|_{z=0}. \]  \hspace{1cm} (3.17)

Boundary condition (iii) implies

\[ \Delta w = \Delta \partial_z w = 0 \quad \text{at} \quad z = 0. \]  \hspace{1cm} (3.18)

Boundary condition (iv) implies

\[ \Delta \{ \eta (\partial_{zz} + k^2) w \} = 0. \]  \hspace{1cm} (3.19)

Finally, boundary equation (v) implies

\[ \Delta \{ [\rho \partial_t - \eta (\partial_{zz} - k^2) + 2\eta k^2] \partial_z w \} = -[(\Delta \rho) G(t) - \sigma k^2] k^2 h. \]  \hspace{1cm} (3.20)

In the above, \( \Delta \) represents the jump in a quantity across the interface from the bottom to the top, \( \text{e.g.} \ \Delta \rho = \rho_2 - \rho_1 \). We will use the formulation (3.15) – (3.20) in Section 3.5 to compute the neutral stability curve numerically.
3.4 Zhang-Viñals description of Faraday waves

A principal feature of the Navier-Stokes formulation of the Faraday problem (3.1) – (3.12) is that it is a free-boundary problem: the location \( h(x, y, t) \) of the fluid interface at which the boundary conditions (ii) – (v) from the previous section are applied is one of the unknowns. This feature complicates the analysis of the equations. In [24], Zhang and Viñals derive from the Navier-Stokes equations reduced equations which do not involve a free boundary. These equations will be the starting point for our perturbation calculations in Chapters 4 and 5. We now describe the derivation of these equations as given in [24].

The derivation begins by assuming that the fluid layers are of infinite depth, i.e. \( h_1 = h_2 = \infty \), and that the upper fluid is a gas of negligible density, such as air. In this case, we may ignore the upper layer. The Navier-Stokes equations for the velocity field \( u \) and the pressure field \( p \) are given by (3.4) evaluated only for the fluid in the bottom layer, i.e. \( j = 1 \). The boundary conditions follow immediately from those in the previous section.

The next step is to reduce the Navier-Stokes equations to the so-called quasipotential equations, as we now explain. For an ideal, initially irrotational fluid, the flow field \( u \) is purely potential. For fluids with sufficiently weak viscous dissipation (i.e. \( \text{Re} \gg 1 \), where the Reynolds number is \( \text{Re} = \frac{\omega_0}{(\nu \tilde{k}^2)} \) with \( \omega_0 \) and \( \tilde{k} \) suitably chosen characteristic time and length scales) the flow in the bulk of the fluid is potential, and viscous effects are confined to a thin layer near the free surface. These viscous effects may be captured in effective boundary conditions at the moving surface. The procedure used in [24] is to assume weak viscous dissipation, and then to decompose
the velocity field into an irrotational and a rotational part, \( i.e. \) \( u = v + V \equiv \nabla \tilde{\Phi} + V \),

where \( \tilde{\Phi}(x, y, z, t) \) is the velocity potential. The resulting problem for \( V \) is approximated inside the thin boundary layer in order to supply the boundary conditions for the harmonic \( \tilde{\Phi} \) problem. This derivation, and the resulting quasipotential equations, are lengthy. They are omitted here but may be found in [24].

The quasipotential equations are fully nonlinear. The only assumption made in their derivation is weak viscous dissipation. The equations may be simplified, however, by neglecting nonlinear viscous terms and thus restricting study to the weakly nonlinear dynamics of the system. It is also assumed that the fluid is initially quiescent. The results of these simplifications are the so-called linear damping quasipotential equations, which take the form

\[
\nabla^2 \tilde{\Phi} = 0 \quad \text{for } z < h(x, y, t) \tag{3.21}
\]

with boundary conditions

\[
\begin{align*}
\partial_t h &= 2 \nu \nabla^2 h - \nabla \cdot \tilde{\Phi} \cdot \nabla h + \partial_z \tilde{\Phi} \quad \text{at } z = h(x, y, t) \tag{3.22a} \\
\partial_t \tilde{\Phi} &= 2 \nu \nabla^2 \tilde{\Phi} - \frac{1}{2} (\nabla \tilde{\Phi})^2 + G(t)h - \sigma \kappa \quad \text{at } z = h(x, y, t) \tag{3.22b} \\
\partial_z \tilde{\Phi} &\to 0 \text{ as } z \to -\infty. \tag{3.22c}
\end{align*}
\]

Here, the horizontal Laplacian operator is \( \nabla \perp = \partial_x^2 + \partial_y^2 \).

The final step is to write (3.22a) and (3.22b), which depend on all three spatial variables, in a nonlocal form which depends only on the two horizontal spatial
coordinates. It will be convenient to define the surface velocity potential

\[ \Phi(x, y, t) = \tilde{\Phi}(x, y, h(x, y, t), t). \]  

Substituting (3.23) and (3.12), the boundary conditions (3.22a) and (3.22b) become

\[
\begin{align*}
\partial_t h &= 2\nu \nabla^2 h + (\partial_n \tilde{\Phi}) \sqrt{1 + (\nabla h)^2} \\
\partial_t \Phi &= 2\nu \nabla^2 \Phi - \frac{1}{2} (\nabla \Phi)^2 + \left[ \frac{(\partial_n \tilde{\Phi}) \sqrt{1 + (\nabla h)^2} + \nabla \Phi \cdot \nabla h}{2[1 + (\nabla h)^2]} \right]^2 \tag{3.24a} \\
&\quad + G(t) h + \frac{\sigma}{\rho} \nabla \cdot \left( \frac{\nabla h}{\sqrt{1 + (\nabla h)^2}} \right) \tag{3.24b}
\end{align*}
\]

where \( \partial_n \tilde{\Phi} = n \cdot (\nabla_\perp + \partial_z e_z) \tilde{\Phi} \). In these equations, all terms excluding those containing \( \partial_n \tilde{\Phi} \) depend only on the horizontal coordinates \( x \) and \( y \). Dependence on the \( z \) coordinate is eliminated by exploiting the fact that \( \tilde{\Phi} \) satisfies Laplace’s equation. The substitution

\[
(\partial_n \tilde{\Phi}) \sqrt{1 + (\nabla h)^2} = \tilde{G}(h) \Phi(x, y, t) \tag{3.25}
\]

is used, where \( \tilde{G}(h) \) is the Dirichlet-Neumann operator. This relation follows from Green’s third identity, which relates the normal derivative of \( \tilde{\Phi} \) on the boundary to its value on the boundary in the form of a boundary integral (a nonlocal relation). Equation (3.25) is substituted into the boundary conditions (3.22a) and (3.22b) and the result is Taylor expanded. Only terms through cubic order in \( h \) and \( \Phi \) are kept. After rescaling the time and spatial coordinates, we arrive at the Zhang-Viñals equations for the surface height \( h(x, \tau) \) and surface velocity potential \( \Phi(x, \tau) \), where \( x \in \mathbb{R}^2 \) is the horizontal coordinate.
The Zhang-Viñas equations are

\[
(\partial_\tau - \gamma \nabla^2) h - \hat{D} \Phi = \mathcal{F}(h, \Phi) \tag{3.26a}
\]

\[
(\partial_\tau - \gamma \nabla^2) \Phi - (\Gamma_0 \nabla^2 - G(\tau)) h = G(h, \Phi), \tag{3.26b}
\]

where the nonlinear terms are

\[
\mathcal{F}(h, \Phi) = -\nabla \cdot (h \nabla \Phi) + \frac{1}{2} \nabla^2 (h^2 \hat{D} \Phi) - \hat{D}(h \hat{D} \Phi) + \hat{D} \left\{ h \hat{D}(h \hat{D} \Phi) + \frac{1}{2} h^2 \nabla^2 \Phi \right\} \tag{3.27}
\]

\[
G(h, \Phi) = \frac{1}{2} (\hat{D} \Phi)^2 - \frac{1}{2} (\nabla \Phi)^2 - (\hat{D} \Phi) \left\{ h \nabla^2 \Phi + \hat{D}(h \hat{D} \Phi) \right\} - \frac{1}{2} \Gamma_0 \nabla \cdot \{ (\nabla h)(\nabla h)^2 \}. \tag{3.28}
\]

(For brevity, we drop the \( h \) and \( \Phi \) dependence of \( \mathcal{F} \) and \( G \) from now on.) The operator \( \hat{D} \) is a nonlocal operator resulting from the use of (3.25). It multiplies each Fourier component of a field by its wave number, e.g. \( \hat{D} e^{i k \cdot x} = |k| e^{i k \cdot x} \). Here time has been scaled by \( \omega \) so that the dimensionless two-frequency acceleration is

\[
G(\tau) = G_0 - [f_m \cos(m \tau) + f_n \cos(n \tau + \phi)] \tag{3.29}
\]

\[
= G_0 - f [\cos(\chi) \cos(m \tau) + \sin(\chi) \cos(n \tau + \phi)].
\]

The damping number (\( \gamma \)), capillarity number (\( \Gamma_0 \)), gravity number (\( G_0 \)), and dimensionless accelerations (\( f_m \) and \( f_n \)) are related to the forcing function (3.3) and the
fluid parameters in Section 3.3 by

\[ \gamma \equiv \frac{2\nu \tilde{k}^2}{\omega}, \quad \Gamma_0 \equiv \frac{\sigma \tilde{k}^3}{\rho \omega^2}, \quad G_0 \equiv \frac{g_0 \tilde{k}}{\omega^2}, \quad f_m \equiv \frac{g_m \tilde{k}}{\omega^2}, \quad f_n \equiv \frac{g_n \tilde{k}}{\omega^2}. \] (3.30)

The spatial coordinate has been scaled by the wave number \( \tilde{k} \), which we choose to satisfy the gravity-capillary wave dispersion relation

\[ g_0 \tilde{k} + \frac{\sigma \tilde{k}^3}{\rho} = \left( \frac{m\omega}{2} \right)^2. \] (3.31)

We choose this scaling for convenience, so that the unforced wave with natural dimensionless frequency \( m/2 \) has a dimensionless wave number \( k = 1 \). Note that (3.30) and (3.31) imply a relationship between the gravity number and the capillarity number, namely

\[ G_0 + \Gamma_0 = \frac{m^2}{4}. \] (3.32)

It will often be convenient to work with an alternate form of (3.26). Following [27], we apply \((\partial_\tau - \gamma \nabla^2)\) to (3.26a) to obtain

\[ (\partial_\tau - \gamma \nabla^2)^2 h - (\partial_\tau - \gamma \nabla^2) \hat{D} \Phi = (\partial_\tau - \gamma \nabla^2) \mathcal{F}. \] (3.33)

In (3.33), we substitute for \((\partial_\tau - \gamma \nabla^2)\Phi\) by using (3.26b). The resulting equation and equation (3.26a) (which we rearrange) constitute the system of equations that we use
in our perturbation analyses in Chapters 4 and 5:

\[
\left\{ (\partial_\tau - \gamma \nabla^2)^2 - \hat{D} [\Gamma_0 \nabla^2 - G(\tau)] \right\} h = (\partial_\tau - \gamma \nabla^2) F + \hat{D} G \tag{3.34a}
\]
\[
\hat{D} \Phi = (\partial_\tau - \gamma \nabla^2) h - F. \tag{3.34b}
\]

The linearization of (3.34b) is a damped Mathieu equation for each surface height Fourier mode \( p_k(\tau)e^{ikx} \):

\[
p_k'' + 2\gamma k^2 p_k' + \left\{ \gamma^2 k^4 + \Omega^2(k) \right\} p_k = kG(\tau)p_k \tag{3.35}
\]

where the natural frequency \( \Omega(k) \) satisfies the dispersion relation

\[
\Omega^2(k) = G_0 k + \Gamma_0 k^3. \tag{3.36}
\]

### 3.5 Linear stability comparison

In this section, we review linear stability results for the Faraday problem. We also compare the critical wave number and forcing amplitude that are predicted by the Navier-Stokes and Zhang-Viñals formulations of the Faraday problem presented in Sections 3.3 and 3.4.

In [78, 79], the linear (in)stability of the flat free surface was determined numerically using the Navier-Stokes formulation as we now describe. Equations (3.15) – (3.20) represent a periodically-forced linear system, and thus are guaranteed to have solutions of Floquet form (see [22] for a detailed introduction to Floquet theory within
the context of ordinary differential equations). For the analysis of Faraday waves, it is convenient to write the periodic portion of the Floquet solution as a Fourier series in time. The vertical velocity $w$ and interface position $h$ take the form:

$$w_j(z,t) = e^{(\mu + i\alpha)t} \sum_{n=-\infty}^{\infty} \tilde{w}_{jn}(z)e^{in\omega t} \quad (3.37a)$$

$$h(t) = e^{(\mu + i\alpha)t} \sum_{n=-\infty}^{\infty} \tilde{h}_n e^{in\omega t}. \quad (3.37b)$$

Here, $\mu + i\alpha$ is the Floquet exponent, and the Floquet multiplier $e^{(\mu + i\alpha)T}$ measures the evolution of the periodic function over one forcing period $T = 2\pi/\omega$.

Since we wish to determine the neutral stability of the problem, we insist that the Floquet multiplier have unit magnitude, i.e. $|e^{(\mu + i\alpha)T}| = 1$. For all known examples, transitions from the flat fluid state to Faraday waves occur only for real-valued Floquet multipliers $\pm 1$ (see [78] for a brief discussion). Thus, we set $\mu = 0$ and $\alpha = 0, \omega/2$. For the case $\alpha = 0$, $w_j(z,t)$ and $h(t)$ have period $2\pi/\omega$, the same as the forcing period $T$. This is called the harmonic case. For the case $\alpha = \omega/2$, $w_j(z,t)$ and $h(t)$ have period $4\pi/\omega$, or twice the forcing period $T$. This is called the subharmonic case. We note that for the harmonic and subharmonic cases, $w_j$ obeys the respective reality conditions $\tilde{w}_{j,-n} = \overline{w}_{j,n}$ and $\tilde{w}_{j,-n} = \overline{w}_{j,n-1}$ (and similar conditions for $h$). For each case, we may substitute (3.37) into (3.15) – (3.20). The coefficients $\tilde{w}_{jn}$ may be related to $\tilde{h}_n$ through the boundary conditions. The result is a generalized eigenvalue problem for the Fourier coefficients $\tilde{h}_n$ which determines the neutral stability curve, i.e. the critical forcing amplitude $g_z$ for each wave number $k$. The primary bifurcation occurs at the minimum $(k_c, g_z^{\text{crit}})$ of the neutral stability curve as described in Section
2.2, and may correspond to either subharmonic or harmonic waves.

For the Zhang-Viñals equations, the procedure is similar. We begin with the linearized equation (3.35) and expand \( p(\tau) \) in a Fourier series which is either \( 2\pi \) or \( 4\pi \) periodic. The resulting generalized eigenvalue problem for the Fourier coefficients determines the critical forcing amplitude as a function of wave number.

For single frequency forcing, the primary instability is typically to waves which are subharmonic with respect to the forcing period. This subharmonic response was first observed by Faraday in [6] and has been determined theoretically and experimentally in many of the previous studies discussed in Section 3.2. (However, it has been observed that a harmonic response is possible for very thin layers of fluid vibrated at low frequency [72, 73, 74] and for certain viscoelastic fluids [71]).

For the case of two-frequency forcing, as given by (3.3), the analysis in [79] showed that the character of the instability depends on the the value of \( \chi \) and the parities of \( m \) and \( n \). For instance, if the \( \cos(m\omega t) \) forcing component is dominant and if \( m \) is even (odd) then the bifurcation will be to waves which are harmonic (subharmonic) with respect to the forcing period \( T = 2\pi \). For \( m \) and \( n \) not both odd, there is a codimension-two point in the \( g_z - \chi \) parameter space (or alternatively, in \( g_m - g_n \) space) at which harmonic and subharmonic instabilities occur simultaneously at different spatial wave numbers. The corresponding value \( \chi = \chi_{bc} \) is called the “bicritical point”.

We now turn to a comparison of the linear results predicted by the two mathematical formulations of the Faraday problem. We have computed the critical wave number and critical forcing amplitude from the linearized Navier-Stokes equations (3.15) – (3.20) and from the linearized Zhang-Viñals equations (3.35). We put the
Navier-Stokes results for onset parameters in dimensionless form, so that we may compare them with those predicted by the Zhang-Viñals equations.

A complete comparison of the linear stability of the two formulations is not feasible because the parameter space is so large. We choose here to focus on an example corresponding to the SL-I superlattice pattern from [10] shown in Figure 1.2a. The fluid parameters used to obtain the pattern were density $\rho = 0.95 \ g/cm^3$, surface tension $\sigma = 20.6 \ dyn/cm$, and kinematic viscosity $\nu = 0.209 \ cm^2/s$. The forcing parameters in (3.3) were $m = 6$, $n = 7$, $\omega/(2\pi) = 16.44 \ s^{-1}$, $\chi = 61^\circ$, and $\phi = 20^\circ$. The fluid depth was $h_1 = 0.3 \ cm$. For these physical parameters, $\tilde{k}$ in (3.31) is approximately $15.5 \ cm^{-1}$. The dimensionless damping parameter is $\gamma = 0.97$ and the dimensionless gravity-capillarity number is $\Gamma_0 = 7.6$. The dimensionless fluid depth is $h^* = h_1\tilde{k} = 4.7$.

For our calculation, we focus on single frequency forcing for simplicity, and so we set $\chi = 0^\circ$. In this case, the period of the forcing function is $T = 2\pi/(6\omega)$ and the primary instability is subharmonic with respect to $T$. In order to assess the effects of fluid depth and viscous damping, we vary $h^*$ for fixed $\gamma$ in the first calculation, and vary $\gamma$ for fixed $h^*$ in the second. For the Navier-Stokes calculation, the upper fluid layer is assumed to be an air column of infinite height. The results are shown in Figure 3.4.

First we focus on Figures 3.4a and b, which show the variation in the dimensionless critical wave number $k_c$ and forcing amplitude $f_c$ as a function of the dimensionless fluid depth $h^*$ for fixed damping parameter $\gamma = 0.97$ and $\gamma = 0.1$. The solid curves are computed from the linearized Navier-Stokes equations (3.15) – (3.20), and the dotted lines respresent the constant value computed from the linearized Zhang-Viñals
Figure 3.4: Onset parameters as computed from the linearized Navier-Stokes equations (3.15) – (3.20) and from the linearized Zhang-Viñals equations (3.35). The dimensionless gravity-capillarity parameter is $\Gamma_0 = 7.6$. For the Navier-Stokes calculation, the upper fluid layer is assumed to be air. The forcing parameters in (3.3) are $m\omega/(2\pi) = 98.64 \, s^{-1}$ ($\chi = 0^\circ$). Solid curves correspond to Navier-Stokes data with the upper fluid assumed to be an air column of infinite height. Dotted curves correspond to Zhang-Viñals data, which assumes an infinite fluid depth $h^*$. (a),(b) Critical dimensionless wave number $k_c$ and critical forcing amplitude $f_c$ versus dimensionless fluid depth $h^*$ for fixed damping parameter $\gamma = 0.97$ and $\gamma = 0.1$. (c),(d) $k_c$ and $f_c$ versus $\gamma$ for fixed $h^* = 4.7$. 
equations (3.35), which assume infinite fluid depth. We see that for both values of $\gamma$, $k_c$ and $f_c$ as computed from the Navier-Stokes equations stay essentially constant for $h^* \gtrsim 3$. Thus, past this value, the fluid depth may be considered to be “infinite” at least for the purposes of linear analysis, and for the chosen fluid parameters. However, for $\gamma = 0.97$, the curve corresponding to the Navier-Stokes data levels off to a value that is significantly different than the constant value computed from the Zhang-Viñals equations, while for $\gamma = 0.1$ the two results are in good agreement. The comparison indicates that the assumptions used in deriving the Zhang-Viñals equations are not justified in the regime where the experiments of [10] were performed, i.e. the damping is too high.

Figures 3.4c and d show the variation in $k$ and $f_c$ as a function of the damping parameter $\gamma$. The solid curve is computed from the Navier-Stokes equations with the dimensionless fluid depth fixed at $h^* = 4.7$ (which we have just seen to be “infinite” depth in the previous paragraph). The dotted curve is computed from the Zhang-Viñals equations, which assume infinite fluid depth. The two curves begin coincidently, but diverge quickly. This indicates that even for deep fluids, we should generally not expect the Zhang-Viñals equations to agree quantitatively with the Navier-Stokes formulation, except at relatively small values of $\gamma$. For $\gamma = 1$ (the maximum value on the horizontal axis) the Zhang-Viñals equations underpredict $k_c$ by approximately 10% and overpredict $f_c$ by approximately 25% with respect to the Navier-Stokes result.

We have shown here that for the parameters corresponding to the experiments in [10], the Navier-Stokes formulation and the Zhang-Viñals formulation lead to different predictions for the linear (in)stability of the Faraday problem. The perturbation anal-
yses performed in the remainder of this dissertation use the Zhang-Viñals equations as a starting point, and thus we expect that the quantitative results will accurately model experiments only for small $\gamma$. Thus, they will not be applicable to the experiments in [10], which are a case of strong viscous damping. However, we expect that many of our other qualitative results, which rely heavily on symmetry arguments, will be applicable to situations that would be more accurately described by the Navier-Stokes equations.
Chapter 4

Weakly damped modes and pattern selection

4.1 Introduction

In this Chapter, we begin our study of Faraday wave nonlinear pattern formation, focusing on the role that weakly damped modes may play in the pattern selection process. The results of this chapter were published in [28].

Many of the experimental [9, 63, 68, 70, 14, 15, 64] and theoretical [24, 61, 86, 87] studies of exotic patterns in the Faraday system attribute their formation near the bicritical (codimension-two) point to resonant triad interactions involving the critical or near-critical modes with different spatial wave numbers. In particular, the focus has been on spatial triads $k_1$, $k_2$ and $k_3 = k_1 + k_2$, where $|k_1| = |k_2|$ is the wave number of one critical mode, and $|k_3|$ is the wave number of the other critical mode. The angle $\theta_r$, which separates $k_1$ and $k_2$, is readily tuned by changing the frequency
components $m\omega$ and $n\omega$ of the two-frequency periodic forcing function (3.3). It has been suggested, for example, that by tuning this angle, different types of exotic wave patterns may be selected [9]. Such a simple mechanism for nonlinear pattern selection, which is based on examining the linear instabilities of the spatially homogeneous state, is naturally attractive, but warrants careful examination.

In [27], it was shown that whether or not resonant triads associated with the bicritical point affect pattern selection depends on the temporal characteristics of the competing instabilities. For instance, the bicritical point of laboratory experiments typically involves a subharmonic mode (Floquet multiplier $-1$) and a harmonic mode (Floquet multiplier $+1$). On the subharmonic side of the bicritical point, the onset pattern selection problem is strongly influenced by the presence of the weakly damped harmonic mode. In contrast, on the harmonic side, the onset pattern selection problem is completely insensitive to the presence of near critical subharmonic modes. These general ideas were demonstrated in [27] through a bifurcation analysis of the Zhang-Viñals Faraday wave equations (3.34).

In this chapter, we extend the bifurcation analysis in [27] to two-dimensional spatially-periodic patterns and to higher forcing frequencies within the two-frequency forcing function. With the experimentally-relevant higher forcing frequencies (e.g. $6\omega$ and $7\omega$) employed in this chapter, we find the new possibility that spatially-resonant triads involving nearly critical harmonic modes may influence the harmonic wave pattern selection problem. This is not an option for the lower forcing frequencies (e.g. $1\omega/2\omega$ and $2\omega/3\omega$) used in previous weakly nonlinear analyses of the two-frequency Faraday problem [86, 27].

We follow the methods in [76, 77, 88, 89] by posing the pattern selection problem
in terms of a symmetry-breaking bifurcation of the trivial fixed-point of a stroboscopic map. By restricting solutions to those that are spatially-periodic on some hexagonal lattice we obtain a finite-dimensional bifurcation problem (analogous to the one discussed in Section 2.4) that can be analyzed using the methods of equivariant bifurcation theory [23]. For a review of this approach to hydrodynamic pattern formation problems, see [44].

This formulation of the bifurcation problem allows us to address recent two-frequency Faraday wave experimental observations [10] of a transition between simple hexagons and the triangular SL-I superlattice wave pattern depicted in Figure 1.2a. Specifically, we follow [25] and consider a bifurcation problem that is equivariant with respect to a 12-dimensional irreducible representation of $D_6 \oplus T^2$ (see Figure 2.6 for a picture of the 12 critical modes). This problem takes the form of a stroboscopic map and is closely related to the continuous time bifurcation problem which we reviewed in Section 2.4. The observed harmonic wave states correspond to primary transcritical branches of the generic bifurcation problem. In order for the observed hexagon-superlattice pattern transition to be reproduced by the bifurcation problem, we must consider a degenerate case in which the quadratic coefficient vanishes. Moreover, the cubic coefficients must satisfy certain inequalities, e.g. certain combinations of nonlinear cross-coupling coefficients must be small compared to the cubic self-coupling coefficient.

