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Decoupling of the general scalar field mode and the solution space for Bianchi type I and V cosmologies coupled to perfect fluid sources

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The scalar field degree of freedom in Einstein's plus matter field equations is decoupled for Bianchi type I and V general cosmological models. The source, apart from the minimally coupled scalar field with arbitrary potential $V(\Phi)$, is provided by a perfect fluid obeying a general equation of state $p=p(\rho)$. The resulting ODE is, by an appropriate choice of final time gauge affiliated to the scalar field, reduced to first order, and then the system is completely integrated for arbitrary choices of the potential and the equation of state. © 2006 American Institute of Physics.
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I. INTRODUCTION

In the past two decades there has been a growing interest in scalar field cosmological models primarily due to the prominent importance of scalar fields for inflationary scenarios.¹ The importance of the coupling between a scalar field and the gravitational field has been further stressed by Madsen,² who has shown that it can have nontrivial consequences for the spontaneous breaking of gauge symmetries. A dynamical systems approach has been extensively used in the study of scalar field cosmologies and their asymptotic behavior.³ See also Ref. 4 for a concise review.

As far as exact solutions of scalar field cosmologies are concerned, Burd and Barrow⁵ have studied homogeneous but anisotropic Bianchi models of types III and VI (as well as Kantowski-Sachs models) and have found exact solutions. Lidsey, and Aguirregabiria *et al.*⁶ have found exact solutions for Bianchi type I models. Feinstein and Ibanez⁷ have found exact solutions for Bianchi models of type III and VI, Moss and Wright, Madsen, and Abreut, Crawford, and Mimoso⁸ have studied exact solutions in the setting of conformal scalar field cosmologies. Paul⁹ has obtained exact solutions of a higher derivative theory in the presence of an interacting scalar field. The discovery of the BTZ black hole has motivated the study of analytic solutions in the context of scalar field cosmology in $(2+1)$ dimensions,¹⁰ while Russo¹¹ has obtained the general solution for

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a scalar field cosmology in d dimensions with exponential potentials and a flat Robertson-Walker metric. An early work on exponential potentials has been done by Salopek and Bond¹² while recent treatments are given by Kehagias and Kofinas¹³ and Neupane.¹⁴

Cosmological models containing both a fluid and a scalar field have also been studied. Chimento and Jakubi¹⁵ have given exact solutions of scalar field cosmologies with a perfect fluid and a viscous fluid, respectively. Mendez¹⁶ has obtained an exact solution for the case of an imperfect fluid in a FRW space-time. In the so-called scaling scalar field cosmologies, the energy density due to the scalar field is proportional to the energy density of the perfect fluid. Thus a number of spatially flat, isotropic models in which the energy density of the scalar field is proportional to that of the perfect fluid have been investigated.¹⁷ Billyard, Coley, and van den Hoogen¹⁸ have studied the stability of these scaling solutions within the class of spatially homogeneous cosmological models with a barotropic fluid matter content. Furthermore, they have studied the qualitative behavior of spatially homogeneous models with a barotropic fluid and a noninteracting scalar field with an exponential potential in the class of Bianchi type B models.¹⁹ Saha²⁰ has obtained exact solutions for a Bianchi type I model with a perfect fluid and dark energy content, while Chimento and Cossarini have studied exact solutions in 1+1 dimensions using an isotropic perfect fluid source.²¹

In a significant paper, Hawkins and Lidsey²² have shown that, for a flat FRW geometry, the dynamics of scalar field cosmologies with a perfect fluid matter content can be described by the nonlinear, Ermakov-Pinney equation (which, in turn, leads to tantalizing analogies with the dynamics of Bose-Einstein condensates), while an early work of Barrow also deserves mention in this context. Exact solutions have been obtained in this description.²³ Another kind of decoupling of the scalar field degree of freedom has been initiated in Ref. 24, where a Robertson-Walker background geometry minimally coupled to the scalar field has been investigated. By use of integrals of the motion and of the Klein-Gordon (KG) equation in the quadratic constraint equation, a single, higher-order nonlinear differential equation for the scalar field was obtained.

In the present work we generalize this decoupling for the case of general Bianchi type I and V geometries in the presence of a general perfect fluid source. The advantage is that the resulting ODE is fully integrated and this is achieved for arbitrary choices of the scalar field potential and the fluid equation of state. The paper is organized as follows. In Sec. II the geometry and the matter description as well as the governing equations for the system are presented. In Sec. III the decoupling of the scalar field degree of freedom is performed for Bianchi type I, the reduction of the resulting ODE is given and the system is completely integrated. The corresponding calculations for Bianchi type V are given in Sec. IV. Finally, the conclusions and a discussion of the results obtained are presented in Sec. V.

Throughout we use geometrized units, i.e., $c=8\pi G=1$, while $g_{\mu\nu}$ has the signature $(-, +, +, +)$.

II. THE GOVERNING EQUATIONS AND THE MATTER CONTENT

Our starting point is the line element for a spatially homogeneous geometry,

$$ds^2 = [N^\alpha(t)N_\alpha(t) - N^2(t)]dt^2 + 2N_\alpha(t)\sigma_i^\alpha(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma_i^\alpha(x)\sigma_j^\beta(x)dx^i dx^j, \quad (2.1)$$

where $\sigma_i^\alpha(x)$ are the basis one-forms satisfying $\sigma_{i,j}^\alpha(x) - \sigma_{j,i}^\alpha(x) = 2C_{\beta\gamma}^\alpha \sigma_j^\beta(x)\sigma_i^\gamma(x)$ (with $C_{\beta\gamma}^\alpha$ the structure constants of the Lie algebra of the corresponding three-dimensional isometry group of motion acting simply transitively on the spatial hypersurfaces of simultaneity), $N_\alpha(t)$ is the shift vector, $N(t)$ is the lapse function, and $\gamma_{\alpha\beta}(t)$ is the scale factor matrix of the Bianchi type examined.

As explained in Ref. 25 there are special general coordinate transformations mixing space and time in the new space coordinates, whose effect on the line element (2.1) is described by

$$\tilde{\gamma}_{\alpha\beta}(t) = \Lambda_\alpha^\mu \Lambda_\beta^\nu a_{\mu\nu}(t),$$

$$\tilde{N}(t) = N(t)$$

$$\tilde{N}_\alpha(t) = \Lambda_\alpha^\beta(t) [N_\beta(t) + P^\rho(t) \gamma_{\rho\beta}(t)], \quad (2.2)$$

where $\Lambda_\nu^\mu(t)$, $P^\rho(t)$ satisfy

$$\Lambda_\mu^\alpha C_{\beta\gamma}^\mu = \Lambda_\beta^\