In this chapter we compute the quadratic and cubic nonlinear coefficients in the bifurcation problem from the Zhang-Viñals equations (3.34) which apply to deep layers of low viscosity fluids subjected to a periodic acceleration. We show that the necessary inequalities for stable SL-I superlattice patterns can be satisfied for
the forcing frequencies employed in the experiments \((6\omega/7\omega)\), and that a resonant triad involving a weakly damped harmonic mode plays a key role in stabilizing the superpattern. Specifically, we find that the presence of a near critical harmonic mode leads to a cancellation in one of the cubic cross-coupling coefficients, causing this coefficient to become small in magnitude as required. This selects a preferred angle \(\theta_r\) for the superlattice patterns. In other words, it suggests which of the countably infinite 12-dimensional irreducible representations of \(D_6^+T_2\), parameterized by the integer pair \((n_1, n_2)\), is most pertinent to this Faraday wave problem.

The chapter is organized as follows. Section 4.2 reviews results from [27] on the influence of spatio-temporally resonant triads on pattern selection. Section 4.3 then formulates the generic bifurcation problem relevant to our investigation and reviews general results pertinent to SL-I superlattice pattern formation. The coefficients of the leading nonlinear terms are evaluated numerically from expressions derived perturbatively from the Zhang-Viñals equations (3.34) in Section 4.4. The bifurcation results are presented in Section 4.5. We consider two different cases. For the first case we consider an example involving forcing frequencies in ratio \(m/n = 2/3\), focusing on differences between the pattern selection problems for subharmonic and harmonic wave onset in a vicinity of the bicritical point. For the second case, we turn to an example involving higher forcing frequencies in ratio \(m/n = 6/7\) which shows how weakly damped harmonic modes can stabilize harmonic SL-I superlattice patterns involving the angle \(\theta_r\) associated with a harmonic wave resonant triad. Finally, Section 4.6 concludes with a brief summary of the results of this chapter.
Figure 4.1: Spatially resonant triads of three wave vectors and their associated resonant angle $\theta_r$. (a) $\theta_r < 2\pi/3$. (b) $\theta_r > 2\pi/3$.

4.2 Spatiotemporally resonant triads

When the hydrodynamic problem is posed on a horizontally unbounded domain there is no preferred direction (in the horizontal) so that each critical wave number from linear analysis actually corresponds to a circle of critical wavevectors as discussed in Section 2.3. There are two such critical circles at the bicritical point, as shown in Figure 4.1. In this situation it has been argued that resonant triads may play a central role in the Faraday wave pattern selection problem [9, 70, 61, 64, 86].

Two examples of spatially resonant triads are shown in Figure 4.1. They satisfy the spatial resonance condition

$$k_1 + k_2 = k_3. \quad (4.1)$$

For now, we focus on the situation where $k_1$ and $k_2$ are wave vectors associated with
one Faraday instability at the bicritical point, and $k_3$ is associated with the other instability. The resonant angle $\theta_r$ satisfies

$$\cos\left(\frac{\theta_r}{2}\right) = \frac{|k_3|}{2|k_1|}. \quad (4.2)$$

For the first case where $0 \leq \theta_r < 2\pi/3$, we have that $|k_3| > |k_1|$. It is also possible to have a resonant triad where $|k_3| < |k_1|$, as shown in the second case where $2\pi/3 < \theta_r < \pi$. We exclude the hexagonal case for which $|k_3| = |k_1|$ and $\theta_r = 2\pi/3$.

To illustrate the potential for resonant triads to influence pattern formation in parametrically excited systems we consider a bifurcation problem involving the three critical Fourier modes associated with the resonant triads of Figure 4.1. Much of this discussion is a review of the key theoretical ideas in [27]. Because of the periodic forcing of the system, it is natural to formulate the bifurcation problem in terms of a stroboscopic map [88]. Specifically, we denote the free surface height $h(x, t)$ ($x \in \mathbb{R}^2$) at time $t = pT$ ($p \in \mathbb{Z}$) by

$$h(x, pT) = A(p)e^{ik_1 \cdot x} + B(p)e^{ik_2 \cdot x} + C(p)e^{i(k_1+k_2) \cdot x} + \text{c.c.} + \ldots. \quad (4.3)$$

Here $A$, $B$ and $C$ are the complex amplitudes of the linear modes that are neutrally stable at the bicritical point and which form a resonant triad. In this discussion we assume that the angle $\theta_r$ between $k_1$ and $k_2$ is not an integer multiple of $\pi/3$, so that the critical modes interact nonlinearly to generate other modes on a rhombic (rather than hexagonal) lattice. These additional modes, denoted by “…” above, are linearly damped at the bicritical point. We may then use the spatial reflection and
translation symmetries to determine the general form of the bifurcation equations that govern the dynamics on the center manifold. Specifically, to cubic order, the codimension-two bifurcation problem takes the form

\begin{align}
A & \rightarrow \sigma A + \alpha BC + (a|A|^2 + b|B|^2 + c|C|^2)A \\
B & \rightarrow \sigma B + \alpha AC + (a|B|^2 + b|A|^2 + c|C|^2)B \\
C & \rightarrow \mu C + \delta AB + (d|A|^2 + d|B|^2 + e|C|^2)C,
\end{align}

where the coefficients are all real. The Floquet multipliers \(\sigma\) and \(\mu\) are either +1 or −1 depending on whether the linear modes \(A\), \(B\), and \(C\) are harmonically or subharmonically excited, respectively.

In deriving (4.4) we considered only the spatial symmetries associated with the resonant triad. Following [88], we enforce the temporal symmetry associated with the triad through a normal form transformation of (4.4). Specifically, there exists a near-identity nonlinear transformation that removes all nonlinear terms in (4.4) which do not commute with \(J^T\), where \(J\) is the Jacobian matrix associated with the linearized problem (see, for example [37]). Here

\[ J = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \mu \end{pmatrix}, \]

where \(|\sigma| = |\mu| = 1\). The normal form symmetry may be interpreted in terms of time-translation. Specifically, advancing by one period in time maps period-doubled
modes to their negatives, e.g. if $\mu = -1$, then advancing one period takes $C \rightarrow -C$.

In the case that $\mu = +1$ ($\sigma = \pm 1$), the bifurcation problem (4.4) is already in normal form. This observation is trivial if $\sigma = +1$. If $\sigma = -1$, then the normal form symmetry is equivalent in action to that associated with the spatial translation symmetry $x \rightarrow x + d$, where $d$ satisfies $k_1 \cdot d = k_2 \cdot d = \pi$.

In contrast, in the case that $\mu = -1$, a normal form transformation removes the quadratic terms in the bifurcation problem (4.4). The normal form of the bifurcation problem, through cubic order, is then

\begin{align*}
A &\rightarrow \sigma A + (a|A|^2 + b|B|^2 + c|C|^2)A \\
B &\rightarrow \sigma B + (a|B|^2 + b|A|^2 + c|C|^2)B \\
C &\rightarrow -C + (d|A|^2 + d|B|^2 + e|C|^2)C.
\end{align*}

We note that $C = 0$ is a dynamically-invariant subspace of (4.6). This is true to all orders of the normal form since $C = 0$ is the fixed point subspace of a (spatio-)temporal symmetry. Specifically, if $\sigma = +1$ then $C = 0$ is the fixed point subspace associated with the time translation by one period, i.e. $(A,B,C) \rightarrow (A,B,-C)$. And if $\sigma = -1$, then $C = 0$ is the fixed-point subspace associated with the spatio-temporal symmetry involving time translation by one period followed by spatial translation by $d$, where again $k_1 \cdot d = k_2 \cdot d = \pi$.

We now examine (4.4) more closely in the case that $\mu = +1$ so that we cannot remove the quadratic nonlinearities by a normal form transformation. We focus on a detuning from the bicritical point such that the $C$ mode is weakly damped, while the $A,B$ modes are neutrally stable. In this case, $|\sigma| = 1$, $\mu < 1$, we can further
reduce the bifurcation problem to one involving the critical modes $A$ and $B$, with $C$ constrained to the center manifold: 

$$C = \frac{\delta}{(1-\mu)}AB + \ldots.$$ 

We then obtain the reduced bifurcation problem

\begin{align*}
A & \to \sigma A + a|A|^2A + \beta(\theta_r)|B|^2A \\
B & \to \sigma B + a|B|^2B + \beta(\theta_r)|A|^2B
\end{align*}

(4.7a) (4.7b)

where the cross-coupling coefficient is

$$\beta(\theta_r) = b + \frac{\alpha\delta}{1-\mu}.$$ (4.8)

We see that in this case, the near critical spatio-temporally resonant mode $C$ in (4.4) can contribute significantly to the cross-coupling coefficient $\beta(\theta_r)$ since $0 < 1 - \mu \ll 1$ in (4.8). For example, for $\mu$ (real or complex) sufficiently close to 1, the second term in (4.8) dominates and $\beta(\theta_r)$ becomes large in magnitude. However, we also point out that if $b$ and $\alpha\delta$ have opposite signs, then $\beta(\theta_r)$ could actually vanish for some $1 - \mu > 0$. Examples of these two very different situations are given in Section 4.5. Recent results for weak damping and forcing [90] address the sign of the product $\alpha\delta$, and its scaling with respect to the amplitude of the periodic forcing function. We will make reference to these results when we present numerical examples in Section 4.5.

We contrast the above with what happens when $\mu = -1$ at the bicritical point. In this case $\alpha = \delta = 0$ in the normal form (4.6) and $C = 0$ is an invariant subspace with associated dynamics of the form (4.7) with $\beta(\theta_r) = b$. In this case, the triad is spatially resonant, but not temporally resonant, and the cross-coupling coefficient is
Insensitive to any parameter proximity to the bicritical point.

These observations about $\beta(\theta_r)$ are important for understanding which patterns might be observable near onset since branching direction and stability of patterns are determined by various nonlinear (cross-coupling) coefficients in the amplitude equations. We discuss this further at the end of Section 4.3.

Finally we note that similar results hold for the case of complex Floquet multipliers. Only damped modes with complex Floquet multipliers sufficiently close to $+1$ are expected to contribute to $\beta(\theta)$.

### 4.3 Hexagonal lattice bifurcation problem

The analysis of the previous section led to certain conclusions about the nonlinear coefficients in the general rhombic lattice bifurcation problem

\begin{align}
 v_1 & \rightarrow \sigma v_1 + (a |v_1|^2 + \beta(\theta) |v_2|^2) v_1 \quad (4.9a) \\
 v_2 & \rightarrow \sigma v_2 + (a |v_2|^2 + \beta(\theta) |v_1|^2) v_2. \quad (4.9b)
\end{align}

Here $v_1, v_2$ are the complex amplitudes of two critical Fourier modes with wavevectors $k_1, k_2 \ (|k_1| = |k_2| = k_c)$ that are separated by an angle $\theta \in (0, \frac{\pi}{2}) \ (\theta \neq \frac{\pi}{3})$. In particular, it follows from (4.8) that if a weakly damped harmonic mode is removed via center manifold reduction, then $\beta(\theta)$ may become large in magnitude when the spatial resonance condition is met, i.e. when $\theta = \theta_r$. This is in contrast to the situation where there are weakly damped subharmonic modes, which have no special influence on the pattern selection problem at onset.
We now lay the framework for examining possible implications of these results for stability of harmonic hexagonal and triangular SL-I superlattice patterns. We follow [25] and introduce the 12-dimensional $D_6T^2$-equivariant bifurcation problems that enable us to determine the relative stability of simple stripe patterns, simple hexagonal patterns, three distinct rhombic patterns, and superhexagon and supertriangle SL-I patterns. We make use of bifurcation results derived in [25, 30, 26], which apply when there is a single critical wave number $k_c$, to demonstrate how the magnitude of the cross-coupling terms are pivotal in determining pattern stability. This framework is similar to that presented in Section 2.4, except that here the formulation takes the form of a stroboscopic map rather than a flow. The modes on the center manifold in the Fourier expansion of the free surface height are

$$h(x, pT) = \sum_{j=1}^{6} z_j(p)e^{iK_j \cdot x} + c.c.$$ (4.10)

at time $t = pT$, where the vectors $K_j \in \mathbb{R}^2$ are given by (2.29); see Figure 2.6.

The general form of the 12-dimensional $D_6T^2$-equivariant mappings is derived in [26]. These mappings are the discrete-time analogues of (2.32). Through cubic order in the amplitudes $z_j$, they take the form
\[ z_1 \to \sigma \left( (1 + \lambda)z_1 + \epsilon \varepsilon_2 \varepsilon_3 + (b_1 |z_1|^2 + b_2 |z_2|^2 + b_3 |z_3|^2 + b_4 |z_4|^2 + b_5 |z_5|^2 + b_6 |z_6|^2) z_1 \right) \]  
\( (4.11a) \)

\[ z_2 \to \sigma \left( (1 + \lambda)z_2 + \epsilon \varepsilon_1 \varepsilon_3 + (b_1 |z_2|^2 + b_2 |z_1|^2 + b_3 |z_3|^2 + b_4 |z_4|^2 + b_5 |z_5|^2 + b_6 |z_4|^2) z_2 \right) \]  
\( (4.11b) \)

\[ z_3 \to \sigma \left( (1 + \lambda)z_3 + \epsilon \varepsilon_1 \varepsilon_2 + (b_1 |z_3|^2 + b_2 |z_1|^2 + b_2 |z_2|^2 + b_4 |z_6|^2 + b_5 |z_4|^2 + b_6 |z_5|^2) z_3 \right) \]  
\( (4.11c) \)

\[ z_4 \to \sigma \left( (1 + \lambda)z_4 + \epsilon \varepsilon_5 \varepsilon_6 + (b_1 |z_4|^2 + b_2 |z_5|^2 + b_2 |z_6|^2 + b_4 |z_6|^2 + b_5 |z_1|^2 + b_6 |z_2|^2) z_4 \right) \]  
\( (4.11d) \)

\[ z_5 \to \sigma \left( (1 + \lambda)z_5 + \epsilon \varepsilon_4 \varepsilon_6 + (b_1 |z_5|^2 + b_2 |z_4|^2 + b_2 |z_6|^2 + b_4 |z_2|^2 + b_5 |z_1|^2 + b_6 |z_3|^2) z_5 \right) \]  
\( (4.11e) \)

\[ z_6 \to \sigma \left( (1 + \lambda)z_6 + \epsilon \varepsilon_4 \varepsilon_5 + (b_1 |z_6|^2 + b_2 |z_4|^2 + b_2 |z_5|^2 + b_4 |z_4|^2 + b_5 |z_2|^2 + b_6 |z_1|^2) z_6 \right) \]  
\( (4.11f) \)

where \( \lambda \) measures the distance from the critical excitation amplitude, and \( \sigma = +1(-1) \) in the case of (sub)harmonic instability. All nonlinear coefficients are real. If \( \sigma = -1 \) then a normal form transformation removes all even terms on the right-hand side of (4.11) and hence \( \epsilon = 0 \). The dependence of the general equivariant bifurcation problem on the lattice vectors \((n_1, n_2)\) (see Section 2.4) does not appear until higher than cubic order in its Taylor expansion [26].

We now recall some basic results pertaining to the bifurcation problem (4.11). In the \( \sigma = +1 \) case the equivariant branching lemma [23] ensures the existence of harmonic wave solution branches in the form of stripes, simple hexagons, rhombs, and super hexagons [30]. A primary solution branch with submaximal isotropy, named super triangles, was also shown to exist in [25]. See Figure 1.2a for an example of this pattern. Table 2.1 gives the general form of these solutions, along with their branching and stability assignments. The general bifurcation results in the case that \( \sigma = -1 \) can be found in [26]; this bifurcation problem differs from the harmonic case in that it possesses an additional \( \mathbb{Z}_2 \) normal form symmetry. The equivariant branching lemma then ensures existence of five additional solution branches to those
listed in Table 2.1 [26].

We recall from our discussion in Section 2.4 that we focus our analysis on the unfolding of the degenerate bifurcation problem $\epsilon = 0$. When $0 < |\epsilon| \ll 1$, it follows from Table 2.1 that a necessary condition for one of the superpatterns to be stable over some range of $\lambda$ values near onset is for

$$b_1 + 2b_2 < -|b_4 + b_5 + b_6| < 0. \quad (4.12)$$

The combination $b_1 + 2b_2$ is independent of the lattice angle $\theta_h$ in (2.30). We compute it from the Zhang-Viñals equations in the next section by considering a bifurcation to simple hexagons. In contrast, the combination $b_4 + b_5 + b_6$ depends on $\theta_h$ and we compute it from the equations by considering the rhombic lattice bifurcation problem (4.9). Specifically, the cross-coupling coefficients $b_4, b_5, b_6$ are

$$b_4 = \beta(\theta_h), \quad b_5 = \beta\left(\theta_h + \frac{2\pi}{3}\right), \quad b_6 = \beta\left(\theta_h - \frac{2\pi}{3}\right), \quad (4.13)$$

where $\theta_h$ is the angle between $K_1$ and $K_4$ given by (2.30). (The function $\beta(\theta)$ may be extended from $\theta \in \left(0, \frac{\pi}{2}\right]$ to angles $\theta \in (0, 2\pi)$ using $\beta(\theta) = \beta(-\theta) = \beta(\theta + \pi)$, identities that follow from the symmetries of the rhombic lattice bifurcation problem.)

The inequality (4.12) will be satisfied (if at all) only for those $\theta_h$ values where $|b_4 + b_5 + b_6|$ is small compared to $|b_1 + 2b_2|$. Moreover, if $b_1 - b_2 < 0$ in addition to (4.12), then simple hexagons become unstable on a given hexagonal lattice when

$$\lambda = -\frac{\epsilon^2(b_4 + b_5 + b_6)}{(b_1 + 2b_2 - b_4 - b_5 - b_6)^2}. \quad (4.14)$$
If \( b_4 + b_5 + b_6 < 0 \) for all \( \theta_h \), then simple hexagons first lose stability with increasing \( \lambda \) to a perturbation in the direction of a superpattern for that value of \( \theta_h \) that minimizes \( |b_4 + b_5 + b_6| \). If \( b_4 + b_5 + b_6 > 0 \) for any \( \theta_h \), then small amplitude simple hexagons are unstable when \( \lambda > 0 \). Thus we expect the stability properties of SL-I superlattice patterns and simple hexagons to be affected by the presence of a weakly damped harmonic mode when \( \theta_h \) or \( \theta_h \pm 2\pi/3 \) is near \( \theta_r \) (or \( \pi \pm \theta_r \)), the resonant triad angle, since it is in this situation that one of the cross-coupling coefficients \( b_4, b_5 \) or \( b_6 \) may suddenly change in magnitude.

### 4.4 Perturbation analysis

Here we outline the computation of the bifurcation coefficients in (4.9) and (4.11) from the equations of Zhang and Viñals (3.34). A multiple-scale perturbation method is used to derive expressions for the coefficients which are then evaluated numerically using a pseudospectral approach. This follows closely the method described in [27] for the onset of one-dimensional patterns and we refer the reader there for further details.

The coefficients can be derived by considering two different calculations, namely the bifurcation problem (2.32) restricted in turn to a rhombic and a simple hexagons subspace. The coefficients of the flow (2.32) may then be related to those of the map (4.11).

In order to compute the coefficient \( a \) and the cross-coupling coefficient \( \beta(\theta) \) in (4.9) we seek solutions which are periodic on a rhombic lattice associated with an angle \( \theta \). We are thereby able to compute the coefficients \( b_1, b_4, b_5, \) and \( b_6 \) in the bifurcation
equations (2.32) since \( b_1 = a, \ b_4 = \beta(\theta_h), \ b_5 = \beta(\theta_h + 2\pi/3), \) and \( b_6 = \beta(\theta_h - 2\pi/3). \)

First we introduce a small parameter \( \eta \), such that

\[
\begin{align*}
\Phi(x, y, \tau) &= \eta \Phi_1(x, y, \tau) + \eta^2 \Phi_2(x, y, \tau) + \eta^3 \Phi_3(x, y, \tau) + \ldots \quad (4.15b)
\end{align*}
\]

in (3.34) where

\[
T = \eta^2 \tau, \quad f = f_c + \eta^2 f_2.
\]

Here \( f_c \) is the critical excitation amplitude. The terms in the expansion for \( h \) and \( \Phi \) may be written in the following separable Floquet-Fourier form:

\[
\begin{align*}
h_1 &= [w_1(T)e^{ik_c x} + w_4(T)e^{ik_c (cx+sy)} + c.c.]p_1(\tau) \quad (4.17a) \\
\Phi_1 &= [w_1(T)e^{ik_c x} + w_4(T)e^{ik_c (cx+sy)} + c.c.]q_1(\tau) \quad (4.17b) \\
h_2 &= [w_1^2(T)e^{2ik_c x} + w_4^2(T)e^{2ik_c (cx+sy)}]p_{2,1}(\tau) \quad (4.17c) \\
&\quad + w_1(T)\overline{w}_4(T)e^{ik_c ((1-c)x-sy)}p_{2,2}(\tau) \\
&\quad + w_1(T)w_4(T)e^{ik_c ((1+c)x+sy)}p_{2,3}(\tau) + c.c. \\
\Phi_2 &= [w_1^2(T)e^{2ik_c x} + w_4^2(T)e^{2ik_c (cx+sy)}]q_{2,1}(\tau) \quad (4.17d) \\
&\quad + w_1(T)\overline{w}_4(T)e^{ik_c ((1-c)x-sy)}q_{2,2}(\tau) \\
&\quad + w_1(T)w_4(T)e^{ik_c ((1+c)x+sy)}q_{2,3}(\tau) + c.c.
\end{align*}
\]

where \( c = \cos \theta, \ s = \sin \theta, \) and \( \theta \) is not a multiple of \( \frac{\pi}{3} \). Here \( p_1 \) and \( q_1 \) are real \( 2\pi \)-periodic functions of the fast time \( \tau \) in the case of harmonic waves; in the case of subharmonic waves they are \( 4\pi \)-periodic in \( \tau \). Additionally, \( p_{2,r} \) and \( q_{2,r} \) \( (r = 1, 2, 3) \)
are real $2\pi$-periodic functions of $\tau$. The wave number $k_c$ is associated with the onset unstable mode.

At $O(\eta)$ we recover the linear problem which determines $k_c$ and $f_c$, as well as the functions $p_1$, $q_1$ to within a multiplicative constant. At $O(\eta^2)$, equations are found which allow us to solve for the functions $p_2$, $r$ and $q_2$, $r$. Finally, at $O(\eta^3)$, we apply a solvability condition, as discussed in Section 2.5, to ensure that a periodic solution exists. This condition leads to the amplitude equations

$$
\begin{align*}
\delta \frac{dw_1}{dT} & = \alpha f_2 w_1 + A|w_1|^2 w_1 + B(\theta)|w_4|^2 w_1 \quad (4.18a) \\
\delta \frac{dw_4}{dT} & = \alpha f_2 w_4 + A|w_4|^2 w_4 + B(\theta)|w_1|^2 w_4 \quad (4.18b)
\end{align*}
$$

where

$$
\begin{align*}
\delta & = \frac{1}{2\pi} \int_0^{4\pi} (p_1' + \gamma k_c^2 p_1) \tilde{p}_1 \, d\tau \quad (4.19a) \\
\alpha & = \frac{k_c}{4\pi} \int_0^{4\pi} \left[ \cos(\chi) \cos(m\tau) + \sin(\chi) \cos(n\tau + \phi) \right] p_1 \tilde{p}_1 \, d\tau \quad (4.19b) \\
A & = \frac{k_c^2}{4\pi} \int_0^{4\pi} \left[ -k_c (p_1^2 q_1)' - \gamma k_c^2 p_1^2 q_1 - 2(q_1 p_2, 1)' - 2\gamma k_c^2 q_1 p_2, 1 \\
& \quad + k_c q_1 p_1 + \frac{3}{2} k_c^3 \Gamma_0 p_1^3 \right] \tilde{p}_1 \, d\tau \quad (4.19c) \\
B(\theta) & = \frac{k_c^2}{4\pi} \int_0^{4\pi} \left[ (1 - c - \sqrt{2 - 2c}) [(p_1 q_{2,2})]' + \gamma k_c^2 p_1 q_{2,2} - k_c q_1 q_{2,2} \\
& \quad + (1 + c - \sqrt{2 + 2c}) [(p_1 q_{2,3})]' - \gamma k_c^2 p_1 q_{2,3} - k_c q_1 q_{2,3} \\
& \quad - (1 - c) [(p_2, 2 q_1)' + \gamma k_c^2 p_2, 2 q_1] - (1 + c) [(p_2, 3 q_1)' + \gamma k_c^2 p_2, 3 q_1] \\
& \quad - (6 - 2\sqrt{2 - 2c} - 2\sqrt{2 + 2c}) [k_c (p_1^2 q_1)' + \gamma k_c^2 p_1^2 q_1 - k_c^2 p_1 q_1^2] \\
& \quad + \Gamma_0 (3c^2 + s^2) k_c^3 p_1^3 \right] \tilde{p}_1 \, d\tau.
\end{align*}
$$
In the above, a prime denotes differentiation with respect to $\tau$ and $\bar{p}_1$ is the equivalent of $p_1$ for the adjoint problem at $O(\eta)$.

The amplitude equations (4.18) may be rescaled and then comparison with the map (4.9) yields

$$a = b_1 = \text{sgn}(A\alpha), \quad \beta(\theta) = \text{sgn}(A\alpha) \frac{B(\theta)}{A}. \quad (4.20)$$

The fixed points of (4.11) restricted to each of the three different rhombic subspaces in turn lead to

$$b_4 = \beta(\theta_h), \quad b_5 = \text{sgn}(\alpha) \frac{B(\theta_h + \frac{2\pi}{3})}{A}, \quad b_6 = \text{sgn}(\alpha) \frac{B(\theta_h - \frac{2\pi}{3})}{A} \quad (4.21)$$

where $\theta_h$ is determined by the particular hexagonal lattice under consideration.

Similarly, we compute the coefficients $\epsilon$ and $b_2$ in the bifurcation equations (4.11) by seeking solutions in the form of simple hexagons. Here we use a three-timing perturbation method, writing the solution as

$$h(x, y, \tau) = \eta h_1(x, y, \tau, T_1, T_2) + \eta^2 h_2(x, y, \tau, T_1, T_2) + \eta^3 h_3(x, y, \tau, T_1, T_2) + \cdots \quad (4.22a)$$

$$\Phi(x, y, \tau) = \eta \Phi_1(x, y, \tau, T_1, T_2) + \eta^2 \Phi_2(x, y, \tau, T_1, T_2) + \eta^3 \Phi_3(x, y, \tau, T_1, T_2) + \cdots \quad (4.22b)$$

where

$$T_1 = \eta \tau, \quad T_2 = \eta^2 \tau \quad (4.23)$$
and

\[
\begin{align*}
  h_1 &= w_1(T_1, T_2)p_1(\tau)[e^{ik_c x} + e^{ik_c(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)} + e^{ik_c(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + c.c.] \quad (4.24a) \\
  \Phi_1 &= w_1(T_1, T_2)q_1(\tau)[e^{ik_c x} + e^{ik_c(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)} + e^{ik_c(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + c.c.] \quad (4.24b) \\
  h_2 &= w_1^2(T_1, T_2)\{p_{2,1}(\tau)[e^{ik_c 2x} + e^{ik_c(-x + \sqrt{3}y)} + e^{ik_c(-x - \sqrt{3}y)} + c.c.]
    + p_{2,2}(\tau)[e^{ik_c x} + e^{ik_c(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)} + e^{ik_c(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + c.c.]
    + p_{2,3}(\tau)[e^{ik_c(\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + e^{ik_c \sqrt{3}y} + e^{ik_c(\frac{3}{2}x + \frac{\sqrt{3}}{2}y)} + c.c.]\} \quad (4.24c) \\
  \Phi_2 &= w_1^2(T_1, T_2)\{q_{2,1}(\tau)[e^{ik_c 2x} + e^{ik_c(-x + \sqrt{3}y)} + e^{ik_c(-x - \sqrt{3}y)} + c.c.]
    + q_{2,2}(\tau)[e^{ik_c x} + e^{ik_c(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)} + e^{ik_c(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + c.c.]
    + q_{2,3}(\tau)[e^{ik_c(\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + e^{ik_c \sqrt{3}y} + e^{ik_c(\frac{3}{2}x + \frac{\sqrt{3}}{2}y)} + c.c.]\}. \quad (4.24d)
\end{align*}
\]

As with the rhombic case, \(p_1, q_1, p_{2,r}, q_{2,r}\) are real. Additionally, we take the amplitude \(w_1(T_1, T_2)\) to be real.

For the harmonic case, at \(O(\eta^2)\) the solvability condition

\[
\delta \frac{\partial w_1}{\partial T_1} = \beta_0 w_1^2 
\]

must be satisfied, where \(\delta\) is given by (4.19a). The quadratic coefficient is

\[
\beta_0 = \frac{k_c^2}{4\pi} \int_0^{4\pi} \left[ -(p_1 q_1)' - \gamma k_c^2 p_1 q_1 + \frac{1}{2} k_c q_1^2 \right] \tilde{p}_1 \, d\tau . \quad (4.26)
\]

There is no solvability condition for subharmonic waves at \(O(\eta^3)\), reflecting the fact that there are no even terms in the amplitude equations (4.11) for this case.

At \(O(\eta^3)\), we again apply a solvability condition to ensure that a periodic solution
exists. This conditions leads to the amplitude equation

$$\delta \frac{\partial w_1}{\partial T_2} = \alpha f_2 w_1 + (A + 2\beta_2)w_1^3. \quad (4.27)$$

The coefficients $\delta$, $\alpha$, and $A$ are given by (4.19a) – (4.19c), and

$$\beta_2 = \frac{1}{4\pi} \int_0^{4\pi} \left[ \left( \frac{3}{2} - \sqrt{3} \right) k_c^2 [(p_1 q_{2,3})'] + \gamma k_c^2 p_1 q_{2,3} - k_c q_1 q_{2,3} \right] \right. + (2\sqrt{3} - 4) k_c^3 [(p_1 q_1)'] + \gamma k_c^2 p_1 q_1 - k_c p_1 q_1^2 \right] + \frac{3}{2} k_c^3 [(p_2 q_{2,3})'] + \gamma k_c^2 p_2 q_{2,3} q_1 - \Gamma_0 k_c^3 p_1^2 \right] \right. - \frac{3}{2} k_c^2 [(p_2 q_{2,2})'] + \gamma k_c^2 p_2 q_{2,2} + (p_2 q_1)' + \gamma k_c^2 p_2 q_1 - k_c q_1 q_{2,2} \right] \right. - \frac{\beta_0}{\delta} [k_c^2 p_1 + 2p_2 q_1 + 2\gamma k_c^2 p_2 q_1] \right. \bar{p}_1 \, d\tau. \quad (4.28)$$

By rescaling $\eta w_j(T_1, T_2) \rightarrow w_j(T)$ and $\eta^2 \alpha f_2 \rightarrow \alpha f_2$, we obtain the reconstituted hexagonal bifurcation equation

$$\delta \frac{d w_1}{dT} = \alpha f_2 q_1 + \beta_0 w_1^2 + (A + 2\beta_2)w_1^3 \quad (4.29)$$

Finally, after rescaling as for the rhombic case, and comparing (4.29) to (4.11) we find that

$$\epsilon = \text{sgn}(\alpha) \frac{\beta_0}{\sqrt{|\alpha A|}}, \quad b_2 = \text{sgn}(A\alpha) \frac{\beta_2}{A}. \quad (4.30)$$
4.5 Results

This section shows explicitly the role of resonant triads and weakly damped harmonic modes in the pattern selection problem for two-frequency forced Faraday waves. We examine how the cubic nonlinear coefficients in (4.11), for the Zhang-Viñals hydrodynamic equations vary as a function of \( \theta_h \), the lattice angle and explain how this can be related to \( \theta_r \), the resonant triad angle. We focus on two examples, involving forcing frequency ratios \( m/n = 2/3 \) and \( 6/7 \). The \( 2/3 \) case demonstrates the basic difference between the pattern selection problems for subharmonic and harmonic instabilities near the bicritical point. Our investigation also reveals a fundamental difference between harmonic wave pattern selection in the \( 2/3 \) and \( 6/7 \) cases, due to the presence of additional harmonic wave resonance tongues for the higher \( 6/7 \) forcing frequencies; see Figure 4.2.

**Example 1: \( m/n = 2/3 \)**

This example demonstrates a result of the general normal form analysis of Section 4.2, namely that proximity to the subharmonic/harmonic bicritical point will strongly influence the pattern selection problem for subharmonic waves, but not for harmonic waves. Specifically, we examine the cross-coupling coefficient \( \beta(\theta) \) in (4.9) as a function of the angle \( \theta \) for onset of both harmonic and subharmonic waves near the bicritical point. We show that only in the subharmonic case does \( |\beta(\theta)| \) become large at the resonant angle \( \theta_r \) in (4.2).

As described in Section 3.5, the primary instability changes from harmonic (Floquet multiplier +1) to subharmonic (Floquet multiplier −1) as \( \chi \) in (3.29) is increased
Figure 4.2: Neutral stability curves computed from (3.35), the linearized Zhang-Viñals equations. Floquet multipliers of +1 (−1) are indicated in grey (black). (a) $m/n = 2/3$, $\phi = 0^\circ$, $\chi = \chi_c = 66.6^\circ$, $\Gamma_0 = 0.53$, $G_0 = 0.47$ and $\gamma = 0.09$ in (3.29) and (3.34). (b) $m/n = 6/7$, $\phi = 0^\circ$, $\chi = \chi_c = 53.0^\circ$, $\Gamma_0 = 7.5$, $G_0 = 1.5$ and $\gamma = 0.08$. 
through the bicritical point $\chi_c$. This transition is determined from the linear hydrodynamic problem (3.35). A numerically computed neutral curve $f(k)$ for $m/n = 2/3$ forcing and $\chi = \chi_c = 66.6^\circ$ is given in Figure 4.2a. The other parameters of this example are $\phi = 0^\circ$, $\Gamma_0 = 0.53$, $G_0 = 0.47$ and $\gamma = 0.09$.

We now vary $\chi$ near $\chi_c$, holding all other parameters fixed, and examine the rhombic lattice cross-coupling coefficient $\beta(\theta)$ in (4.9) for onset subharmonic/harmonic waves, as appropriate. We have scaled the amplitudes $v_1$ and $v_2$ in (4.9) so that $a = -1$. We note that in the harmonic case $\beta$ diverges as $\theta \to 60^\circ$, i.e. when the rhombic lattice approaches the hexagonal one and there is an additional mode associated with the center manifold dynamics. This is in contrast to the subharmonic case, for which there is a normal form symmetry that ensures existence of a dynamically invariant subspace spanned by a pair of subharmonic modes separated by $60^\circ$. Thus in the subharmonic case $\beta$ remains finite at $\theta = 60^\circ$.

For $\chi > \chi_c$ the primary instability is to subharmonic waves. For instance, for $\chi = 66.7^\circ$ the minimum of the neutral curve occurs at wave number $k_{c,s} = 1.415$ with forcing amplitude $f_c = 0.842$, and is associated with a Floquet multiplier $\sigma = -1$. The nearly critical harmonic resonance tongue has its minimum at $(k, f) = (0.962, 0.846)$. In this case, there is a spatio-temporally resonant triad composed of the weakly damped harmonic mode and, from (4.2), two subharmonic modes separated by $\theta_r = 39.9^\circ$. It follows from our general analysis of Section 4.2 that $\beta(\theta)$ will be large in magnitude for $\theta$ near $\theta_r$. Figure 4.3a shows $\beta(\theta)$ for this case, and indeed, the nonlinear coefficient exhibits a large dip centered at $\theta = \theta_r = 39.9^\circ$. At this angle, $|\beta(\theta)|$ takes on its largest value. Similar observations have been made by Zhang and Viñals [86] for forcing frequencies in ratio $m/n = 1/2$. 
We mention here a prediction of [90], which demonstrates how the assumption of an (approximate) Hamiltonian structure may be exploited to deduce the sign of the product $\alpha \delta$ in (4.7). In particular, when (4.7) applies to a harmonic instability near the bicritical point, it is found that $\alpha \delta < 0$. This agrees with the numerical result presented above, namely that the contribution to $\beta (\theta_r)$ is negative, and thus results in a dip, rather than a spike. [90] also makes predictions regarding the scaling of $\alpha \delta$ for weak damping and forcing; we postpone a discussion of these results until Chapter 5.

In contrast, when $\chi < \chi_c$, so that the first instability to occur with increasing $f$ is harmonic, we find that the weakly damped subharmonic mode leaves no signature in the plot $\beta (\theta)$. For instance, for $\chi = 66.5^\circ$ the primary instability is to harmonic waves at wave number $k_{c,h} = 0.963$ and forcing amplitude $f_c = 0.841$. The subharmonic resonance tongue has a minimum at $(k, f) = (1.415, 0.843)$. While there is a \textit{spatially} resonant triad involving two critical harmonic modes, which by (4.2) are separated by $\theta_r = 85.7^\circ$, the triad of modes is not \textit{spatio-temporally} resonant. Figure 4.3b shows the cross-coupling coefficient $\beta (\theta)$ for this case (with the region near 60° removed). As anticipated, there is no signature of the weakly damped subharmonic mode in the plot. Similar observations have been made in the setting of one-dimensional surface wave patterns [27].

\textbf{Example 2: $m/n = 6/7$}

This example demonstrates a fundamental difference between harmonic wave pattern selection for low forcing frequencies (\textit{e.g.} $2 \omega / 3 \omega$) and for high forcing frequencies (\textit{e.g.}
Figure 4.3: Cross-coupling coefficients $\beta(\theta)$ in (4.9) for the case $m/n = 2/3$ and $\phi = 0^\circ$ in (3.29). The fluid parameters used are given in the caption of Figure 4.2a. (a) $\chi = 66.7^\circ > \chi_c$, when the bifurcation is to subharmonic waves. Note the dip at $\theta = \theta_r = 39.9^\circ$. (b) $\chi = 66.5^\circ < \chi_c$, when the bifurcation is to harmonic waves. Because the (nearly) critical modes are not in temporal resonance, $\beta(\theta)$ shows no special structure at $\theta = \theta_r = 85.7^\circ$. We have removed from this plot the region near $\theta = 60^\circ$, where $\beta(\theta)$ diverges.
$6\omega/7\omega$). This difference is due to the presence of multiple harmonic resonance tongues in the neutral curve associated with the higher forcing frequencies; see Figure 4.2. In particular, these resonance tongues suggest the possibility that weakly damped harmonic modes may influence the harmonic wave pattern selection problem. This is in contrast to the $m/n = 2/3$ example of the previous section, for which only subharmonic wave pattern competition was affected by weakly damped harmonic waves. In this section we also demonstrate that the weakly damped harmonic modes may stabilize harmonic wave superpatterns at a lattice angle $\theta_h \approx \theta_r$, due to a near cancellation of the two terms that contribute to $\beta(\theta)$ given by (4.8) as described in Section 4.2.

We set $m/n = 6/7$ and focus on a bifurcation to harmonic waves for $\chi = 52.4^\circ$, which is close to the bicritical value $\chi_c = 53.0^\circ$. The remaining parameters are $\phi = 0^\circ$, $\Gamma_0 = 7.5$, $G_0 = 1.5$ and $\gamma = 0.08$. We note that while the forcing frequency ratio $m/n = 6/7$ coincides with that used in the experiments in [10] which produced the SL-I superlattice pattern in Figure 1.2a, the remaining parameters do not coincide with the experiment. One problem with using the experimental parameters in the Zhang-Viñals equations is that the primary instability then moves to a subharmonic resonance tongue at very small wave number, i.e., the first resonance tongue of Figure 4.2b. This is because the Zhang-Viñals model does not accurately capture the damping at small $k$ that is due to finite depth effects; see Section 3.5 for a more detailed discussion.

In this example we find two prominent features in the plot of the cross-coupling coefficient $\beta(\theta)$ in Figure 4.4a: a large dip at $\theta = 67.6^\circ$ and a small spike at $\theta = 22.2^\circ$. We now discuss the origin of these two features.
The large dip around $\theta = 67.6^\circ$ is not a consequence of two-frequency forcing. Specifically, the dip remains in $\beta(\theta)$ even for purely $6\omega$ forcing (*i.e.* in the limit $\chi \to 0$); cf. plots of $\beta(\theta)$ in Figures 4.4a and 4.4c which are obtained with $\chi = 52.4^\circ$ and $\chi = 0^\circ$, respectively. Thus this feature may be understood in the context of single frequency forcing, and has in fact already been investigated by Zhang and Viñals [24] in that setting. Specifically, if $\chi = 0^\circ$ then the forcing period is $T' = \frac{T}{6} = \frac{2\pi}{6}$ and the primary instability is to subharmonic waves with period $2T'$. A plot of the corresponding neutral curve is given in Figure 4.4d, with the primary harmonic resonance tongue from Figures 4.2a and b superimposed on it. In this single-frequency setting the feature at $67.6^\circ$ is understood as being due to the damped harmonic mode around $k = 1.7$ in Figure 4.4d. Perhaps more relevant to this discussion is our observation that this feature, which leads to a large value of $|b_4 + b_5 + b_6|$, is *destabilizing* for SL-I superlattice patterns. To see this, we refer to the discussion surrounding equation (4.12) and to Figure 4.4b, which shows that

\[
0 > b_1 + 2b_2 > b_4 + b_5 + b_6 = \beta(\theta_h) + \beta\left(\theta_h + \frac{2\pi}{3}\right) + \beta\left(\theta_h - \frac{2\pi}{3}\right),
\]

\[
= \beta(\theta_h) + \beta\left(\frac{\pi}{3} - \theta_h\right) + \beta\left(\frac{\pi}{3} + \theta_h\right) \text{ for } \frac{\pi}{3} + \theta_h \approx 67.6^\circ. \tag{4.31}
\]

In contrast the spike at $\theta = 22.2^\circ$ in Figure 4.4a minimizes $|b_4 + b_5 + b_6|$ at $\theta_h \approx 22.2^\circ$, as shown in Figure 4.4b. As we show below, this feature can lead to a stabilization of SL-I superlattice patterns and a destabilization of the simple hexagons. First we provide strong evidence that the spike is due to a resonance between the primary harmonic instability ($\sigma = 1$) and a weakly damped harmonic mode with a real Floquet multiplier $\mu$ that is close to 1 (see (4.4) and (4.8) of Section 4.2). In
order to show this we must first compute the Floquet multipliers $\mu(k)$ at the critical forcing amplitude $f_c$ to determine the wave numbers $k$ at which $\mu \approx 1$ at the onset of instability.

We determine the Floquet multipliers $\mu(k)$ at $f = f_c = 1.552$ numerically from the linear problem (3.35). These are presented in Figures 4.5 and 4.6. We find that the multipliers are well approximated away from the two primary resonance tongues by considering the unforced problem ($f = 0$ in equation 3.35), for which

$$\mu_{\pm} = e^{2\pi \lambda_{\pm}}, \quad \lambda_{\pm} = -\gamma k^2 \pm i\Omega k.$$  \hspace{1cm} (4.32)

Figures 4.5a and b show the magnitude $\xi$ and the phase $\psi$ of the Floquet multipliers $\mu = \xi e^{i\psi}$ both as computed numerically from (3.35) (solid line) and approximated by (4.32) (dotted line). Figure 4.6a shows the real part of the Floquet multipliers, $\xi \cos \psi$, versus wave number $k$. The “bubbles” in this plot correspond to wave numbers at which the Floquet multipliers are real (as opposed to a complex conjugate pair). Weakly damped harmonic modes are associated with bubbles near a Floquet multiplier of $+1$. Numerically we find that there are small bubbles of real Floquet multipliers whenever the phase $\psi$ is a multiple of $\pi$; this is demonstrated in Figure 4.6b. In particular, we find a bubble at wave number $k = 0.383$, with associated real Floquet multiplier $\mu = 0.93$. This mode is weakly damped and forms a resonant triad with primary harmonic modes separated by $\theta_r = 22.2^\circ$. (Here $k_{c,h} = 0.997$ for the primary instability, which corresponds to $k_1$ in (4.2), with $k_3 = 0.393$ determined by the weakly damped harmonic mode. The length scale ratio is approximately 2.5, which is close to $\sqrt{7} \approx 2.6$) Here we have focused on the wave numbers associated
Figure 4.4: (a) Cross-coupling coefficient $\beta(\theta)$ in (4.9) computed from (3.34) for the case $m/n = 6/7$, $\phi = 0^\circ$ and $\chi = 52.4^\circ < \chi_c$ in (3.29), and for fluid parameters given in the caption of Figure 4.2b. (b) Plots of $b_1 + 2b_2$ (dashed) and $b_4 + b_5 + b_6$ (solid) versus $\theta_h$. We note that $\theta_h$ only takes on the discrete values satisfying (2.30). (c) Cross-coupling coefficient $\beta(\theta)$ for $6\omega$ forcing only; we have used the same parameters as in (a) except that now $\chi = 0^\circ$. (d) Neutral curve for single frequency forcing. Floquet multipliers of $+1$ ($-1$) are indicated by solid (dashed) lines, and are computed relative to the period $T' = 2\pi/6$. The primary harmonic resonance tongue from the two-frequency case of Figure 4.2b is superimposed as a grey dotted line.
with real Floquet multipliers near $\mu = +1$ since weakly damped modes with complex Floquet multipliers do not form a spatio-temporally resonant triad with the primary harmonic modes.

We now present some hexagonal lattice bifurcation results for the specific parameters of this example, which are given in Figure 4.2b. The computation of the quadratic and cubic coefficients in the bifurcation problem (4.11) is described in Section 4.4. We scale the amplitudes $z_j$ in (4.11) so that $b_1 = -1$, in which case we find that $\epsilon = 0.00014$ and $b_2 = -2.72$. Thus we expect results of Section 4.3, which focused on the unfolding of the degenerate bifurcation problem $\epsilon = 0$, to apply. (The small value of $\epsilon$ for this example may in fact be understood from a result in [90], in which symmetry arguments are used to deduce the scaling of $\epsilon$ with respect to $\gamma$, $m$, and $n$.)

We find that simple hexagons, SL-I super hexagons and super triangles all bifurcate transcritically with the subcritical branch turning around in a saddle-node bifurcation. The stripes and rhombs solutions arise in supercritical pitchfork bifurcations. These claims are true for all lattice angles $\theta_h$ since the cubic coefficients $b_1, \ldots, b_6$ in (4.11) are always negative; see Figure 4.4a. Moreover, we find that simple hexagons are always stabilized in a saddle-node bifurcation and that they do not lose stability until after they reach the supercritical regime $\lambda > 0$. In contrast, super hexagons and super triangles are always unstable at $\lambda = 0$, since at that point the sign of the second eigenvalue in Table 2.1 is determined by $\text{sgn}(b_1 + 2b_2 - 3b_4 - 3b_5 - 3b_6)$, which is positive for all $\theta$ (see Figure 4.4b). Thus, as $\lambda$ is increased through 0, we expect a jump to finite amplitude simple hexagons as the other primary branches of (4.11) are unstable.
Figure 4.5: Floquet multipliers $\mu = \xi e^{i\psi}$ computed from (3.35) for the parameters used in Figure 4.2b and for critical forcing amplitude $f = f_c = 1.552$. (a) Magnitude $\xi$, and (b) phase $\psi$ vs. wave number $k$. The solid lines are computed numerically, while the dotted lines are obtained by considering the unforced problem $f = 0$; see (4.32).
Figure 4.6: Floquet multipliers $\mu = \xi e^{i\psi}$ computed numerically from (3.35) for the parameters used in Figure 4.2b and for critical forcing amplitude $f = f_c = 1.552$. (a) Numerically computed real part $\xi \cos \psi$ of the Floquet multipliers. The “bubbles” correspond to real-valued Floquet multipliers. The boxed region, shown blown up in (b), reveals a tiny “bubble” around $k = 0.383$, with real Floquet multiplier $\mu = 0.93$. 
We find that simple hexagons eventually lose stability as $\lambda$ increases since the following two expressions from Table 2.1 change sign to positive (at least for some $\theta_h$)

\[ \text{sgn}(-\epsilon x + (b_1 - b_2)x^2), \quad \text{sgn}(-\epsilon x + (b_4 + b_5 + b_6 - b_1 - 2b_2)x^2). \]  

(4.33)
as the amplitude $x$ of simple hexagons grows with $\lambda$. The first quantity changes from negative to positive at $\lambda \approx 3.2 \times 10^{-8}$. The second quantity changes sign with increasing $\lambda$ only for those values of $\theta_h$ where $b_4 + b_5 + b_6 - b_1 - 2b_2 > 0$, a condition which is met for $\theta_h \geq 11.5^\circ$. Figure 4.7a shows the value of $\lambda$ where the expressions of (4.33) change sign as a function of $\theta_h$. It follows that simple hexagons lose stability first on the lattice with angle $\theta_h \approx 22.2^\circ$. This instability has an associated eigenvector in the direction of SL-I super hexagon/triangles, and at this value of $\lambda$, super hexagons (or triangles) are stable. These results are summarized in Figure 4.7b, which shows part of the bifurcation diagram computed for the hexagonal lattice with $(n_1, n_2) = (3, 2)$, which corresponds to an angle $\theta_h = 21.8^\circ$. Note that when simple hexagons lose stability, both rhombs (Rh) and an SL-I superlattice pattern are stable. Because the instability that first destabilizes the simple hexagons is in the direction of an SL-I superlattice pattern with $\theta_h \approx 22.2^\circ$, we expect that the transition would be a hysteretic one involving the simple hexagons and a superpattern, at least in the absence of noise and other imperfections. We cannot determine whether the superlattice pattern is hexagonal or triangular from our calculations, since this requires knowledge of fifth order terms in the bifurcation problem [26].
Figure 4.7: (a) $\lambda$ value at which the first (dashed) and second (solid) eigenvalue expressions in (4.33) turn positive versus the lattice angle $\theta_h$. Note that simple hexagons (H) first lose stability to perturbations in the SL-I super hexagon/triangle (SH/ST) direction at $\theta_h \approx 22.2^\circ$. (b) Schematic bifurcation diagram for the $(n_1, n_2) = (3, 2)$ lattice of Figure 2.6, which corresponds to $\theta_h = 21.8^\circ$. Stable (unstable) solutions are indicated by a solid (dotted) line. We do not show secondary branches or primary branches that are never stable. The stable rhombs solution (Rh) corresponds to one with an angle of $81.8^\circ$, which is the rhombs solution closest to $90^\circ$ for this hexagonal lattice. The other two rhombs solutions are unstable.
4.6 Conclusions

In this chapter we have examined the effect of spatio-temporally resonant triads on two-dimensional pattern selection in parametrically excited systems. Using a normal form transformation to enforce temporal symmetry and center manifold reduction, we have argued that weakly damped harmonic modes can strongly influence pattern selection by causing certain cubic cross-coupling coefficients in a 12-dimensional $D_6 + T^2$-equivariant bifurcation problem to suddenly vary in magnitude for certain lattice angles $\theta_h$. This suggests an important consideration in choosing one over another of the countable set of 12-dimensional representations relevant to hexagonal bifurcation problems. Weakly damped subharmonic modes, on the other hand, do not have such an effect.

Our general analysis applies to any parametrically excited pattern forming system, but in particular is relevant to the interpretation of many recent experiments on two-frequency forced Faraday waves. In such experiments, a bicritical point exists where subharmonic and harmonic instabilities are simultaneously excited. On one side of the bicritical point, a subharmonic mode is excited and there is a weakly damped harmonic mode, while on the other, it is the harmonic mode which is excited and the subharmonic mode which is weakly damped. We showed that this weakly damped subharmonic mode does not influence the harmonic wave pattern selection problem.

We have derived the quadratic and cubic coefficients in the rhombic and hexagonal bifurcation equations describing the onset of patterns from the hydrodynamic equations (3.34) of Zhang and Viñals. We presented results for two different sets of parameters. In the first case, the two forcing frequencies are in the ratio 2/3 and the
modes near the bicritical point are the only ones of relevance. As expected from our normal form analysis, for subharmonic waves, the weakly damped harmonic mode affects the cross-coupling coefficients, while for harmonic waves, the weakly damped subharmonic mode had no effect.

In the second case of 6/7 forcing we have shown that, in addition to the modes near the bicritical point, there are other harmonic modes that are important. These modes are not close to onset in the sense that they only become critical at a much higher value of the excitation amplitude, but are weakly damped and thus must be taken into account. We demonstrate that they can have a stabilizing effect on SL-I superlattice patterns at a lattice angle approximately equal to the angle of the harmonic-harmonic resonance. This can occur if the contribution of these weakly damped modes to the nonlinear cross-coupling coefficient nearly cancels the other contributions to this term, and hence is a subtle effect that depends on certain details of the nonlinear problem, as well as the results of the linear analysis which identifies the near critical modes. For the parameters we have chosen, the onset pattern is simple hexagons, but upon a further increase of the forcing, there is an instability to an SL-I superlattice pattern associated with a hexagonal lattice with \((n_1, n_2) = (3, 2)\).

The experiments of Kudrolli, Pier and Gollub [10] found a superlattice pattern near the bicritical point which sits on a lattice with \((n_1, n_2) = (3, 2)\). The work in this chapter suggests that the observation of this pattern could be explained by the interaction of the primary harmonic instability and weakly damped harmonic modes. This issue is pursued further in the next chapter.
Chapter 5

Resonances and superlattice pattern stabilization

5.1 Introduction

In Chapter 4, we examined the role that weakly damped modes may play in Faraday wave pattern selection. A result of our analysis was that a weakly damped harmonic mode may help stabilize an SL-I superlattice pattern. For higher values of the integers \( m \) and \( n \) in the forcing function (3.3), there may be several weakly damped harmonic modes (see, for example, Figure 4.2b). The motivation for this chapter is the following question: which weakly damped harmonic modes are the most important in terms of SL-I superlattice pattern stabilization?

We recall that a defining characteristic of SL-I superlattice patterns is that they contain structure on two length scales. For the case of Faraday waves, one of the length scales is set (approximately) by the dominant frequency component in (3.3).
As we witnessed in Chapter 4, the second length scale in the pattern is not set directly by the other forcing frequency. For instance, a numerical linear stability analysis (see Section 3.5) using the parameters corresponding to the experimental SL-I pattern from [10] in Figure 1.2a indicates that the two critical wave numbers are in a ratio of 1.22 at the bicritical point, but the ratio of the two length scales in the pattern is actually observed to be $\sqrt{7} \approx 2.65$. This observation provides another framing of our motivation for this chapter, namely the identification of a mechanism for the selection of the second length scale in the pattern. The results of this chapter are published in [29].

As in the last chapter, we investigate the role weakly damped modes play in pattern selection. We use symmetry considerations to explain the special importance of particular weakly damped harmonic modes in terms of their contributions to cubic cross-coupling coefficients in the relevant bifurcation equations. Specifically, we find that the most important weakly damped modes are i) the mode oscillating with dominant frequency $m\omega$, which is the temporal harmonic of the Faraday-unstable mode, ii) the “difference frequency mode” oscillating with dominant frequency $|m - n|\omega$, and iii) the “sum frequency mode” oscillating with dominant frequency $(m + n)\omega$. Here, we have assumed without loss of generality that the $\cos(m\omega t)$ component in (3.3) is the dominant one. Additionally, we exclude the forcing frequency ratios $m/n$ with $m + n \leq 5$ which yield stronger resonances. The symmetry arguments are borne out in a weakly nonlinear analysis of the Zhang-Viñals equations (3.34). For weak damping and forcing and one-dimensional surface waves, we perform a perturbation expansion through fourth order which yields analytical expressions for onset parameters and the cubic self-interaction coefficient that determines wave amplitude as a
function of forcing amplitude near onset. This method is in contrast to that of Chapter 4, in which the perturbation expressions could only be evaluated numerically. For stronger damping and forcing we use the methods of Chapter 4 to compute these same parameters numerically as well as the cubic cross-coupling coefficient for competing waves oriented at an angle $\theta$ relative to each other. From the analytical expressions for the one-dimensional case, we are able to quantify the effect of the key resonances and see how their existence depends on the forcing frequency ratio $m/n$. For the two dimensional case, our numerical results show that the resonance effects follow the same scaling laws as in the one-dimensional case. A simple argument, valid for weak damping and forcing and relying only on the inviscid dispersion relation, allows us to predict the spatial angles at which the resonances occur, and to see how their existence depends on $m, n$ and a dimensionless fluid gravity-capillarity parameter. A bifurcation analysis reveals that the difference frequency resonance can help stabilize an SL-I pattern whose large-scale periodicity depends on the wavelength associated with the difference frequency mode. We note that the difference frequency mode has been observed to play an important role in experiments where other complex Faraday wave patterns are formed [13, 14, 15].

This chapter is organized as follows. In Section 5.2 we review basic ideas about resonant triad interactions and their potential for affecting SL-I pattern selection. We then use symmetry arguments to identify which weakly damped modes we expect to be most important in terms of their contributions to the cross-coupling coefficient. Section 5.3 contains the weakly nonlinear analysis of the Zhang-Viñals equations for weak damping and forcing and one-dimensional waves, which leads to approximate formulas for the critical forcing and wave number, and the cubic self-interaction co-
efficient. The perturbation results for onset parameters are discussed in Section 5.4. The expression for the self-interaction coefficient and numerical results for the cross-coupling coefficient are examined in Section 5.5, with special attention given to the role played by resonant triads and the implications for SL-I pattern selection. We summarize our main results in Section 5.6.

5.2 Symmetry analysis

Resonant triads and pattern stability

We begin by again focusing on amplitude equations for a resonant triad of standing waves. Since we will make use of continuous time symmetries, rather than the discrete-time normal form symmetry used in Chapter 4, we now formulate the problem in terms of differential equations. We denote the free surface height \( h(x, t) \) \((x \in \mathbb{R}^2)\) by

\[
h(x, t) = Z_1 e^{i k_1 \cdot x} + Z_2 e^{i k_2 \cdot x} + Z_3 e^{i(k_1+k_2)\cdot x} + c.c. + \ldots .
\] (5.1)

The amplitude equations take the form

\[
\dot{Z}_1 = \Lambda_1 Z_1 + \alpha_1 Z_2 Z_3 + (A|Z_1|^2 + b|Z_2|^2 + C|Z_3|^2)Z_1 \quad (5.2a)
\]
\[
\dot{Z}_2 = \Lambda_1 Z_2 + \alpha_1 Z_1 Z_3 + (A|Z_2|^2 + b|Z_1|^2 + C|Z_3|^2)Z_2 \quad (5.2b)
\]
\[
\dot{Z}_3 = \Lambda_2 Z_3 + \alpha_2 Z_1 Z_2 + (D|Z_1|^2 + D|Z_2|^2 + E|Z_3|^2)Z_3. \quad (5.2c)
\]

All coefficients are real-valued. The argumentation is completely analogous to that of Section 4.2. If the \( Z_3 \) mode is in fact damped (i.e. \( \Lambda_2 < 0 \)) and the \( Z_1 \) and \( Z_2 \) modes
are neutrally stable (i.e. $\Lambda_1 = 0$) then a further center manifold reduction may be performed to the critical $Z_1$ and $Z_2$ modes. Then $Z_3$ satisfies

$$Z_3 = -\frac{\alpha_2}{\Lambda_2} Z_1 Z_2 + \ldots$$ (5.3)

and the (unfolded) bifurcation problem, to cubic order, is

$$\frac{dZ_1}{dT} = \Lambda_1 Z_1 + A|Z_1|^2 Z_1 + B(\theta_r)|Z_2|^2 Z_1$$ (5.4a)

$$\frac{dZ_2}{dT} = \Lambda_1 Z_2 + A|Z_2|^2 Z_2 + B(\theta_r)|Z_1|^2 Z_2$$ (5.4b)

where

$$B(\theta_r) = b - \frac{\alpha_1 \alpha_2}{\Lambda_2}.$$ (5.5)

In [90], symmetry arguments are used to derive a scaling law for the quadratic coefficients $\alpha_1$ and $\alpha_2$ in (5.2a) when the resonant triad applies to the bicritical point, and for the case of weak damping and forcing. For instance, it is shown that for $m$ odd and $n$ even, and for the case that the $\cos(m\omega t)$ forcing frequency component in (3.3) dominates, $\alpha_1$ and $\alpha_2$ are each proportional to $g_m^{\frac{n-2}{2}} g_n^{\frac{m-1}{2}}$, where $g_m$ and $g_n$ are the forcing amplitudes in (3.3). Thus for $m + n \geq 5$, the quadratic terms in (5.2a) are quite small for weak forcing and damping. Their contribution to the pattern selection problem can only be made significant by getting sufficiently close to the bicritical point where $\Lambda_2 \to 0$.

Here we will focus on resonant triads other than those associated with the bicritical point. In particular, we will identify resonant triads for which the quadratic terms scale (at most) linearly with $g$ in (3.3) and for which the scaling is independent of $m$.
and \( n \). Thus away from the bicritical point and for weakly forced waves, we expect these triads to play a more important role in pattern formation if \( m + n > 5 \).

**Determination of important resonances for weak damping and forcing**

Our goal here is to examine resonant triads from a symmetry perspective with a special emphasis on temporal symmetries. Without loss of generality, we assume (unless otherwise specified) that the \( \cos(m \omega t) \) forcing is of greater significance than the \( \cos(n \omega t) \) forcing. For weak damping, the critical Faraday waves oscillate with a dominant frequency component of \( m \omega / 2 \). We also consider weakly damped waves of frequency \( \Omega > 0 \) \( (\Omega \neq m \omega / 2) \), to be determined such that they lead to the largest possible contribution to the cross-coupling coefficient \( B(\theta) \) when slaved away; cf. (5.3) – (5.5).

We follow [90] and focus on travelling waves, on which the action of time-translation is transparent. The travelling wave bifurcation equations are then reduced to those describing the standing wave problem. Specifically, we expand the fluid surface height \( h(\mathbf{x}, t), \mathbf{x} \in \mathbb{R}^2 \), in terms of the following six travelling waves:

\[
\begin{align*}
    z_1 e^{i(k_1 \cdot \mathbf{x} + \frac{1}{2} m \omega t)} &+ w_1 e^{i(k_1 \cdot \mathbf{x} - \frac{1}{2} m \omega t)} + z_2 e^{i(k_3 \cdot \mathbf{x} + \frac{1}{2} m \omega t)} \\
    + w_2 e^{i(k_2 \cdot \mathbf{x} - \frac{1}{2} m \omega t)} &+ z_3 e^{i(k_3 \cdot \mathbf{x} + \Omega t)} + w_3 e^{i(k_3 \cdot \mathbf{x} - \Omega t)} + c.c.
\end{align*}
\]

(5.6)

Here, \( z_j \) and \( w_j, j = 1, 2, 3 \), are the slowly varying amplitudes of the travelling waves. The wave vectors \( k_1 \ldots k_3 \) are assumed to satisfy the spatial resonance condition
(4.1). The frequency $\Omega$ and the wave number $|k_3|$ are related by a dispersion relation. In writing (5.6) we have assumed the problem is posed on an unbounded horizontal domain and then restricted our attention to solutions that are periodic on a rhombic lattice. Spatial translation symmetry acts on $(z_j, w_j), j = 1, 2, 3,$ as

$$T(\Theta_1, \Theta_2): (z_j, w_j) \rightarrow (z_j, w_j)e^{i\Theta_j}, \quad \Theta_3 \equiv \Theta_1 + \Theta_2$$

(5.7)

where $(\Theta_1, \Theta_2) \in T^2$. A rotation by $\pi$, denoted by $R$, acts as

$$R: (z_j, w_j) \rightarrow (\overline{w}_j, \overline{z}_j), \quad j = 1, 2, 3.$$  

(5.8)

and a reflection in the plane containing $k_3$, denoted by $\kappa$, acts as

$$\kappa : (z_1, w_1) \leftrightarrow (z_2, w_2)$$

$$ (z_3, w_3) \rightarrow (z_3, w_3).$$

(5.9)

Furthermore, there is a time-translation symmetry which acts on the forcing parameters in (3.3) and on the complex travelling wave amplitudes in (5.6):

$$T_{\Delta t} : (z_1, z_2, z_3) \rightarrow (z_1 e^{i \frac{3}{2} m \omega \Delta t}, z_2 e^{i \frac{3}{2} m \omega \Delta t}, z_3 e^{i \Omega \Delta t})$$

$$ (w_1, w_2, w_3) \rightarrow (w_1 e^{-i \frac{3}{2} m \omega \Delta t}, w_2 e^{-i \frac{3}{2} m \omega \Delta t}, w_3 e^{-i \Omega \Delta t})$$

$$ (G_m, G_n) \rightarrow (G_m e^{i m \omega \Delta t}, G_n e^{i m \omega \Delta t}).$$

(5.10)

We now determine which quadratic terms will be allowed in the travelling wave amplitude equations, anticipating that these terms will lead to contributions to $B(\theta)$. 
For example, from the spatial translation symmetry (5.7), the only quadratic terms that are allowed in the $\dot{z}_1$ equation are $\bar{z}_2 z_3$, $\bar{z}_2 w_3$, $\bar{w}_2 z_3$ and $\bar{w}_2 w_3$.

We now consider the restrictions placed by the time translation symmetry (5.10) in order to determine which $\Omega$ are allowed. We expect the largest contributions to $B(\theta)$ in (5.4) to occur when the coefficients of quadratic terms are independent of the forcing amplitudes $G_m$ and $G_n$ at leading order, at least for small forcing. In this case, there is only one quadratic term that is permitted, namely

(i) $\bar{w}_2 w_3$ with $\Omega = m\omega$.

The next largest contributions to $B(\theta)$ occur when the coefficients of the quadratic terms in (5.2) are proportional to one power of $G_m$ or $G_n$ at leading order. In this case, the permitted equivariant terms in the $\dot{z}_1$ equation are

(ii) $\overline{G}_m \bar{z}_2 z_3$ with $\Omega = 2m\omega$

(iii) $G_n \bar{z}_2 z_3$ with $\Omega = (m - n)\omega$ if $m > n$

(iv) $G_n \bar{z}_2 w_3$ with $\Omega = (n - m)\omega$ if $n > m$

(v) $\overline{G}_n \bar{z}_2 z_3$ and $G_n \bar{w}_2 w_3$ with $\Omega = n\omega$.

We may immediately dispense with several of these cases. The resonance in case (ii) is not relevant for our investigation of Faraday waves because the weakly damped mode oscillating with frequency $2m\omega$ is at sufficiently high wave number that the spatial resonance condition (4.1) cannot be satisfied for the inviscid dispersion relation. The resonance in case (v) does not result in a contribution to $B(\theta)$ at linear order.
in $G_m$, $G_n$. This may be understood by considering the effects of an approximate time reversal symmetry and an approximate Hamiltonian structure [90, 91] and has been verified by an explicit perturbation calculation similar to those performed in Section 5.3.

We refer to (iiiia) and (iiiib) as cases of “difference frequency resonance.” We now examine the amplitude equations for case (iiiia) (case (iiiib) is analogous) which are determined by the symmetries (5.7) - (5.10). The cubic truncation takes the form

\begin{align}
\dot{z}_1 &= \lambda_1 z_1 + \delta G_m w_1 + G_n \beta_2 z_2 z_3 + r_0 w_1 z_2 \bar{w}_2 \\
&\quad + (r_1 |z_1|^2 + r_2 |z_2|^2 + r_3 |z_3|^2 + r_4 |w_1|^2 + r_5 |w_2|^2 + r_6 |w_3|^2) z_1 \\
\dot{z}_3 &= \lambda_2 z_3 + G_n \beta_2 z_1 z_2 \\
&\quad + (r_7 |z_1|^2 + r_7 |z_2|^2 + r_8 |z_3|^2 + r_9 |w_1|^2 + r_9 |w_2|^2 + r_{10} |w_3|^2) z_3.
\end{align}

Related equations for $\dot{z}_2$, $\dot{w}_1$, $\dot{w}_2$, and $\dot{w}_3$ can be obtained from the discrete spatial symmetries (5.8) and (5.9). The $\delta G_m w_1$ term in (5.11a) is the usual parametric forcing term. We have dropped linear and quadratic terms that scale higher than linearly in $G_m$ and $G_n$. We have also dropped any cubic terms whose coefficients depend on these parameters.

Since the resonant $z_3$ mode is damped ($\lambda_2 < 0$), we may slave it away so that (5.11a) becomes

\begin{align}
\dot{z}_1 &= \lambda_1 z_1 + \delta G_m w_1 + r_0 w_1 z_2 \bar{w}_2 \\
&\quad + \left\{r_1 |z_1|^2 + \left(r_2 - \frac{|G_n|^2 \beta_1 \beta_2}{\lambda_2}\right) |z_2|^2 + r_4 |w_1|^2 + r_5 |w_2|^2\right\} z_1.
\end{align}
and equations related by the discrete spatial symmetries (5.8) and (5.9).

For sufficiently large forcing $|G_m|$, the trivial solution of (5.12) loses stability. A center manifold reduction to standing wave equations of the form (5.4) may be performed at the critical forcing strength. The cross-coupling coefficient $B(\theta)$ in (5.4) then includes a contribution proportional to $|G_n|^2/(\text{Re} \lambda_2)$ that results from slaving the difference frequency mode.

Similar arguments can be made for case iv., in which the resonant mode oscillates at the so-called “sum frequency” $(m+n)\omega$. For this case, too, the slaved mode results in a contribution to $B(\theta)$ that is proportional to $|G_n|^2/(\text{Re} \lambda)$, where $\text{Re} \lambda$ represents the damping of the slaved resonant mode.

Case (i) corresponds to the well-known 1:2 temporal resonance, which is present for single frequency forcing [24]. An analysis similar to that performed above reveals that slaving of the damped mode oscillating with dominant frequency $m\omega$ results in a contribution to $B(\theta)$ which is independent of $G_m$ and $G_n$, and is inversely proportional to $\text{Re} (\lambda)$. Thus we expect that this contribution will be larger than that due to the sum or difference frequency resonances.

For weak damping and forcing, then, we expect the standing wave modes with frequency $m\omega$, $|m-n|\omega$, and $(m+n)\omega$ to be the most important weakly damped modes in terms of their contributions to $B(\theta)$ when $m+n > 5$. Due to the constraints imposed by temporal symmetries, any other resonant modes will necessarily have higher powers of $g$ in front of the quadratic terms in their travelling wave equations, and thus will result in smaller contributions to $B(\theta)$ for weak forcing. The exception is the wave oscillating with dominant frequency $\frac{1}{2}n\omega$. Because this wave is forced directly by the $n\omega$ forcing component, its damping becomes arbitrarily small as the
bicritical point is approached, and its slaving can result in a large contribution to $B(\theta)$ as demonstrated in [28]. Since our analysis assumes that the resonant modes have finite damping (i.e. we are bounded away from the bicritical point), this case is excluded here.

5.3 Perturbation analysis for one-dimensional waves

Using the ideas discussed in the previous section, we now perform a perturbation analysis on the Zhang-Viñals Faraday wave equations (which we introduced in Section 3.4) to obtain quantitative results for Faraday waves in one spatial dimension for $m + n > 5$.

Since there is no spatial angle $\theta$ to tune in one dimension, the only possible resonant triad interaction is the 1:2 interaction which occurs when a standing wave with critical wave number $k$ and its spatial harmonic with wave number $2k$ fulfill one of the temporal resonance conditions from Section 5.2. This situation may be achieved by varying fluid parameters in the dispersion relation. In particular, in our calculations we vary a dimensionless capillarity parameter. Near those special values of the capillarity parameter where a temporal resonance occurs, we expect additional contributions to the cubic self-interaction coefficient $A$ in the standing wave bifurcation equation

$$\frac{dZ_1}{dT} = \Lambda_1 Z_1 + A|Z_1|^2 Z_1$$

which is simply (5.4) restricted to one spatial dimension.
Outline of the calculation

We calculate from the Zhang-Viñals equations (3.34) systems of ODEs for travelling waves in one spatial dimension that are valid for the different cases of spatiotemporal resonance described in Section 5.2. Our perturbation calculations are performed for small amplitude waves in the limit of weak damping and forcing ($\gamma, f_m, f_n \ll 1$).

For our calculations, we focus on counter-propagating travelling waves having critical wave number $k$ (to be determined) which are assumed to be subharmonic to the dominant forcing component $\cos(m\tau)$ and thus, to leading order, have frequency $m/2$. We refer to these waves as the “basic waves.” Furthermore, we insist that $f_n$ not exceed the critical value at which standing waves of dominant frequency $n/2$ bifurcate. Thus, we do not include these waves in our nonlinear calculation, and our nonlinear analysis is restricted to the parameter region

$$0 < f_m \ll 1, \quad 0 < f_n < f_n^{\text{crit.}} \ll 1 \quad (5.14)$$

which is bounded away from the bicritical point.

To facilitate our analysis, we perform four separate calculations, each of which pertains to a different possible case of spatiotemporal resonance. The first case is a nonresonant case, for which we retain only the basic waves in the leading order solution of the perturbation problem. Then we consider cases in which the basic waves are (nearly) temporally resonant with their spatial harmonics having wave number $2k$. Based on the arguments in Section 5.2, the three resonant frequencies we consider are $m$ (1:2 temporal resonance), $|m - n|$ (difference frequency resonance) and $m + n$ (sum frequency resonance). Each resonance may be achieved by choosing
a particular value of the capillarity parameter $\Gamma_0$. For each of these resonant cases, we retain both the basic waves and the resonant waves at leading order. For all cases, resonant terms at subsequent orders in the perturbation calculation lead to solvability conditions, from which we obtain travelling wave amplitude equations, and by a further center manifold reduction, standing wave amplitude equations.

No resonance

We use the following scaling:

$$
\gamma = \epsilon \gamma_1, \quad f_n = \epsilon f^1_n, \quad f_m = \epsilon f^1_m + \epsilon^3 f^3_m + \ldots
$$

$$
k = k_0 + \epsilon^2 k_2 + \ldots, \quad \partial_\tau \rightarrow \partial_\tau + \epsilon \partial_{T_1} + \epsilon^2 \partial_{T_2} + \epsilon^3 \partial_{T_3} + \ldots
$$

$$
h = \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 + \ldots, \quad \Phi = \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \epsilon^3 \Phi_3 + \ldots
$$

where $0 < \epsilon \ll 1$. The fields $h$ and $\Phi$ are functions of the spatial variable $x$, the fast time $\tau$, and the slow times $T_j$.

The expressions for $f_m$ and $k$ indicate expansions of the critical wave number and forcing value. We find that terms proportional to $\epsilon^2$ in $f_m$ and $\epsilon$ in $k$ are not necessary. The wave number, forcing, time derivative, and the two fields are expanded through $\mathcal{O}(\epsilon^3)$ because we carry out the perturbation calculation to $\mathcal{O}(\epsilon^4)$. This higher order calculation is needed since $A$, the cubic coefficient in the standing wave equation (5.13), turns out to be an $\mathcal{O}(\epsilon)$ quantity. (It was shown in [90] that this scaling may be understood by considering the effect of a time-translation symmetry $\tau \rightarrow -\tau$ which is present for undamped waves, and whose influence is still felt for waves of
sufficiently weak damping. This symmetry restricts the form of the Taylor expansion in $\gamma$ of the coefficients in the bifurcation equations.)

At $\mathcal{O}(\epsilon)$, (3.34a) is

$$L_0 h_1 = 0 \quad (5.16)$$

where

$$L_0 \equiv \partial^2_\tau + D(G_0 - \Gamma_0 \partial^2_x). \quad (5.17)$$

Equation (5.16) has an infinite-dimensional solution space consisting of all plane waves $e^{ik_0 x + i\Omega(k_0) \tau}$, where $\Omega$ is the natural frequency given by the dispersion relation (3.36). Thus, $h_1$ should consist of a superposition of these plane waves, i.e.

$$h_1 = \sum_{k_0} z(k_0) e^{ik_0 x + i\Omega(k_0) \tau},$$

where the wave number $k_0$ may be any wave number that fits into our periodic domain. However, at $\mathcal{O}(\epsilon^2)$, all of the amplitudes $z(k_0)$ are damped on the slow time scales, except for the case $\Omega(k_0) = m/2$, $k_0 = 1$. Using this a posteriori justification, we choose $h_1$ to include only those solutions which may grow on the slow time scales. Therefore, $h_1$ consists of one set of counter-propagating waves:

$$h_1 = z_1 e^{ikx + im/2 \tau} + w_1 e^{ikx - im/2 \tau} + \text{c.c.} \quad (5.18)$$

where $z_1$ and $w_1$ are functions of $T_1$, $T_2$, and $T_3$.

At $\mathcal{O}(\epsilon^2)$, $\mathcal{O}(\epsilon^3)$ and $\mathcal{O}(\epsilon^4)$ we apply solvability conditions which yield the respec-
tive equations

\[
\begin{align*}
\frac{\partial z_1}{\partial T_1} &= -\gamma_1 z_1 + i\eta_1 w_1 \\
\frac{\partial z_1}{\partial T_2} &= i\nu_2 z_1 + ic_1|z_1|^2 z_1 + ic_2|w_1|^2 z_1 \\
\frac{\partial z_1}{\partial T_3} &= -\gamma_3 z_1 + i\eta_3 w_1 + c_3|z_1|^2 z_1 + c_4|w_1|^2 z_1 \\
&\quad + ic_5|w_1|^2 z_1 + ic_6|w_1|^2 w_1 + ic_7 z_1^2 w_1
\end{align*}
\]

and similar equations for \( w_1 \) which are related by the spatial reflection symmetry

\[
x \to -x : (z_1, w_1) \to (w_1, z_1).
\]

The coefficients in (5.19) are given in Appendix A.

We reconstitute the time derivative and amplitudes in the travelling wave equations by multiplying (5.19a), (5.19b), and (5.19c) by \(\epsilon^2, \epsilon^3, \) and \(\epsilon^4\) respectively, adding the results, and letting \(\epsilon z_1 \to z_1, \epsilon w_1 \to w_1, \) and \(\epsilon \partial_{T_1} + \epsilon^2 \partial_{T_2} + \epsilon^3 \partial_{T_3} \to \partial_T.\) We obtain

\[
\frac{dz_1}{dT} = (-\hat{\gamma} + i\nu)z_1 + i\eta w_1 + (d_1 + ic_1)|z_1|^2 z_1 + (d_2 + ic_2)|w_1|^2 z_1 \\
&\quad + id_3|z_1|^2 w_1 + id_4|w_1|^2 w_1 + id_5 z_1^2 w_1
\]

and a similar equation for \(\frac{dw_1}{dT}\) related by (5.20). The coefficients in the reconstituted equations are

\[
\hat{\gamma} = \epsilon\gamma_1 + \epsilon^3\gamma_3, \quad \nu = \epsilon^2\nu_2, \quad \eta = \epsilon\eta_1 + \epsilon^3\eta_3, \quad d_1 = \epsilon c_3 \\
d_2 = \epsilon c_4, \quad d_3 = \epsilon c_5, \quad d_4 = \epsilon c_6, \quad d_5 = \epsilon c_7.
\]
(Note that $\epsilon\gamma_1 = \gamma$, etc. so that the $\epsilon$’s drop out of the final equation. Note also that the reconstituted equations are not necessarily equivalent to the order-by-order equations, and that it should be checked whether they contain “extra” solutions due to improper balance of terms.)

Now we solve for $k_2$, the correction to the critical wave number, and $f_m^1$ and $f_m^3$ in (5.15), the forcing amplitudes associated with onset. The condition for neutral stability of the flat state follows from (5.21) and is

$$\eta^2 = \gamma^2 + \nu^2. \quad (5.23)$$

At leading order, $O(\epsilon^2)$, we solve (5.23) for $f_m^1$, to find that

$$f_m^1 = 2m\gamma_1. \quad (5.24)$$

At $O(\epsilon^4)$, we find that

$$f_m^3 = \frac{k_2^2(m^2 + 8\Gamma_0)^2}{16m\gamma_1} + \frac{k_2(f_n^1)^2(m^2 + 8\Gamma_0)}{4m\gamma_1(n^2 - m^2)}$$

$$- \frac{\gamma_1(f_n^1)^2(7n^2m^2 + n^4 - 4m^4)}{2mn^2(n^2 - m^2)^2} + \frac{\gamma_1k_2(7m^2 - 8\Gamma_0)}{4m}$$

$$+ \frac{(f_n^1)^4}{4m\gamma_1(n^2 - m^2)^2} - \frac{9\gamma_1^3}{4m} \quad (5.25)$$

which is minimized at

$$k_2 = -\frac{2\gamma_1^2(7m^2 - 8\Gamma_0)}{(m^2 + 8\Gamma_0)^2} - \frac{2(f_n^1)^2}{(m^2 + 8\Gamma_0)(n^2 - m^2)} \quad (5.26)$$
and so

\begin{equation}
\begin{aligned}
f_m^3 &= -\frac{\gamma_1^2(29m^4 + 16m^2\Gamma_0 + 320\Gamma_0^2)}{2m(m^2 + 8\Gamma_0)^2} \\
\quad &+ \frac{2\gamma_1(f_n^1)^2 m(m^4 + 8m^2\Gamma_0 - 16n^2\Gamma_0 - 2n^4)}{n^2(m^2 + 8\Gamma_0)(n^2 - m^2)^2}.
\end{aligned}
\end{equation}

To reconstitute the expressions for critical forcing and wave number, we recall (5.15) to obtain

\begin{equation}
\begin{aligned}
f_m^c &= 2m\gamma - \frac{\gamma_1^2(29m^4 + 16m^2\Gamma_0 + 320\Gamma_0^2)}{2m(m^2 + 8\Gamma_0)^2} \\
\quad &+ \frac{2\gamma(f_n)^2 m(m^4 + 8m^2\Gamma_0 - 16n^2\Gamma_0 - 2n^4)}{n^2(m^2 + 8\Gamma_0)(n^2 - m^2)^2} \\
k &= 1 - \frac{2\gamma_2(7m^2 - 8\Gamma_0)}{(m^2 + 8\Gamma_0)^2} - \frac{2(f_n)^2}{(m^2 + 8\Gamma_0)(n^2 - m^2)^2}.
\end{aligned}
\end{equation}

The superscript \(c\) indicates that \(f_m\) has been set to its critical value.

Due to the restrictions (5.14) which we placed on the forcing amplitudes, the calculation performed above gives us information only about one side of the linear stability boundary. In order to obtain expressions for the entire linear stability boundary in \(f_m - f_n\) space, we perform a similar linear calculation for the case that the dominant
forcing component is $\cos(n\tau)$. The critical forcing and wave number in this case are

$$f_n^c = 2\gamma nk_n$$

$$k_n = k_n + \frac{(f_m)^2k_n^2}{2(G_0 + 3k_n^2\Gamma_0)(n^2 - m^2)} + \frac{\gamma^2k_n^3(G_0k_n + 3k_n^2\Gamma_0 - 2n^2)}{2(6G_0\Gamma_0k_n^2 + G_0^2 + 9\Gamma_0^2k_n^4)}.$$  

Here $k_n$ satisfies

$$\Omega^2(k_n) = \left(\frac{n}{2}\right)^2,$$

where $\Omega(k)$ represents the natural frequency given by the dispersion relation (3.36). These linear results are discussed in Section 5.4.

We now return to the case that the $\cos(m\tau)$ forcing dominates and continue our calculation in order to determine the cubic coefficient in the standing wave equation (5.13). We reduce (5.21) to a bifurcation equation for the amplitude $Z_1$ of the critical mode to obtain the standing wave amplitude equation:

$$\frac{dZ_1}{dT} = \Lambda Z_1 + A_{nonres}|Z_1|^2Z_1.$$  

The cubic coefficient $A_{nonres}$ is calculated through its leading order, namely $O(\epsilon)$. We find

$$A_{nonres} = 2\epsilon \left( c_3 + c_4 - c_5 - c_6 + c_7 - \frac{\nu_2(c_1 + c_2)}{\gamma_1} \right).$$
We substitute for $\nu_2$ and $c_1 \ldots c_7$ in (5.34) and reconstitute to obtain

$$A_{\text{nonres}} = \frac{-3\gamma(5m^2 + 2\Gamma_0)}{2m^2} + \frac{181\gamma m^2}{10(m^2 + 8\Gamma_0)} - \frac{28\gamma m^2}{m^2 + 12\Gamma_0} + \frac{37\gamma m^2}{5(m^2 - 12\Gamma_0)} - \frac{16\gamma m^4}{(m^2 - 12\Gamma_0)^2}. \quad (5.35)$$

Note that $A_{\text{nonres}}$ diverges as $\Gamma_0 \to m^2/12$. This divergence reflects the fact that the second spatial harmonic of the critical mode is resonantly excited when $\Gamma_0 = m^2/12$. We perform the necessary calculation for this case next.

### 1:2 spatiotemporal resonance

Now we perform a calculation to handle the case of resonance involving the temporal harmonic; this resonance occurs when the spatial harmonic of the basic waves oscillates with frequency $m$. The condition is

$$\Omega^2(2k_0) = m^2 \quad (5.36)$$

where $\Omega$ is given by the dispersion relation (3.36). Solving (5.36) for $\Gamma_0$, we see that the 1:2 spatiotemporal resonance occurs for $\Gamma_0 = \Gamma_{1.2} = m^2/12$, which is the value of $\Gamma_0$ at which the nonresonant calculation diverges.

The analysis here is similar to that of the nonresonant case, except that we now include the resonant mode in our calculation. Thus,

$$h_1 = z_1 e^{ikx + im \tau} + w_1 e^{ikx - im \tau} + z_3 e^{2ikx + im \tau} + w_3 e^{2ikx - im \tau} + \text{c.c.}. \quad (5.37)$$
Additionally, since we are interested only in the parameter region near the resonance, we expand around the resonant value of the capillarity number:

\[ \Gamma_0 = \Gamma_{1:2} + \epsilon \hat{\Gamma}_{1:2}. \] (5.38)

A solvability condition at \( \mathcal{O}(\epsilon^2) \) yields

\[ \frac{\partial z_1}{\partial T_1} = -\gamma_1 z_1 + i\eta_1 w_1 + ie_1 \bar{z}_1 z_3 \] (5.39a)

\[ \frac{\partial z_3}{\partial T_1} = (-\gamma_4 + i\nu_4) z_3 + ie_2 z_1^2 \] (5.39b)

and equations related by a spatial reflection symmetry similar to (5.20). The coefficients are given in Appendix A.

The leading order term in the standing wave cubic coefficient \( A_{1:2} \) depends only on terms in (5.39) and thus may be determined without carrying the perturbation calculation any further. A reduction of (5.39) to the standing wave equation (5.13) reveals that the leading term in \( A_{1:2} \) is an \( \mathcal{O}(\epsilon^{-1}) \) quantity given by

\[ A_{1:2} = -\frac{2e_1 e_2 \gamma_4}{\epsilon (\gamma_4^2 + \nu_4^2)}. \] (5.40)

We substitute for the coefficients to obtain

\[ A_{1:2} = -\frac{\gamma m^4}{9(\Gamma_0 - \Gamma_{1:2})^2 + 16\gamma^2 m^2} \] (5.41)

which is valid for \( \Gamma_0 \) sufficiently close to \( \Gamma_{1:2} \).
Difference frequency resonance

Now we perform a calculation to handle the case of resonance involving the difference frequency mode, which occurs when the spatial harmonic of the basic waves oscillates with frequency $|m - n|$. This condition may be written as

$$
\Omega^2(2k_0) = (m - n)^2
$$

(5.42)

where $\Omega$ is given by the dispersion relation (3.36). Solving (5.42) for $\Gamma_0$, we see that the difference frequency resonance occurs for

$$
\Gamma_0 = \Gamma_{diff} = \frac{1}{6}n^2 - \frac{1}{3}nm + \frac{1}{12}m^2.
$$

(5.43)

The calculation is similar to the previous one. We let

$$
\Gamma_0 = \Gamma_{diff} + \epsilon \tilde{\Gamma}_{diff}.
$$

(5.44)

Now, $h_1$ is given by

$$
h_1 = z_1 e^{ikx + i\frac{m}{2} \tau} + w_1 e^{ikx - i\frac{m}{2} \tau} + z_3 e^{2ikx + i(m-n)\tau} + w_3 e^{2ikx - i(m-n)\tau} + c.c.
$$

(5.45)
The solvability conditions at $O(\epsilon^2)$ and $O(\epsilon^3)$ yield

\[
\frac{\partial z_1}{\partial T_1} = -\gamma_1 + z_1 + i\eta_1 w_1 \tag{5.46a}
\]

\[
\frac{\partial z_3}{\partial T_1} = (-\gamma_4 + i\tilde{\nu}_4)z_3 \tag{5.46b}
\]

\[
\frac{\partial z_1}{\partial T_2} = iv_2z_1 + ir_1\bar{z}_1z_3 + ic_1|z_1|^2z_1 + ic_2|w_1|^2z_1 \tag{5.46c}
\]

\[
+ ic_8z_3 + ic_9|w_3|^2z_1
\]

\[
\frac{\partial z_3}{\partial T_2} = iv_5z_3 + ir_2\bar{z}_1z_3 + ic_{10}|z_1|^2z_3 + ic_{11}|w_1|^2z_3 \tag{5.46d}
\]

\[
+ ic_{12}|z_3|^2z_3 + ic_{13}|w_3|^2z_3.
\]

and equations for $w_1$ and $w_3$ related by a spatial reflection symmetry. The coefficients are given in Appendix A. The values for $\nu_2$, $c_1$, and $c_2$ are given by (A.2), (A.3), and (A.4) evaluated at $\Gamma_0 = \Gamma_{diff}$.

It is not necessary to carry the perturbation calculation further to determine the standing wave cubic coefficient $A_{diff}$ at leading order. The coefficient $A_{diff}$ has two types of contributions. One type is unrelated to the slaved difference frequency mode and is equal to $A_{nonres}$ evaluated at $\Gamma_0 = \Gamma_{diff}$. The other type is due to the quadratic terms in (5.46) and results from the slaving of the damped difference frequency mode. We find that

\[
A_{diff} = A_{nonres}(\Gamma_0 = \Gamma_{diff}) + \hat{A}_{diff} \tag{5.47}
\]

where

\[
\hat{A}_{diff} = \frac{2\epsilon\gamma_4r_1r_2}{\gamma_4^2 + \tilde{\nu}_4^2}. \tag{5.48}
\]
We substitute for the coefficients to obtain

\[ \hat{A}_{\text{diff}} = \frac{m \gamma (f_n)^2 (m^2 - 4mn + 2n^2)^2}{n^2 [16 \gamma^2 (n - m)^2 + 9(\Gamma_0 - \Gamma_{\text{diff}})^2] (n - m)(n - 2m)^2}. \] (5.49)

**Sum frequency resonance**

The condition for the sum frequency resonance is

\[ \Omega^2(2k_0) = (m + n)^2 \] (5.50)

where \( \Omega \) is given by the dispersion relation (3.36). Solving (5.50) for \( \Gamma_0 \), we see that the sum frequency resonance occurs for

\[ \Gamma_0 = \Gamma_{\text{sum}} = \frac{1}{6} n^2 + \frac{1}{3} nm + \frac{1}{12} m^2. \] (5.51)

The calculation is almost identical to that of the previous section, and the result may be obtained by letting \( n \to -n \) (and thus \( \Gamma_{\text{diff}} \to \Gamma_{\text{sum}} \)) in (5.49) to find \( \hat{A}_{\text{sum}} \). We find

\[ A_{\text{sum}} = A_{\text{nonres}}(\Gamma_0 = \Gamma_{\text{sum}}) + \hat{A}_{\text{sum}}. \] (5.52)

where

\[ \hat{A}_{\text{sum}} = \frac{-m \gamma (f_n)^2 (m^2 + 4mn + 2n^2)^2}{n^2 [16 \gamma^2 (n + m)^2 + 9(\Gamma_0 - \Gamma_{\text{sum}})^2] (n + m)(n + 2m)^2}. \] (5.53)
5.4 Linear results

We now discuss results that apply to the linear stability of the trivial solution of the Faraday wave problem, i.e. the flat interface state. Figures 5.1 and 5.2 contain sample results for the case $m = 4$, $n = 9$, $\Gamma_0 = 2$, and various values of the damping parameter $\gamma$. The data are computed both numerically and from the analytical expressions (5.28) – (5.31). Figure 5.1 shows the linear stability boundary in $f_m - f_n$ space. Figure 5.2 shows the critical wave number as a function of the quantity $\chi$. Note that increasing $\chi$ corresponds to marching counterclockwise around the linear stability boundary of Figure 5.1.

The expressions for critical acceleration and wave number were derived in Section 5.3 by performing a perturbation expansion on the Zhang-Viñals equations (3.34) for small amplitude waves and weak damping and forcing. For arbitrary damping and forcing, the linearization of (3.34) is (3.35), a damped Mathieu equation for each Fourier mode $p_k(\tau)e^{ikx}$, as discussed in Section 3.4. Here we focus on (5.28) and (5.29) which apply when the bifurcation is due to the $\cos(m\tau)$ forcing. This bifurcation corresponds to crossing through the right side of the linear stability region. Similar statements hold for crossing through the top of the linear stability region, when the bifurcation is due to the $\cos(n\tau)$ forcing, in which case (5.30) - (5.31) are the relevant quantities.

At leading order, the critical forcing (5.28) is proportional to the damping $\gamma$. There are two correction terms. One correction term is proportional to $\gamma^3$ and is independent of $f_n$. This term always has an overall negative sign and hence lowers $f_m$. The other correction term is proportional to $\gamma(f_n)^2$ and is due to the second
Figure 5.1: Linear stability boundary in $f_m - f_n$ space, the parameter space of the two acceleration amplitudes in (3.29). For a given value of the damping parameter $\gamma$, the flat interface state is unstable above and to the right of the corresponding curve. Dotted lines are numerical data; solid lines correspond to the analytical expressions (5.28) and (5.30). The two are distinguishable on this graph only for $\gamma = 0.2$. The other parameters are $m = 4$, $n = 9$, $\phi = 0$ in (3.29), and $\Gamma_0 = 2$ in (3.34).
Figure 5.2: Critical wave number $k_c$ as a function of $\chi$, shown here for $\chi < \chi_{bc}$. The critical wave number decreases as the bicritical point is approached. Dotted lines are numerical data; solid lines correspond to the analytical expression (5.29). The parameters used are the same as those in Figure 5.1.
forcing component. The overall sign of this term is determined by the quantity

\[ s \equiv m^4 + 8m^2\Gamma_0 - 16n^2\Gamma_0 - 2n^4. \]  

(5.54)

If \( s < 0 \), the \( \gamma(f_n)^2 \) term has an overall negative sign, and thus the second forcing component \( \cos(n\tau) \) is destabilizing; that is to say, it pushes the bifurcation to occur at a smaller value of \( f_m \). However, if \( s > 0 \), the second forcing component actually stabilizes the flat fluid surface beyond those values of \( f_m \) where it would have otherwise gone unstable.

By analyzing the expression for \( s \), remembering that \( \Gamma_0 \) is restricted to the interval \( 0 < \Gamma_0 < m^2/4 \), we see that there are three possible cases:

1. If \( m/n < \sqrt{2} \approx 1.919 \), the second frequency component is destabilizing for all values of \( \Gamma_0 \).

2. If \( \sqrt{2} < m/n < \sqrt{2/3 + \frac{1}{3}\sqrt{10}} \approx 1.31 \), the second frequency component is stabilizing for \( \Gamma_0 < \Gamma_c = \frac{m^4-2n^4}{16n^2-8m^2} \).

3. If \( m/n > \sqrt{2/3 + \frac{1}{3}\sqrt{10}} \), the second frequency component is stabilizing for all values of \( \Gamma_0 \).

In short, the secondary forcing component is stabilizing if it is at sufficiently low frequency compared to the dominant forcing component. (However, for weak damping and forcing, the effect of the \( \gamma f_n^2 \) term is quite small.)

The bicritical point \( \chi_{bc} \) may be determined from (5.28) and (5.30), the expressions
for critical forcing. To leading order, it is given by the simple expression

$$\chi_{bc} = \arctan \left( \frac{nk_n}{m} \right)$$

(5.55)

where $k_n$ is determined by the dispersion relation (5.32). Note that to leading order, $\chi_{bc}$ depends on $m$, $n$, and the capillarity number $\Gamma_0$, and is independent of damping and forcing. Using the bounds on $k_n$ that are set by the dispersion relation, we see that for a given ratio $m/n$, $\chi_{bc}$ takes on extreme values of

$$\chi_{bc}^1 = \arctan \left[ \left( \frac{n}{m} \right)^{3/2} \right] \text{ at } \Gamma_0 = 0 \quad \text{(pure gravity waves)}$$

(5.56a)

$$\chi_{bc}^2 = \arctan \left[ \left( \frac{n}{m} \right)^{5/3} \right] \text{ at } \Gamma_0 = m^2/4 \quad \text{(pure capillary waves).}$$

(5.56b)

For $m < n$, $\chi_{bc}^1$ is a maximum and $\chi_{bc}^2$ is a minimum; the reverse is true for $m > n$. As $\Gamma_0$ is changed, $\chi_{bc}$ varies smoothly and monotonically between the two extrema. Examples are shown in Figure 5.3 for $m = 4$ and various values of $n$.

The critical wave number, to leading order, is 1. This is simply the dimensionless wave number determined by the dispersion relation $\Omega^2(k) = (m/2)^2$, where $\Omega(k)$ is given by (3.36). There are two correction terms. One is proportional to $\gamma^2$ and has an overall negative sign. The other is proportional to $f_{\eta}^2$. The overall sign of this term is given by the sign of $m - n$. Therefore, the presence of the second forcing component shifts the wave number in such a way as to “repel” it from the other instability associated with the bicritical point. This effect was observed in the experiments in [13] and can also be seen in Figure 5.2, which shows the critical wave number for $\chi < \chi_{bc}$. 
Figure 5.3: Bicritical point $\chi_{bc}$ (in degrees) versus capillarity parameter $\Gamma_0$ for $m = 4$. Lines correspond to the expression in (5.55). Symbols correspond to a numerical computation with damping $\gamma = 0.1$ ("o") and $\gamma = 0.4$ ("x").
Figure 5.4: Magnitudes of the nine most significant fast-time frequency components in a neutrally stable Faraday mode near the bicritical point. The vertical axis (note log scale) shows the magnitude $|a_j|$ of the frequency component $e^{ij\tau}$, normalized so that the largest component has magnitude one. The horizontal axis shows the Fourier index $j$. The components have been arranged in decreasing order of their magnitude. Squares correspond to data from a numerical computation. Circles follow from the perturbation analysis in Section 5.3. The component $j = 2$ is captured at leading order, the components $j = 7, 11, 6$ at second order, and the components $j = 3, 16, 15, 20, 10$ at third order. The parameters used are $m = 4, n = 9, f_n = 3.61$, and $\phi = 0$ in (3.29), and $\gamma = 0.1$ and $\Gamma_0 = 2$ in (3.34). These data are for the critical mode, with wave number $k$; the spatially resonant mode with wave number $2k$ will be dominated by a different frequency component determined by the dispersion relation (3.36) (e.g. $|m - n|$ if $\Gamma_0 = \Gamma_{diff}$).
Finally, we discuss the fast-time dependence of the Faraday-unstable mode, which we write as $p(\tau)$. As demonstrated in [79], $p(\tau)$ will be harmonic or subharmonic relative to the frequency of the forcing function (3.29). As discussed in Section 3.5, previous work has depended on a numerically determined (truncated) Fourier series at some point in the linear or nonlinear analysis. For instance, in [28, 78, 79, 27] the time dependence of the critical mode is written as

$$p(\tau) = \sum_{j=-N}^{N} a_j e^{ij\tau}$$

for the harmonic case. Then, the coefficients $a_j$ are determined numerically. This method assumes no a priori information about the relative importance of the frequency components kept in the expansion.

Our analysis determines the relative importance of the frequency components in $p(\tau)$ for arbitrary $m$ and $n$ in (3.29). For our perturbation expansion in Section 5.3 we assumed that at leading order, the Faraday waves have frequency $\frac{1}{2}m$. At second order in the expansion we captured the frequency components $|n - \frac{1}{2}m|$, $n + \frac{1}{2}m$, and $\frac{3}{2}m$. At third order, we captured the components $|n - \frac{3}{2}m|$, $|2n - \frac{1}{2}m|$, $n + \frac{3}{2}m$, $2n + \frac{1}{2}m$, and $\frac{5}{2}m$.

These results are consistent with what we find numerically. Figure 5.4 shows the nine temporal Fourier coefficients $|a_j|$ that are largest, arranged in decreasing order of their magnitude. Note that even near the bicritical point, where this data was obtained, the $\frac{1}{2}m$ frequency component is order $1/\gamma$ larger than any of the other components.
5.5 Nonlinear results

One spatial dimension

In this section we discuss the nonlinear results of Section 5.3 for the cubic coefficient $A$ in (5.13). We have checked our perturbation results with numerical computations. Figure 5.5 shows a sample result of $A$ versus the capillarity parameter $\Gamma_0$ for $m/n = 4/9$. The solid line corresponds to an expression which matches $A_{\text{nonres}}$ in (5.35) to $A_{\text{diff}}$ in (5.47) and thus is valid for all values of $\Gamma_0$ away from the 1:2 resonance. This expression diverges at $\Gamma_0 = \Gamma_{1:2} = m^2/12$ as discussed in section 5.3. The dotted line corresponds to the expression $A_{1:2}$ in (5.41). Additionally, we have calculated the relative error in the perturbation results for $A$ as a function of the damping $\gamma$.

For instance, for $m/n = 4/9$ and $\chi = 75^\circ < \chi_{bc}$, as $\gamma$ is varied from 0.05 to 0.25, the relative error in $A(\Gamma_0 = \Gamma_{\text{diff}})$ increases from 0.001 to 0.25, and the relative error in $A(\Gamma_0 = \Gamma_{1:2})$ increases from 0.05 to 0.43.

$A_{\text{nonres}}$, the value of the cubic coefficient away from the 1:2, difference, and sum frequency resonances, was computed in Section 5.3 and is given by (5.35). $A_{\text{nonres}}$ is proportional to the damping parameter $\gamma$. Furthermore, $A_{\text{nonres}}$ is always negative indicating that in the nonresonant regime, the bifurcation from the flat state is always supercritical.

$A_{1:2}$, the value of the cubic coefficient near the 1:2 temporal resonance, is given by (5.41). This quantity was derived for $|\Gamma_0 - \Gamma_{1:2}| \simeq \mathcal{O}(\gamma)$. Thus, in the region of validity, $A_{1:2} \simeq \mathcal{O}(\gamma^{-1})$. This is significantly larger in magnitude than $A_{\text{nonres}}$, which is $\mathcal{O}(\gamma)$. Furthermore, $A_{1:2}$ is negative, again indicating a supercritical bifurcation.
Figure 5.5: Cubic coefficient $A$ in (5.13) as a function of the capillarity parameter $\Gamma_0$. The dots correspond to a numerical computation. The dotted line corresponds to the expression for $A_{1:2}$ in (5.41). The solid line corresponds to an expression which asymptotically matches $A_{\text{nonres}}$ and $A_{\text{diff}}$ (details not given). The large dip at $\Gamma_0 \approx \Gamma_{1:2} = 4/3$ is due to the 1:2 resonance discussed in Section 5.2. The small bump around $\Gamma_0 \approx \Gamma_{\text{diff}} = 17/6$, at which the one-dimensional waves have their largest amplitude, is due to the difference frequency resonance, also discussed in Section 5.2. The parameters used are $m = 4$, $n = 9$, $\chi = 75^\circ$ and $\phi = 0$ in (3.29), and $\gamma = 0.05$ in (3.34).
This large negative contribution is manifest as the large dip around \( \Gamma_0 \approx \Gamma_{1:2} = \frac{4}{3} \) in Figure 5.5. \( A_{1:2} \) has a global minimum at \((\Gamma_0, A_{1:2}) = (\Gamma_{1:2}, -\frac{m^2}{16\gamma})\), so that exactly at the 1:2 resonance, the value of the coefficient \( A \) is inversely proportional to the damping, as predicted by the symmetry arguments of Section 5.2. Thus, near the 1:2 resonance, one-dimensional waves will decrease significantly in amplitude.

\( A_{\text{diff}} \), the value of the cubic coefficient near the difference frequency resonance, is given by (5.47). The condition for difference frequency resonance is \( \Gamma_0 = \Gamma_{\text{diff}} \), where \( \Gamma_{\text{diff}} \) is given by (5.43). Since \( \Gamma_0 \) is restricted to the range \([0, m^2/4]\), this condition can only be met for certain \( m/n \) ratios. Specifically, \( \Gamma_{\text{diff}} \in [0, m^2/4] \) only for

\[
m/n \in M_1 \cup M_2, \quad M_1 = [\sqrt{2} - 1, 2 - \sqrt{2}], \quad M_2 = [2 + \sqrt{2}, \infty). \tag{5.58}
\]

Thus, while the 1:2 resonance was relevant for all possible forcing frequency ratios \( m/n \), this is not the case for the difference frequency resonance. The difference frequency resonance results in a contribution to \( A \), namely \( \hat{A}_{\text{diff}} \) given by (5.49), and thus \( A_{\text{diff}} \) has a local extremum at \( \Gamma_0 = \Gamma_{\text{diff}} \). The sign of \( \hat{A}_{\text{diff}} \) is given by the sign of \( n - m \). If the secondary forcing component is at a higher frequency than the primary, \( i.e. \) if \( m/n \in M_1 \), then the difference frequency resonance results in a positive contribution to \( A \). The extremum is a local maximum, and the amplitude of the supercritical waves increases as the resonance is approached. This is demonstrated by the small bump around \( \Gamma_0 \approx \Gamma_{\text{diff}} = 17/6 \) in Figure 5.5. If \( m/n \in M_2 \), then the contribution is negative. The extremum is a local minimum, and the amplitude of the waves decreases. In either case, the extra contribution to \( A(\Gamma_0 = \Gamma_{\text{diff}}) \) that is due to the difference frequency resonance is proportional to \( (f_n)^2/\gamma \) as predicted by
the symmetry arguments of Section 5.2, and thus is a significantly smaller effect than the 1:2 resonance.

For the case that \( m/n \in M_1 \), when \( A \) has a local maximum, it is possible for this maximum to actually cross the \( A = 0 \) axis and become positive, thus causing the bifurcation to become subcritical. An example of a subcritical bifurcation may be obtained with the parameters \( m/n = 49/100 \), \( \Gamma_0 = \Gamma_{\text{diff}} = 2801/12 \approx 233.4 \), \( \gamma_1 = 0.01 \) and \( f_n = 3.9 < 3.94 \approx f_n^c \), in which case \( A = 0.57 > 0 \).

Now we turn to the results for the sum frequency resonance. \( A_{\text{sum}} \) is given by (5.52). The condition for sum frequency resonance is \( \Gamma_0 = \Gamma_{\text{sum}} \), where \( \Gamma_{\text{sum}} \) is given by (5.51). Similar to the difference frequency resonance case, this condition will only be met for certain \( m/n \) ratios. Specifically, the sum frequency mode resonance is possible only for

\[
\frac{m}{n} \geq \sqrt{2} + 1.
\] (5.59)

Thus, the sum frequency resonance can only be realized when the second forcing component is at sufficiently low frequency. The sum frequency resonance results in a contribution to \( A \), namely \( \hat{A}_{\text{sum}} \), which is given by (5.53). \( A_{\text{sum}} \) has a local extremum at \( \Gamma_0 = \Gamma_{\text{sum}} \). Like the difference frequency case, the contribution to \( A \) due to the sum frequency resonance is proportional to \( (f_n)^2/\gamma \). Unlike the difference frequency case, the \( \hat{A}_{\text{sum}} \) contribution always has a negative sign. However, this contribution is generally not significant because the algebraic prefactor in \( \hat{A}_{\text{sum}} \) is small for typical values of \( m/n \) for which the sum frequency resonance is possible.

A partial summary of the results for 1-d resonances may be found in Figure 5.6. This number line shows the regions of forcing frequency ratio \( m/n \) in which each
Figure 5.6: Regions of forcing frequency ratio $m/n$ in which the 1:2, difference, and sum frequency resonances are possible for one-dimensional waves. The plus (+) and minus (−) signs indicate whether the resonance results in a positive or negative contribution to the cubic coefficient $A$ in (5.13). In the case of (−) the bifurcation to one-dimensional waves is necessarily supercritical. Note that only points corresponding to rational numbers on the number line are meaningful.

type of resonance is possible. The plus (+) and minus (−) signs indicate whether the resonance results in a positive or negative contribution to the coefficient $A$, and hence whether it makes the supercritical waves larger (+) or smaller (−) in amplitude.

Two spatial dimensions

We now present nonlinear results for Faraday waves in two spatial dimensions. We have computed the cross-coupling coefficient $B(θ)$ in (5.4) using the method in Chapter 4. We interpret features of $B(θ)$ in light of the resonances discussed in Section 5.2. Many of these features may be understood by means of a simple argument which is valid for weak damping and forcing. We simply solve the spatial resonance condition (4.1) for $θ_{1:2}$, $θ_{\text{sum}}$ or $θ_{\text{diff}}$, which are the angles at which the 1:2, sum frequency, and difference frequency resonances occur. To do this, we must set $|k_3| = k(Ω)$ where $k(Ω)$ is the inverse of the dispersion relation (3.36) and $Ω = m$, $m + n$ or $|m − n|$ depending on the resonance under consideration. A number of results immediately follow:

- The 1:2 resonance is possible only for $Γ_0 ≥ m^2/12 = Γ_{1:2}$. 
• The difference frequency resonance is possible only for

\[ m - \frac{1}{2}\sqrt{2m^2 + 24\Gamma_0} \leq n \leq m + \frac{1}{2}\sqrt{2m^2 + 24\Gamma_0}. \]  \hspace{1cm} (5.60)

• The sum frequency resonance is possible only for \( n \leq -m + \frac{1}{2}\sqrt{2m^2 + 24\Gamma_0} \).

From these statements, we also see that

• The ranges of \( \theta_{1:2}, \theta_{\text{sum}} \) and \( \theta_{\text{diff}} \) are restricted.

• There are some forcing frequency ratios \( m/n \) for which the sum and difference frequency resonances are not possible for any value \( \Gamma_0 \).

An example is given in Figure 5.7, which shows the cross-coupling coefficient \( B(\theta) \) computed for forcing frequency ratios \( m/n = 8/9, 8/11 \) and \( 8/21 \) for fixed fluid parameters; \( \chi \) is chosen in each case to obtain a harmonic instability near the bicritical point. The large dip at \( \theta = \theta_{1:2} \approx 70^\circ \) is a consequence of the 1:2 resonance. At this angle, there is a resonant triad comprised of two Faraday-unstable modes with dominant frequency \( m/2 \) and the weakly damped mode oscillating primarily with the harmonic frequency \( m \). As expected from the analysis of Section 5.2, near this angle, the weakly damped mode contributes to \( B(\theta) \), which here is manifest as the large dip. This phenomenon is similar to the 1:2 resonance in one spatial dimension, which resulted in a large dip in the cubic self-interaction coefficient \( \Lambda \) in (5.13). It follows from the dispersion relation that \( \theta_{1:2} \) will depend on \( m \) and \( \Gamma_0 \) but will be largely independent of the parameters \( n, f_n, \) and \( \gamma \). The independence with respect to \( n \) is evident in Figure 5.7, in which the dip occurs at the same angle for \( m/n = 8/9, 8/11, \) and \( 8/21 \).
Figure 5.7: Cross-coupling coefficient $B(\theta)$ in (5.4). The solid line corresponds to $m/n = 8/9$ in (3.29); the dotted and dashed lines correspond to $m/n = 8/11$ and $8/21$ respectively. For each curve, the parameter $\chi$ is chosen to obtain a harmonic instability near the bicritical point. The other parameters are $\phi = 0$ in (3.29), and $\Gamma_0 = 14$ and $\gamma = 0.1$ in (3.34). The large dip at $\theta = \theta_{1:2} \approx 70^\circ$ is due to the 1:2 temporal resonance discussed in Section 5.2 and its position is independent of the second forcing component. The small spike is due to the difference frequency resonance discussed in Section 5.2. We have removed from this plot the region near $\theta = 60^\circ$ where $B(\theta)$ necessarily diverges; a calculation for the hexagonal lattice is required here.

For all numerical calculations that we performed, the 1:2 resonance resulted in a large negative contribution to the cross-coupling coefficient $B(\theta)$. As discussed in
Section 4.3, this type of contribution is destabilizing for superlattice patterns with characteristic angles $\theta_h$ near $\theta_{1:2}$. Our numerical results indicate that the magnitude of the dip caused by the 1:2 resonance follows the scaling law that we deduced from symmetry considerations in Section 5.2, and that we derived for one-dimensional waves: namely, that the contribution from the weakly damped mode scales like $1/\gamma$; see Figure 5.8.

The sum frequency resonance angle $\theta_{sum}$ may also be predicted by the weak damping argument. However, unlike the 1:2 resonance described above and the difference frequency resonance described below, the sum frequency resonance for two dimensional waves is quite difficult to detect numerically for typical values of $m/n$ and for small $\gamma$. This is consistent with the result for one spatial dimension, and consistent with the fact that the mode oscillating at the sum frequency has a larger wave number and thus is more strongly damped.

Finally, we turn to results for the difference frequency resonance. The effect of the difference frequency resonance may be seen in Figure 5.7, and is manifest as a spike in the plot of $B(\theta)$. Let us first concentrate on the solid curve in Figure 5.7, which corresponds to a forcing frequency ratio of $m/n = 8/9$. For this case, at $\theta = \theta_{diff} \approx 17^\circ$ there is a resonant triad composed of two modes with dominant frequency $m/2$ and the weakly damped mode oscillating with dominant frequency $|n - m|$. As expected from the analysis of Section 5.2, near this angle, the weakly damped mode contributes to $B(\theta)$, which causes the spike. This phenomenon is similar to the difference frequency resonance in one spatial dimension, which resulted in a contribution to the cubic self-interaction coefficient $A$. 
Figure 5.8: $B(\theta_{1:2})$, the value of the cubic cross-coupling coefficient at the angle of 1:2 resonance, versus $1/\gamma$, the reciprocal of the damping parameter. Results are shown for $m = 8$ and $m = 10$ in (3.29). Best-fit lines are also shown. The other parameters are $n = 9$ and $\phi = 0$ in (3.29), and $\Gamma_0 = 14$ in (3.34). The parameter $\chi$ is chosen to obtain a harmonic instability near the bicritical point $\chi_{bc}$. As expected based on the symmetry arguments of section 5.2, $B(\theta_{1:2})$ scales as $1/\gamma$. 


As with the case of 1:2 resonance, the simple argument we have used to predict the resonance angle $\theta_{\text{diff}}$ relies only on the dispersion relation. We thus expect that $\theta_{\text{diff}}$ will depend on $m$, $n$, and $\Gamma_0$ but will be largely independent of the parameters $f_n$ and $\gamma$. The dependence on the second forcing frequency $n$ is evident in Figure 5.7, in which shifting from $n = 9$ to $n = 11$ causes the spike to shift from $\theta_{\text{diff}} \approx 17^\circ$ to $\theta_{\text{diff}} \approx 47^\circ$.

Figure 5.9 shows the angle of spatial resonance $\theta_{\text{diff}}$ versus the capillarity parameter $\Gamma_0$ for the forcing frequency ratios $m/n = 8/9$ and $8/11$ and for various values of $\gamma$. The solid lines represent the prediction of $\theta_{\text{diff}}$ based on the dispersion relation, while the points represent data from a full numerical computation of $B(\theta)$.

Another result that follows from the dispersion relation is that if the second forcing frequency $n$ is sufficiently different from $m$, the difference frequency resonance will not be possible for any value of $\Gamma_0$. This phenomenon is demonstrated in Figure 5.7. The forcing frequency ratio $m/n = 8/21$ violates the condition (5.60) for all allowed $\Gamma_0$, and the corresponding $B(\theta)$ curve (dashed line) displays only the 1:2 resonance effect.

Now we discuss the magnitude and direction of the difference frequency mode resonance effect. In contrast to the 1:2 resonance, we find that the difference frequency resonance may result in a spike or a dip. (To recall our terminology, “spike” refers a feature on the plot that points towards $B = 0$, while “dip” refers to a feature pointing away from $B = 0$.) Limited numerical results for the sign of the resonance effect agree with the result for one spatial dimension. In particular, we have performed computations at $\gamma = 0.1$ for the forcing frequency ratios $m/n = 8/7, 8/9, 8/11, 10/7, 10/9$ and $10/11$, each for values of $\Gamma_0$ ranging between 0 and $\Gamma_{\text{max}} = m^2/4$. In all
Figure 5.9: Angle of difference frequency resonance $\theta_{\text{diff}}$ versus capillarity number $\Gamma_0$. Lines correspond to a prediction of $\theta_{\text{diff}}$ based on the dispersion relation (3.36) and on the trigonometric relation (4.2). Symbols correspond to a numerical calculation of $\theta_{\text{diff}}$: $\gamma = 0.2$ ("x"), $\gamma = 0.8$ ("□"). The other parameters are $m = 8$, $\chi = 50^\circ$ and $\phi = 0$ in (3.29).
cases, we observe that if \( n < m \) then the difference frequency resonance results in a dip at \( \theta = \theta_{\text{diff}} \); if \( n > m \), it results in a spike. Our numerical findings here regarding the sign of the difference frequency resonance effect for weak damping are consistent with those that follow from assuming that the problem has a Hamiltonian structure for \( \gamma = 0 \) [91].

As in the one-dimensional case, the magnitude of the difference frequency resonance effect follows the scaling law that we deduced from symmetry considerations in Section 5.2, namely that the contribution from the weakly damped mode scales like \((f_n)^2/\gamma\). This scaling may be seen in Figure 5.10. We compute the magnitude of the effect by finding \( B(\theta_{\text{diff}}) - B_{f_n=0}(\theta_{\text{diff}}) \), where \( B_{f_n=0}(\theta_{\text{diff}}) \) is the value of the cross-coupling coefficient at the resonant angle computed without the second forcing component. We plot the size of the spike versus \((f_n)^2/\gamma\).

As discussed in Section 5.2, a spike occurring at spatial angle \( \theta = \theta_{\text{diff}} \) will help stabilize SL-I patterns with characteristic angles \( \theta_h \) near \( \theta_{\text{diff}} \). To demonstrate this effect, we consider an example for \( m/n = 8/11 \) forcing, with \( \gamma = 0.2 \), and \( \Gamma_0 = 13 \) and focus on the case of a harmonic instability. These dimensionless parameters can be realized, for instance, by a fluid with surface tension \( \Gamma = 4.2 \text{ dyn/cm} \), density \( \rho = 1.0 \text{ g/cm}^3 \), and kinematic viscosity \( \nu = 0.01 \text{ cm}^2/\text{s} \) being forced with base frequency \( \omega/(2\pi) = 16.2 \text{ Hz} \). (The fluid properties here are similar to those of water, but with lower surface tension. This situation might be achieved by the use of a surfactant).

When \( \chi = 60.5^\circ \), there is a spike in \( B(\theta) \) at \( \theta_{\text{diff}} = 46.9^\circ \), which is close to the value of \( 47.0^\circ \) that is predicted by the dispersion relation. We have performed a limited bifurcation analysis similar to that in Chapter 4. The stability of super hexagon and
Figure 5.10: $B(\theta_{\text{diff}}) - B_{f_n=0}(\theta_{\text{diff}})$, the magnitude of the difference frequency spike, versus $(f_n)^2/\gamma$. The damping parameter $\gamma$ is varied between 0.01 and 0.1, and the strength of the second forcing frequency $f_n$ is varied between 0 (which corresponds to single frequency forcing) and $f_n^c$. Best-fit lines are also shown. The other parameters are $m = 8$ and $\phi = 0$ in (3.29), and $\Gamma_0 = 14$ in (3.34).
Figure 5.11: SL-I super hexagon (left) and super triangle (right) patterns with characteristic angle $\theta_h \approx 47^\circ$. For $m/n = 8/11$ forcing with $\gamma = 0.2$ and $\Gamma_0 = 13$, both patterns are unstable for $\chi = 0^\circ$. For $\chi = 60.5^\circ < \chi_{bc}$, one of these patterns is stabilized by the difference frequency resonance effect; a higher order calculation is needed to determine which one. The patterns shown were created by an appropriate superposition of the 12 critical Fourier modes as discussed in Section 2.4.

super triangle SL-I patterns is investigated within the setting of a 12-dimensional bifurcation problem. An SL-I pattern with lattice angle $\theta_h \approx 47^\circ$ is stable for a small range of $f$ above onset; examples of the SL-I pattern are shown in Figure 5.11. A higher order calculation is necessary to determine whether it is the super hexagon or super triangle variety of SL-I pattern that is stabilized (these two different types of SL-I patterns have different phases associated with the complex amplitudes).
5.6 Conclusions

In this chapter we have examined the role that weakly damped modes play in the pattern selection process for Faraday waves forced with frequency components $m\omega$ and $n\omega$. Our symmetry arguments predict that the modes oscillating primarily with the frequency $m\omega$, the difference frequency $|n-m|\omega$, and the sum frequency $(n+m)\omega$ will be the most important in terms of their contribution to the cubic coefficients $A$ and $B(\theta)$ in the standing wave equations (5.4). The symmetry considerations also provided scaling laws for the magnitude of these resonance effects.

Starting with the Zhang-Viñals Faraday wave equations, we performed a weakly nonlinear analysis for weak damping and forcing and for $m + n > 5$ in order to calculate expressions for the self-interaction coefficient $A$. We obtained expressions for the critical forcing and wave number, and analyzed them to elucidate the role played by the secondary forcing component in the linear instability. We also were able to identify the most important frequency components in the unstable eigenmode in terms of the integers $m$ and $n$. We then analyzed the expression for the cubic coefficient $A$, and determined the sign and scaling of contributions due to the various resonance effects.

We then used the Zhang-Viñals equations to numerically calculate the cross-coupling coefficient $B(\theta)$ according to the method in [28]. The predictions of the symmetry arguments were manifest. The results for $B(\theta)$ are of particular interest since this coefficient is crucial in determining the stability of SL-I patterns like those observed in [10]. We made use of an argument, valid for weak damping, which relies only on the dispersion relation to successfully predict the resonant angle. While our
symmetry arguments predict the scaling of the resonance effects and the dispersion relation predicts the angle, neither argument predicts the sign of the contribution to $B(\theta)$. Our numerical calculation revealed that the 1:2 resonance results in a dip, and thus is destabilizing for SL-I patterns. However, the difference frequency resonance in some cases results in a spike, which can help stabilize SL-I patterns with characteristic angles near the resonant angle $\theta_{diff}$. This was demonstrated by means of a simple bifurcation example.

We may now speculate on the role of the bicritical point in stabilizing SL-I patterns. It has been observed that SL-I patterns occur in experiments only for parameters near the bicritical point. It is tempting to believe, then, that the weakly damped mode associated with the secondary forcing component is somehow responsible for the pattern. Here we have shown that this interpretation is not necessarily the correct one. Proximity to the bicritical point (i.e., making $f_n$ as large as possible before switching over to the other instability) maximizes the strength of the difference frequency mode. As we have seen, this mode can help stabilize the SL-I pattern.
Chapter 6

Enhanced stabilization of SL-I patterns

6.1 Introduction

In Chapters 4 and 5, we used symmetry arguments and weakly nonlinear analysis to demonstrate how a resonance between the critical modes and the difference frequency mode could lead to the stabilization of an SL-I superlattice pattern in Faraday waves forced with two rationally-related frequency components. In this chapter, our goal is to look more in depth at how to exploit and enhance this effect.

As a starting point, we return to the work of Chapter 4. In Section 4.3, we discussed necessary conditions for SL-I pattern stability within the framework of the 12-dimensional $D_6$-$T^2$-equivariant bifurcation problem (4.11) [25]. In Section 4.5, a bifurcation analysis for a particular fluid and forcing function demonstrated that an SL-I pattern with lattice angle $\theta_h \approx 21.8^\circ$ was stable for a very small range of
the control parameter $4.52 \times 10^{-9} \leq \lambda \leq 1.86 \times 10^{-8}$. The relevant mechanism for this stabilization was a resonance between the pattern modes and a weakly damped harmonic mode, which caused one of the nonlinear coefficients, namely $b_4$, to become small in magnitude when evaluated at the angle of spatial resonance $\theta_r \approx \theta_h$. In Chapter 5 we showed that such a “spike” in the nonlinear coefficient at $\theta_r$ can be obtained when the weakly damped harmonic mode is the so-called “difference frequency” mode, oscillating with dominant frequency equal to the difference of the two frequencies in the forcing function.

We now return to an examination of the eigenvalues of (4.11), whose signs are given in Table 2.1. The goal is to determine what conditions might lead to an SL-I pattern that is stable for a larger range of $\lambda$ than for our previous example. We focus on conditions for which SL-I patterns are stable for $\lambda > \lambda_{\text{crit}}$ (at least, for values of $\lambda$ for which the weakly nonlinear description is valid) where $\lambda$ is the control parameter in (4.11) and $\lambda_{\text{crit}}$ is a quantity to be determined. When this situation is achieved, we say that the SL-I pattern is “eventually” stable.

For some particular SL-I pattern to be eventually stable, the conditions that follow directly from Table 2.1 are (4.12) along with

$$b_1 - b_2 < 0 \quad (6.1a)$$

$$4(b_1 - b_2)^2 - 2(b_4 - b_5)^2 - 2(b_4 - b_6)^2 - 2(b_5 - b_6)^2 > 0. \quad (6.1b)$$

Condition (4.12) follows from the first two SL-I stability quantities in Table 2.1, and conditions (6.1) follow from the $\zeta_1$ and $\zeta_2$ quantities.

The remainder of this chapter is devoted to exploiting the results of Chapters 4
and 5 in order to produce a situation in which an SL-I pattern is eventually stable. Sections 6.2 and 6.3 focus on making $b_4$, $b_5$, and $b_6$ small in magnitude so that (4.12) and (6.1a) are satisfied. In Section 6.2 we demonstrate how a forcing function with more than two frequency components may be used to accomplish this goal, namely by exciting multiple difference frequency modes. In Section 6.3 we demonstrate how we may cause a spike in these coefficients to become even larger by direct forcing of the difference frequency mode. Section 6.4 focuses on condition (6.1). We discuss a crude method for keeping the coefficient $b_2$ greater than $b_1$. In Section 6.5, we combine the aforementioned techniques to create a forcing function for which the sufficient conditions (4.12) and (6.1a) are satisfied. Thus, we produce a situation in which the SL-I pattern selected by the engineered forcing function is, in fact, stable for $\lambda > \lambda_{\text{crit}}$.

6.2 Multiple frequency forcing

In Chapters 4 and 5 we demonstrated how a resonance between the critical modes and the damped difference frequency mode could cause a spike in the cross-coupling coefficient $\beta(\theta)$ in the general rhombic lattice bifurcation problem (4.9). This spike helped stabilize an SL-I pattern with lattice angle $\theta_h$ near the spatial resonance $\theta_r$. The cross-coupling coefficient $\beta(\theta)$ is related to the cubic coefficients $b_4, b_5, b_6$ in (4.11) by (4.13). Thus, a single spike in $\beta(\theta)$ causes one of $b_4, b_5, b_6$ to become small in magnitude. In this section, we demonstrate how a multiple frequency forcing function leads to multiple difference frequency resonances, which cause multiple spikes in $\beta(\theta)$. If we can arrange for these spikes to occur at angles $\theta, 60^\circ - \theta, \text{ and } 60^\circ + \theta$, then $b_4, b_5, \text{ and } b_6$ for the desired SL-I pattern will all be small in magnitude, and we will be
closer to satisfying the condition (4.12).

Since we wish to obtain three spikes in $\beta(\theta)$, we will use the four frequency forcing function

$$f [r_m \cos(m\tau) + r_n \cos(n\tau + \phi_n) + r_p \cos(p\tau + \phi_p) + r_q \cos(q\tau + \phi_q)] .$$

(6.2)

Here, $m \ldots q$ are relatively prime integers, $r_m \ldots r_q$ measure the relative strength of the four frequency components, and we insist that $r_m^2 + r_n^2 + r_p^2 + r_q^2 = 1$ for analogy to the two-frequency case (3.29). Without loss of generality we assume that the $\cos(m\tau)$ frequency component is the dominant one.

We demonstrated in Section 5.5 that the angle of difference frequency resonance can be approximated using the dispersion relation (3.36) (at least, for weak damping and forcing). We take advantage of this fact here to find $m, n, p, q$ in (6.2) and $\Gamma_0$ in (3.34) such that the three difference frequency spikes line up appropriately, as discussed above.

The critical wave number, due primarily to the $\cos(m\tau)$ forcing, is estimated by

$$\Omega^2(k_m/2) = \left(\frac{m}{2}\right)^2$$

(6.3)

where $\Omega$ is the natural frequency in the dispersion relation (3.36). Because of the scaling of the Zhang-Viñals equations (3.34), $k_m/2 = 1$. The three wave numbers
corresponding to the three difference frequency modes are estimated by

\[ \Omega^2(k_{|n-m|}) = (n-m)^2 \]  
\[ \Omega^2(k_{|p-m|}) = (p-m)^2 \]  
\[ \Omega^2(k_{|q-m|}) = (q-m)^2. \]

The three angles of spatial resonance \( \theta_{|n-m|}, \theta_{|p-m|}, \theta_{|q-m|} \) are determined by using \(|k_1| = k_m/2\) and \(|k_3| = k_{|n-m|}, k_{|p-m|}, k_{|q-m|}\) respectively in (4.1). In practice, we find parameters \((m, n, p, q, \Gamma_0)\) such that \(\theta_{|n-m|}, \theta_{|p-m|}, \theta_{|q-m|}\) line up appropriately by performing a numerical search of parameter space. We also wish to choose \(r_m \ldots r_q\) so as to obtain the largest possible spikes in \(\beta(\theta)\). Our symmetry analysis from Section 5.2 showed that for the case of the two-frequency forcing function (3.29), the spike size is proportional to \((f_n)^2/\gamma\). Thus, for that case, we should make \(f_n\) as large as possible, i.e. we should choose the parameters in (3.29) to lie as close as possible to the bicritical point. By analogy, for the four-frequency forcing (6.2), we should choose the parameters to lie as close as possible to the “quad-critical” point.

We have produced an example of this situation for \((m, n, p, q) = (8, 9, 11, 13), (r_m, r_n, r_p, r_q) = (0.287, 0.325, 0.539, 0.722)\) and \((\phi_n, \phi_p, \phi_q) = (0, 0, 0)\) in (6.2) and \(\gamma = 0.25\) and \(\Gamma_0 = 12.4\) in (3.34). The linear stability diagram is shown in Figure 6.1. The primary instability is harmonic, and the onset parameters are \((k_c, f_0) = (0.968, 12.6).\) For this example, we find that \(\epsilon = 0.402, b_1 = -1,\) and \(b_2 = -1.98\) in (4.11). The cross-coupling coefficient \(\beta(\theta)\) is shown in Figure 6.2. As expected, \(\beta(\theta)\) contains three spikes due to the three difference frequency resonances. The wave numbers of the difference frequency modes predicted by (6.4) are \((k_{|n-m|}, k_{|p-m|}, k_{|q-m|}) = \ldots \)
(0.234, 0.792, 1.187) and the predicted resonance angles are \((\theta_{|m|}, \theta_{|p-m|}, \theta_{|q-m|}) = (13.4^\circ, 46.6^\circ, 72.8^\circ)\). The actual spikes occur not precisely at the predicted angles, but rather at \((\theta_{|m|}, \theta_{|p-m|}, \theta_{|q-m|}) = (13.5^\circ, 46.7^\circ, 74.2^\circ)\). The slight discrepancy occurs because \(\gamma\) and \(f\) are large, so the dispersion relation gives a less accurate estimate.

By focusing on lattice angle \(\theta_h\) such that \(\theta_h, 60^\circ - \theta_h,\) and \(60^\circ + \theta_h\) are near \((\theta_{|m|}, \theta_{|p-m|}, \theta_{|q-m|}) = (13.5^\circ, 46.7^\circ, 74.2^\circ),\) we may obtain a situation where \(b_4, b_5, b_6\) are small in magnitude. For instance, for \(\theta_h \approx 46.5,\) \(\theta_h\) is close to \(\theta_{|p-m|},\) \(60^\circ - \theta_h\) is close to \(\theta_{|m|},\) and \(60^\circ + \theta_h\) is close to \(\theta_{|p-m|}\). (The last statement holds from the identities \(\beta(\theta) = \beta(-\theta) = \beta(\theta + \pi),\) which follow from the symmetries of the rhombic lattice bifurcation problem; see Section 4.3.) This case of \(\theta_h \approx 46.5\) corresponds to a lattice with \((n_1, n_2) = (39, 34).\) For this angle, we find that \(b_4 = -1.05, b_5 = -1.40,\) and \(b_6 = -0.139 \) in (4.11). Based on the analysis of Section 4.5 we expect that the stability of SL-I patterns may be enhanced. (This has been confirmed by a bifurcation analysis, but we omit the details here since a more interesting example is examined in Section 6.5.)

In this section we have seen that a four-frequency forcing function may be engineered such that multiple difference frequency resonances conspire to favor the same SL-I pattern. Nonetheless, the spikes in the coupling coefficient \(\beta(\theta)\) do not cause \(b_4, b_5,\) and \(b_6\) to become extremely small in magnitude. We performed our calculation as close as possible to the quad-critical point, and thus we expect that the sizes of the difference frequency spikes are already maximized for the forcing function (6.2). In the next section, however, we demonstrate how direct forcing of a difference frequency modes gives us greater control over the size of a spike in \(\beta(\theta).\)
Figure 6.1: Linear stability diagram for fluid parameters $\gamma = 0.25$ and $\Gamma_0 = 12.4$ in (3.34). The parameters in the forcing function (6.2) are forcing frequencies $(m, n, p, q) = (8, 9, 11, 13)$, relative forcing amplitudes $(r_m, r_n, r_p, r_q) = (0.287, 0.325, 0.539, 0.722)$ and phases $(\phi_m, \phi_n, \phi_p, \phi_q) = (0, 0, 0, 0)$. (Sub)harmonic tongues are shown in gray (black). The primary instability is harmonic and the parameters are near a “quad-critical” point in parameter space.
Figure 6.2: Cross-coupling coefficient $\beta(\theta)$ in (4.9) computed for the same parameters as in Figure 6.1. The three spikes at $(\theta_{|n-m|}, \theta_{|p-m|}, \theta_{|q-m|}) = (13.5^\circ, 46.7^\circ, 74.2^\circ)$ are due to resonances with the weakly damped modes oscillating primarily with the difference frequencies $|n-m|, |p-m|$ and $|q-m|$. 
6.3 Direct forcing of difference frequency mode

In this section we demonstrate how we may force the difference frequency mode directly in order to exert greater control over the size of a spike in the cross-coupling coefficient $\beta(\theta)$. We begin by recalling that the basic resonance in the Faraday wave system is subharmonic, i.e. modes oscillate with a dominant frequency equal to one-half of the forcing frequency responsible for their excitation. Thus, for the two-frequency forcing function (3.29), we must add a component oscillating with frequency $2|n - m|$ in order to directly force the difference frequency mode. That is to say, we use the three-frequency forcing function

$$f \left[ r_m \cos(m\tau) + r_n \cos(n\tau + \phi_n) + r_{2|n-m|} \cos(2|n - m|\tau + \phi_{2|n-m|}) \right]. \quad (6.5)$$

(We note that this type of forcing has been used in an experiment in [13]).

We have produced an example using $(m, n) = (6, 7)$ in (6.5) and $\gamma = 0.08$ and $\Gamma_0 = 7.5$ in (3.34). The fluid parameters and the $6/7$ forcing frequency ratio are the same as for the example in Figures 4.2b and 4.4a. The difference here is the addition of the $\cos(2\tau)$ frequency component in (6.5). We have chosen $(r_m, r_n, r_{2|n-m|}) = (0.61, 0.79, 0.064)$ and $(\phi_n, \phi_{2|n-m|}) = (0, 0)$ in (6.5).

The primary instability is harmonic and the onset parameters are $(k_c, f_0) = (0.997, 1.553)$. The linear stability diagram is shown in Figure 6.3. The onset parameters are nearly identical to those for the example in Figure 4.2b, and the linear stability diagram looks quite similar. The difference here is the presence of additional resonance tongues due to the inclusion of the $\cos(2\tau)$ forcing. We are close to a
tri-critical point in parameter space involving the large harmonic and subharmonic tongues in the center of the figure, which are due to the \( \cos(6\tau) \) and \( \cos(7\tau) \) forcing, and the thin harmonic tongue near \( k = 0.4 \), which is due to the \( \cos(2\tau) \) forcing.

The cross coupling coefficient \( \beta(\theta) \) is shown as the solid line in Figure 6.4. Note the similarity to Figure 4.4a. As we expect, there is a spike at \( \theta \approx 22^\circ \) which is caused by the difference frequency mode resonance. In contrast to the case in Figure 4.4a, we have forced the difference frequency mode directly here, resulting in a larger spike. In fact, we have chosen the forcing amplitudes in (6.5) such that this spike reaches precisely to \( \beta(\theta) = 0 \), and thus the corresponding coefficient \( b_4 \) in (4.11) will be zero at this angle. The size of this spike can be controlled by varying the forcing amplitudes. For instance, by setting \( (r_m, r_n, r_{2|n-m|}) = (0.61, 0.79, 0.072) \) in (6.5), we obtain an even larger spike. This case is shown as the dotted line in Figure 6.4.

Alternatively, the size of the spike can be controlled by varying the phase of the component which directly forces the difference frequency mode, i.e. \( \phi_{2|n-m|} \) in (6.5), rather than by controlling its magnitude. This effect has been predicted by symmetry arguments [91]. (The phase dependence of some other resonant contributions to the cross-coupling coefficient \( \beta(\theta) \) is discussed in [90].) The \( \pi \)-periodic variation in \( \beta(\theta_{|n-m|}) \), the value of the cross-coupling coefficient evaluated at the angle of difference frequency resonance, with respect to \( \phi_{2|n-m|} \) is shown in Figure 6.5.

In this section we have demonstrated how direct forcing of the difference frequency mode enables us to control the size of the spikes in the cross-coupling coefficient. We anticipate that we will combine this technique with the technique of the previous section in order to realize the condition (4.12). We postpone this exercise for the time being, though, and now turn to a brief examination of how the condition (6.1a)
Figure 6.3: Linear stability diagram computed for \((m, n, 2|n - m|) = (6, 7, 2), (r_m, r_n, r_{2|n-m|}) = (0.61, 0.79, 0.064)\) and \((\phi_n, \phi_{2|n-m|}) = (0, 0)\) in (6.5) and \(\gamma = 0.08\) and \(\Gamma_0 = 7.5\) in (3.34). (Sub)harmonic tongues are shown in gray (black). The primary instability occurs at the tip of the large harmonic tongue near \(k = 1\). The parameters are near a “tri-critical” point in parameter space.
Figure 6.4: Cross-coupling coefficient $\beta(\theta)$ in (4.9). The solid line is computed for the same parameters as in Figure 6.3. The dotted line is computed for the same parameters, except with $(r_m, r_n, r_{2|n-m|}) = (0.61, 0.79, 0.072)$ in (6.5).
Figure 6.5: The value of the cross-coupling coefficient evaluated at the angle of difference frequency resonance, $\beta(\theta_{|n-m|})$, versus the phase $\phi_{2|n-m|}$ in (6.5). The parameters are the same as those used in Figure 6.3, except that $\phi_{2|n-m|}$ is now varied between $0^\circ$ and $360^\circ$. 
might be met.

6.4 Hexagonal resonance

In this brief section we describe how we may attempt to satisfy the condition (6.1b). In Chapter 5, we saw that the coefficient $A$ in (5.13) changed suddenly in magnitude around those values of the capillarity parameter $\Gamma_0$ at which the 1:2 spatial resonance also satisfied the 1:2, sum, or difference frequency temporal resonance. In this section, we demonstrate the analogous case for the 1:$\sqrt{3}$ spatial resonance (the $\sqrt{3}$ length scale follows from solving (4.2) for $\theta_r = 60^\circ$). In particular, we expect that around those values of $\Gamma_0$ at which the 1:$\sqrt{3}$ resonance also satisfies one of the aforementioned temporal resonances, the cubic hexagonal interaction coefficient $b_2$ in (4.11) will change suddenly in magnitude.

As in Chapter 5, we may estimate the resonant values of $\Gamma_0$ using the Zhang-Viñals dispersion relation. We consider the two-frequency forcing function (3.29) and as before assume that the $\cos(m\tau)$ component is the dominant one. The condition for the 1:2 temporal resonance is

$$\Omega^2(\sqrt{3}) = m^2$$

where $\Omega$ is given by (3.36). We solve for $\Gamma_0$ to obtain

$$\Gamma_0 = \Gamma_{1:2} = \frac{m^2 \sqrt{3}}{24}(4 - \sqrt{3}).$$
Similar calculations for the resonant values $\Gamma_{\text{sum}}$ and $\Gamma_{\text{diff}}$ yield

$$
\Gamma_{\text{sum}} = \frac{\sqrt{3}}{24} \left\{ (4 - \sqrt{3})m^2 + 8mn + 4n^2 \right\} \quad (6.8a)
$$

$$
\Gamma_{\text{diff}} = \frac{\sqrt{3}}{24} \left\{ (4 - \sqrt{3})m^2 - 8mn + 4n^2 \right\} \quad (6.8b)
$$

We have not at this time calculated an analytical expression for the coefficient $b_2$ analogous to that for $A$ in Chapter 5. Nonetheless, we may compute $b_2$ numerically using the methods of Chapter 4 and interpret features in light of the resonance arguments we have made.

A sample result is shown in Figure 6.6, which shows $b_2$ in (4.11) as a function of $\Gamma_0$ for $m = 6$, $n = 7$, $\chi = 51^\circ$ and $\phi = 0^\circ$ in (3.29) and $\gamma = 0.08$ in (3.34). A large dip due to the 1:2 temporal resonance is present at $\Gamma_0 = \Gamma_{1:2} \approx 6.1$ which is close to the predicted value of $\Gamma_{1:2} \approx 5.9$. This is the only remarkable feature; the sum and difference frequency resonances cannot be realized for the allowed values $0 < \Gamma_0 < m^2/4$. While we do not have an analytical expression as proof, our limited investigation agrees with the analytical result for the coefficient $A$ and with numerical results for $\beta(\theta)$ in that the 1:2 resonance always seems to cause the coefficient to become larger in magnitude, i.e. more negative. Thus, in trying to satisfy condition (6.1b), we should choose a value of $\Gamma_0$ that is not close to $\Gamma_{1:2}$. For the present example, we might choose $\Gamma_0 = 3$, at which $b_2 = -0.073$ takes on its smallest magnitude.

For other forcing functions, it may be possible to realize the sum and/or difference frequency resonances, and these may help us to achieve the condition (6.1a). For now, our technique for attempting to meet this condition is a crude one, namely avoiding the 1:2 resonance zone.
Figure 6.6: The bifurcation coefficient $b_2$ in (4.11) as a function of $\Gamma_0$ for $m = 6$, $n = 7$, $\chi = 51^\circ$ and $\phi = 0^\circ$ in (3.29) and $\gamma = 0.08$ in (3.34). A large dip due to the 1:2 temporal resonance is present at $\Gamma_0 = \Gamma_{1:2} \approx 6.1$ which is close to the predicted value of $\Gamma_{1:2} \approx 5.9$. 
6.5 Combined methods

In this section we use the techniques of the previous three sections to design a forcing function for which the conditions (4.12) and (6.1) are met. In particular, we present an example of a multi-frequency forcing function which leads to multiple difference frequency spikes, and which forces the difference frequency modes directly to make the spikes larger. Furthermore, we select our parameters to avoid the hexagonal 1:2 temporal resonance, and thus make the cubic hexagonal coefficient small in magnitude.

We use the five-frequency forcing function

\[ f[r_8 \cos(8\tau) + r_{10} \cos(10\tau) + r_{11} \cos(11\tau) + r_4 \cos(4\tau) + r_6 \cos(6\tau)]. \]  

The \( \cos(8\tau) \) forcing will be the dominant one for our example. We think of the \( \cos(10\tau) \) and \( \cos(11\tau) \) components as secondary components which will lead to two difference frequency spikes in the cross-coupling coefficient \( \beta(\theta) \), as in Section 6.2. The difference frequencies are \( 10 - 8 = 2 \) and \( 11 - 8 = 3 \). We think of the \( \cos(4\tau) \) and \( \cos(6\tau) \) as tertiary components which are added to directly force the difference frequency modes, and thus give us greater control over the size of the spikes as in Section 6.3. As we will see in the bifurcation analysis below, additional components leading to a third difference frequency spike are not necessary.

The value \( \Gamma_{1:2} \) at which the hexagonal 1:2 temporal resonance occurs is determined by (6.6) and for this case is \( 32\sqrt{3}/3 - 8 \approx 10.5 \). For our present example we set \( \Gamma_0 = 5.24 \) which is far from \( \Gamma_{1:2} \). We obtained the integers 8, 10, and 11 in (6.9).
and the chosen value of $\Gamma_0$ by numerically searching parameter space for a situation where the difference frequency spikes would line up appropriately, as in Section 6.2.

The dispersion relation (3.36) gives an estimate of $k_4 = 1$ for the critical wave number. The two wave numbers corresponding to the two difference frequency modes are estimated by

$$\Omega(k_2)^2 = 4 \quad (6.10a)$$
$$\Omega(k_3)^2 = 9 \quad (6.10b)$$

where $\Omega$ is given by (3.36). The wave numbers are $(k_2, k_3) = (0.351, 0.682)$. The two angles of spatial resonance $\theta_2$ and $\theta_3$ are estimated by using $|k_1| = 1$ and $|k_3| = k_2, k_3$ respectively in (4.1). We find that the predicted values are $(\theta_2, \theta_3) = (20.2^\circ, 39.9^\circ)$.

We set $\gamma = 0.1$ in (3.34) and search the $(r_8, r_{10}, r_{11}, r_4, r_6)$ parameter space to obtain a situation where the two difference frequency spikes extend approximately to $\beta(\theta) = 0$. We find $(r_8, r_{10}, r_{11}, r_4, r_6) = (0.376, 0.580, 0.694, 0.0696, 0.190)$. With these parameters, the harmonic instability occurs at $(k_c, f_0) = (0.993, 4.20)$. The linear stability diagram is shown in Figure 6.7, and is close to the quint-critical point in parameter space.

The cross-coupling coefficient $\beta(\theta)$ is shown in Figure 6.8. The notable features are two very sharp spikes due to the two difference frequency resonances. They occur at angles $(\theta_2, \theta_3) = (20.3^\circ, 39.9^\circ)$ which are nearly equal to the predicted values. We have chosen our forcing amplitudes in (6.9) such that the spike at $\theta_2$ extends almost precisely to $\beta(\theta) = 0$. Since the sharp spikes do not line up exactly to stabilize the same SL-I pattern, i.e. $\theta_3 \neq 60^\circ - \theta_2$, we also choose our parameters such that the
Figure 6.7: Linear stability diagram computed for parameters $(r_8, r_{10}, r_{11}, r_4, r_6) = (0.376, 0.580, 0.694, 0.0696, 0.190)$ in (6.9) and $\gamma = 0.1$ and $\Gamma_0 = 5.24$ in (3.34). (Sub)harmonic tongues are shown in gray (black). The primary instability occurs at the tip of the large harmonic tongue near $k = 1$. The parameters are near a “quint-critical” point in parameter space.
Figure 6.8: Cross-coupling coefficient $\beta(\theta)$ for the same parameters as in Figure 6.7. The two sharp spikes occur at angles $(\theta_2, \theta_3) = (20.3^\circ, 39.9^\circ)$.

spike at $\theta_3$ extends past $\beta(\theta) = 0$, which causes $\beta(60^\circ - \theta_2 = 39.7)$ to be small in magnitude.

We focus on bifurcation results for an $(n_1, n_2) = (16, 13)$ hexagonal lattice with characteristic angle $\theta_h \approx 39.7^\circ$. We scale the amplitudes $z_j$ in (4.11) so that $b_1 = -1$, in which case we find that $\epsilon = 0.00863$ and $b_2 = -0.0667$. Note that, as desired, $|b_2| \ll 1$. The remaining coefficients are equal to $(b_4, b_5, b_6) = (0.0555, 0.156, -0.0410)$. 


Figure 6.9: Bifurcation diagram for the parameters in Figure 6.7 and an \((n_1, n_2) = (16, 13)\) lattice, which corresponds to \(\theta_h \approx 39.7^\circ\) in (2.30). Stable (unstable) solutions are indicated by a solid (dotted) line. We do not show secondary branches or primary branches that are never stable. Hexagons (H) are never stable in the supercritical regime. A super hexagon or super triangle (SH/ST) SL-I pattern is stable for \(\lambda > \lambda_{\text{crit}}\), where \(\lambda_{\text{crit}}\) is the value of the control parameter at which the SL-I branch turns around in a saddle-node bifurcation.
With this example, we have satisfied the conditions (4.12) and (6.1), and thus have obtained the desired SL-I stabilization.

The bifurcation diagram is given in Figure (2.32). Rolls and all rhombs solutions associated with the chosen lattice are never stable, so we do not show those branches on the diagram. Hexagons and the SL-I superpattern bifurcate subcritically and turn around in saddle-node bifurcations. Hexagons are stable only for a range of the subcritical regime, namely \(-1.64 \times 10^{-5} < \lambda < -7.39 \times 10^{-6}\). In contrast, the SL-I pattern is stabilized at the saddle-node bifurcation and remains stable for a semi-infinite range of the control parameter \(\lambda > 1.93 \times 10^{-5}\). Thus, within the framework of the bifurcation problem we have studied, we expect that the transition from the trivial state would be a hysteretic one in which the solution jumps to the only stable pattern at onset, namely the SL-I pattern. We mention again that we cannot determine whether the superlattice pattern is hexagonal or triangular from our calculations, since this requires knowledge of fifth order terms in the bifurcation problem (4.11) [26].
Chapter 7

Conclusion

7.1 Review of dissertation

In this dissertation, we have investigated pattern formation in Faraday waves forced with two rationally-related frequency components. Specifically, by focusing on the constraints imposed by temporal symmetries, we were able to assess the effect that weakly damped modes have on the pattern selection problem. We performed weakly nonlinear analyses on the Zhang-Viñals equations [24] to demonstrate the general symmetry results, and to obtain quantitative results for Faraday waves. We paid special attention to the implications that our results have for the formation of complex superlattice patterns of the SL-I type, which contain two length scales and have recently been observed in Faraday wave laboratory experiments [10].

In Chapter 3, we presented two different mathematical formulations for the Faraday wave problem. The first was the well-known Navier-Stokes equations with free boundary. The second was the Zhang-Viñals equations, which were derived from the
Navier-Stokes equations in [24] and which describe small-amplitude Faraday waves on deep layers of weakly viscous fluids. We compared the linear stability predictions made by the two formulations, and saw good agreement for small $\gamma$, though reasonable agreement was not obtained for the relative large value $\gamma = 0.97$ relevant for the experiment in [10]. The analysis performed in Chapters 4 and 5 use the Zhang-Viñals equations as a starting point, and thus we expect that our quantitative results will apply to experiments only for small $\gamma$. Many of our qualitative results, however, rely heavily on symmetry arguments, and we expect that they will be applicable to situations that would be more accurately described by the Navier-Stokes equations.

In Chapter 4, we studied a bifurcation problem describing the competition of stripe patterns, simple hexagons, three distinct rhombic patterns, and SL-I super hexagon and super triangle patterns [25]. The bifurcation problem, presented in [25], took the form of a stroboscopic map. We used a discrete-time normal form symmetry to demonstrate how weakly damped harmonic modes may affect pattern selection. Specifically, they may enter into resonant triad interactions with the pattern modes, which lead to resonant contributions to the bifurcation coefficients. To demonstrate this effect explicitly, we performed a perturbation calculation, valid for arbitrary damping and forcing near onset, on the Zhang-Viñals equations to obtain expressions for the bifurcation coefficients. These expressions were evaluated numerically. A sample bifurcation analysis showed how a weakly damped harmonic mode can help stabilize an SL-I pattern.

In Chapter 5, we focused on the case of weak damping and forcing, and used continuous-time temporal symmetries to identify which particular weakly damped modes are the most important in terms of SL-I pattern stabilization. For weak
damping and forcing and one-dimensional waves, we derived from the Zhang-Viñals equations analytical expressions for the critical forcing and wave number, and for one of the bifurcation coefficients. We quantified the predicted resonance effects and determined how their existence depends on the forcing frequency ratio $m/n$. For two-dimensional waves and stronger damping and forcing, we computed onset parameters and all of the bifurcation coefficients numerically, as before, and confirmed the symmetry results. Bifurcation analysis revealed that in practice, the mode oscillating with dominant frequency equal to the difference of the two forcing frequencies can lead to a stabilization of an SL-I pattern. Thus, while the dominant forcing component determined one length scale in the SL-I pattern, we demonstrated that the “difference frequency” may provide the other length scale, and thus determine the large-scale periodicity of the pattern. This length scale can be well-predicted by a simple dispersion relation which depends only on $|n−m|\omega$ and the physical properties of the fluid.

In Chapter 6, we applied the results of Chapters 4 and 5 to “engineer” forcing functions with the goal of SL-I pattern stabilization. In particular, we showed how using appropriately chosen forcing functions with more than two frequency components could lead to drastically enhanced SL-I stabilization.

### 7.2 Future research

The results of this dissertation suggests several directions for further research.

The experiments in [10] found, near the bicritical point, an SL-I superlattice pattern which sits on a particular lattice. In Chapter 4, we suggested that the observation
of this pattern could be explained by the interaction of the primary harmonic instability and weakly damped harmonic modes. However, as discussed in Chapter 3, the Zhang-Viñals equations are not valid in the parameter regime where this experiment was performed, and thus a weakly nonlinear study of the Navier-Stokes equations, in the spirit of [87], is necessary to confirm this conjecture.

In Chapter 6, we suggested the use of multi-frequency forcing functions as a mechanism for engineering a particular SL-I pattern. We did not implement any phase differences between our forcing components, and in practice, we had to be proximate to a multi-critical point in parameter space to accomplish significantly enhanced SL-I stabilization. By generalizing the result of [91] as discussed in Section 6.3, by varying the phase of the forcing component that directly forces the difference frequency mode, it might be possible to achieve a similar effect, but further from the multi-critical point. A further avenue of research would be to extend our work to see if forcing functions might be engineered to stabilize types of patterns other than the SL-I, including both the simple patterns that tile the plane as well as more complex patterns, such as quasipatterns.

Our primary results concerning pattern stabilization come from weakly nonlinear analysis. We would like to test the ideas gleaned from our analysis within the setting of a numerical simulation of the Zhang-Viñals equations. We have already taken a first step in this direction by writing a simulation code and testing the validity of the weakly nonlinear analysis for one-dimensional waves; see Appendix B for details.

In addition to the nonlinear aspects of Faraday wave pattern formation, there are issues related to linear stability that may warrant further investigation. One result of the analysis in Chapter 5 was that (at least for the weak damping/forcing
regime where our calculation was performed) the addition of a second forcing frequency component may have either a stabilizing or destabilizing effect with respect to the instability caused by the primary frequency component. That is to say, depending on system parameters, the addition of the second frequency component in the forcing function can push the bifurcation to occur at either higher or lower values of acceleration than it otherwise would. This issue could be examined further. For instance, it might be tested experimentally, for fluids of higher viscosity. It could also be studied in a broader context than Faraday waves. Calculations on model equations might reveal general criteria under which additional frequency components stabilize or destabilize the flat state.

Finally, it would be interesting to apply the techniques used in this dissertation to other periodically-forced pattern-forming systems, such as forced chemical systems. Recent experimental work [92] produced complex frequency-locked patterns in a variant of the Belousov-Zhabotinsky chemical reaction-diffusion system, in which the photosensitive reaction rate is modulated via a time-periodic sequence of light pulses. Recent work has used symmetry arguments [93, 94] or numerical simulation of simple chemical models [92] to investigate the frequency-locked patterns. The patterns here differ from the Faraday wave patterns in several key ways, and their study will pose interesting new challenges. For instance, the patterns observed in the chemical system are observed far from onset, and are not spatially periodic. Conducting an analytical study, using techniques similar to those in Chapter 5 and adopting simple chemical models such as the Brusselator [31, 32] or the Oregonator [95] with an added periodic forcing term as a starting point, may yield further insight.
Appendix A

Traveling wave equation coefficients

We now give the expressions for the coefficients in the travelling wave equations (5.19), (5.39) and (5.46) which we computed in Section 5.3.

\[
\eta_1 = -\frac{f_m^1}{2m} \tag{A.1}
\]

\[
\nu_2 = \frac{k_2(8\Gamma_0 + m^2)}{4m} + \frac{3(f_m^1)^2}{8m^3} + \frac{(f_n^1)^2}{2m(n^2 - m^2)} \tag{A.2}
\]

\[
c_1 = \frac{2m^4 - 15m^2\Gamma_0 + 36\Gamma_0^2}{2m(m^2 - 12\Gamma_0)} \tag{A.3}
\]

\[
c_2 = -\frac{2m^4 + 15m^2\Gamma_0 + 36\Gamma_0^2}{m(m^2 + 12\Gamma_0)} \tag{A.4}
\]

\[
\gamma_3 = 2\gamma_1 k_2 \tag{A.5}
\]

\[
\eta_3 = -\frac{9(f_m^1)^3}{32m^5} + \frac{f_m^1(f_n^1)^2(m^4 - m^2n^2 - n^4)}{2n^2m^3(n^2 - m^2)^2} - \frac{f_m^1k_2(8\Gamma_0 + 3m^2)}{4m^3} - \frac{f_m^3}{2m} \tag{A.6}
\]
\[ c_3 = \frac{-\gamma_1 (7m^4 - 48m^2 \Gamma_0 + 144 \Gamma_0^2)}{(m^2 - 12 \Gamma_0)^2} \quad (A.7) \]
\[ c_4 = \frac{6\gamma_1 (m^2 + 4 \Gamma_0)}{m^2 + 12 \Gamma_0} \quad (A.8) \]
\[ c_5 = \frac{3f_m^1 (4m^8 - 47m^6 \Gamma_0 + 516m^4 \Gamma_0^2 + 2160m^2 \Gamma_0^3 + 8640 \Gamma_0^4)}{4m^3 (m^2 + 12 \Gamma_0)^2 (m^2 - 12 \Gamma_0)^2} \quad (A.9) \]
\[ c_6 = \frac{3f_m^1 (4m^6 - 63m^4 \Gamma_0 - 240m^2 \Gamma_0^2 - 720 \Gamma_0^3)}{8m^3 (m^2 + 12 \Gamma_0)^2 (m^2 - 12 \Gamma_0)^2} \quad (A.10) \]
\[ c_7 = \frac{-f_m^1 (4m^6 - 39m^4 \Gamma_0 + 144m^2 \Gamma_0^2 + 432 \Gamma_0^3)}{8m^3 (m^2 + 12 \Gamma_0)^2 (m^2 - 12 \Gamma_0)^2} \quad (A.11) \]
\[ e_1 = \frac{m}{2} \quad (A.12) \]
\[ \gamma_4 = 4\gamma_1 \quad (A.13) \]
\[ \nu_4 = \frac{3\hat{\Gamma}_{1:2}}{m} \quad (A.14) \]
\[ e_2 = \frac{m}{4} \quad (A.15) \]
\[ \tilde{\nu}_4 = -\frac{3\hat{\Gamma}_{eff}}{n-m} \quad (A.16) \]
\[ r_1 = \frac{e^{i\phi} f_n^1 (2n^2 - 4nm + m^2)}{2n(m - n)(2m - n)} \quad (A.17) \]
\[ c_8 = \frac{48n^6 - 72n^3 m^3 + 204m^2 n^4 - 176n^5 m}{4nm(m - n)(m^2 - 10nm + 6n^2)} \]
\[ \quad - \frac{m^6 + 8n^5 - 8m^2 n^2}{4nm(m - n)(m^2 - 10nm + 6n^2)} \quad (A.18) \]
\[ c_9 = \frac{-48n^6 - 2200n^3 m^3 + 1324m^2 n^4 - 400n^5 m}{4m(6n^2 - 14nm + 5m^2)(2m^2 - 3nm + n^2)} \]
\[ \quad - \frac{143m^6 - 824nm^5 + 1912m^4 n^2}{4m(6n^2 - 14nm + 5m^2)(2m^2 - 3nm + n^2)} \quad (A.19) \]
\[ \nu_5 = \frac{9 \Gamma_{diff}^2}{2(m-n)^3} - \frac{(f_1^1)^2}{(3m-2n)(m-2n)(m-n)} - \frac{(f_m^1)^2}{(2m-n)(2m-3n)(m-n)} + \frac{k_2(7m^2 - 22nm + 11n^2)}{6(m-n)} \]

\[ r_2 = \frac{e^{-i\phi} m f_n^1 (m^2 - 4nm + 2n^2)}{4n(m-n)^2(2m-n)} \]

\[ c_{10} = \frac{c_8 m}{m-n} \]

\[ c_{11} = \frac{c_9 m}{m-n} \]

\[ c_{12} = \frac{2(2n^4 - 8n^3m + 9m^2n^2 - 2m^3n + m^4)}{(n-m)(3n^2 - 6nm + m^2)} \]

\[ c_{13} = \frac{4(119m^2n^2 - 62m^3n + 11m^4 + 22n^4 - 88n^3m)}{(n-m)(5n^2 - 10nm + 3m^2)} \]
Appendix B

Numerical simulation

In this appendix, we present the details of a numerical algorithm for simulating the Zhang-Viñals Faraday wave equations. We use essentially the same Fourier-Galerkin method as described in [82], and we review the main points here. We begin with the governing equations (3.26), which are introduced in Section 3.4, and which we now write in the following form:

\begin{align}
\partial_\tau h &= L_1 h + L_2 \Phi + \mathcal{F} \quad \text{(B.1a)} \\
\partial_\tau \Phi &= L_1 \Phi + L_4(\tau)h + \mathcal{G} \quad \text{(B.1b)}
\end{align}

The linear operators are

\begin{align}
L_1 &= \gamma \nabla^2 \quad \text{(B.2a)} \\
L_2 &= \mathcal{B} \\
L_4(\tau) &= \Gamma_0 \nabla^2 - G_0 + f(\tau) \quad \text{(B.2c)}
\end{align}

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and $f(\tau)$ is the time-dependent forcing function.

We may now take the Fourier transform of (B.1) to obtain equations for the amplitudes of each Fourier mode $e^{i\mathbf{k} \cdot \mathbf{x}}$:

\[
\begin{align*}
\partial_\tau \hat{h}_k &= \hat{L}_1 \hat{h}_k + \hat{L}_2 \hat{\Phi}_k + \hat{F}_k \\
\partial_\tau \hat{\Phi}_k &= \hat{L}_1 \hat{\Phi}_k + \hat{L}_4(\tau) \hat{h}_k + \hat{G}_k
\end{align*}
\] (B.3a) (B.3b)

where the linear operators in Fourier space are

\[
\begin{align*}
\hat{L}_1 &= -\gamma |k|^2 \\
\hat{L}_2 &= |k| \\
\hat{L}_4(\tau) &= -\Gamma_0 |k|^2 - G_0 + f(\tau).
\end{align*}
\] (B.4a) (B.4b) (B.4c)

We follow [82] and discretize (B.3) in space and time, using a trapezoidal scheme for the linear terms and a second order Adams-Bashforth scheme for the nonlinear terms to obtain the following equations at each time step $n$:

\[
\begin{align*}
\frac{\hat{h}_k^{n+1} - \hat{h}_k^n}{\Delta \tau} &= \frac{1}{2} \hat{L}_1 (\hat{h}_k^{n+1} + \hat{h}_k^n) + \frac{1}{2} \hat{L}_2 (\hat{\Phi}_k^{n+1} + \hat{\Phi}_k^n) \\
&\quad + \frac{1}{2} (3\hat{F}_k^n - \hat{F}_k^{n-1}) \\
\frac{\hat{\Phi}_k^{n+1} - \hat{\Phi}_k^n}{\Delta \tau} &= \frac{1}{2} (\hat{L}_1 \hat{h}_k^{n+1} + \hat{L}_4(\tau) \hat{h}_k^n) + \frac{1}{2} \hat{L}_1 (\hat{\Phi}_k^{n+1} + \hat{\Phi}_k^n) \\
&\quad + \frac{1}{2} (3\hat{G}_k^n - \hat{G}_k^{n-1}).
\end{align*}
\] (B.5a) (B.5b)

From now on, we drop the subscripts, and it is implied that every equation represents a set of equations for the Fourier mode amplitudes.
After some algebra, (B.5) may be written as

\[ M_1 h^{n+1} - M_2 \Phi^{n+1} = R_1 \] (B.6a)
\[ M_1 \Phi^{n+1} - M_4^{n+1} h^{n+1} = R_2. \] (B.6b)

Here, the right-hand sides are

\[ R_1 = M_r h^n + M_2 \Phi^n + \frac{\Delta \tau}{2} (3\hat{F}^n - \hat{F}^{n-1}) \] (B.7a)
\[ R_2 = M_r \Phi^n + M_4 h^n + \frac{\Delta \tau}{2} (3\hat{G}^n - \hat{G}^{n-1}) \] (B.7b)

and the operators are

\[ M_1 = 1 - \frac{\Delta \tau}{2} \hat{L}_1 \] (B.8a)
\[ M_2 = \frac{\Delta \tau}{2} \hat{L}_2 \] (B.8b)
\[ M_r = 1 + \frac{\Delta \tau}{2} \hat{L}_1 \] (B.8c)
\[ M_4^n = \frac{\Delta \tau}{2} \hat{L}_4 (\tau) \] (B.8d)
\[ M_4^{n+1} = \frac{\Delta \tau}{2} \hat{L}_4 (\tau + \Delta \tau). \] (B.8e)

Solving (B.6), we find that

\[ h^{n+1} = P(R_1 + M_4^{-1} M_2 R_2) \] (B.9a)
\[ \Phi^{n+1} = P(M_4^{n+1} M_1^{-1} R_1 + R_2) \] (B.9b)
where

\[ P = (M_1 - M_4^{n+1}M_1^{-1}M_2)^{-1}. \]  \hspace{1cm} (B.10)

In the above, \( X^{-1} \) indicates the operator inverse of \( X \), \( e.g. \)

\[ M_1^{-1} = \frac{1}{1 + \frac{\Delta r \gamma}{2}|k|^2}. \]  \hspace{1cm} (B.11)

We have solved the system in a one-dimensional domain of length \( 8\lambda_c \) and in a two-dimensional box of size \( 8\lambda_c \times 8\lambda_c \), where \( \lambda_c = 2\pi/k_c \) is the critical wavelength determined from linear stability analysis. We use 8 modes per wavelength (the same spatial resolution as in [82]) for a total of 64 modes per axis. The initial conditions for this two-step algorithm are obtained as follows. We draw two random initial conditions for \( h \) from a normal distribution with mean 0 and variance \( 10^{-4} \) and then manually set the spatial average of each one to be 0 by subtracting an appropriate constant uniformly over the domain. We set the two initial conditions for \( \Phi \) to be uniformly zero. The algorithm is run in MATLAB, and the nonlinear terms are evaluated pseudospectrally using MATLAB’s built-in fast Fourier transform function. (See [96, 97] for introductions to pseudospectral methods.)

In [82] it was observed that oscillations at high wave numbers appear after sufficiently long integration times (see that work for a brief discussion). For the integration times we have used thus far, we have not observed this phenomenon, but nonetheless we implement the technique used in [82] and apply a low pass filter. (Our results given here have been computed both with and without the filter, and no perceptible difference was found.) At the end of each forcing period, the amplitude of each
Fourier mode in the \( h \) and \( \Phi \) fields is multiplied by a constant \( \Lambda(|k|) \) which is given by

\[
\Lambda(k) = \begin{cases} 
1 & \text{if } |k| \leq 3 \\
4(1 - \frac{|k|}{4}) & \text{if } 3 < |k| \leq 4 \\
0 & \text{if } |k| > 4.
\end{cases} 
\]  

(B.12)

When we implement the algorithm in MATLAB, we find that small errors accumulate due to roundoff error from the numerous fast Fourier transforms that are taken. To correct this problem, once per period we set the amplitude of the zero-wavenumber mode to be zero (this is equivalent to insisting that the fluid interface undergo no net translation). Additionally, once per period we take the inverse transform of the solution fields and discard the small imaginary part that has accumulated.

We have performed a comparison between the numerical simulation and the weakly nonlinear theory for one-dimensional waves; see Figure B.1. Weakly nonlinear theory predicts that the amplitude of the one-dimensional rolls will be equal to

\[
|Z|_{\text{roll}} = \sqrt{-\frac{\alpha \epsilon f_0}{A}}. 
\]  

(B.13)

Here, \( f_0 \) is the critical forcing amplitude computed from linear stability analysis and \( \alpha \) and \( A \) are coefficients in the amplitude equations (4.18) which are computed in Chapter 4. The parameter \( \epsilon = (f - f_0)/f_0 \) measures the distance above onset. Our computation was performed for single frequency forcing, i.e. \( m = 1 \) and \( \chi = 0^\circ \) in (3.29). The other parameters used were \( \Gamma_0 = 0.125 \) and \( \gamma = 0.2, 0.4 \) in (3.26). We used a time step of size \( \Delta \tau = 0.05 \). Checks were performed with twice as many time
Figure B.1: Comparison of roll amplitude $|Z|_{\text{roll}}$ versus control parameter $\epsilon$ as predicted by weakly nonlinear theory (lines) and as computed from numerical simulation (dots). The parameters are $m = 1$ and $\chi = 0^\circ$ in (3.29) and $\Gamma_0 = 0.125$ and $\gamma = 0.2, 0.4$ in (3.26).
steps to verify convergence.

We have also performed preliminary computations in two spatial dimensions, though as of yet we have not performed any systematic investigation. Our future work will include testing the predictions of Chapters 4 and 5 using our two-dimensional numerical simulation. See Chapter 7 for more details.
References


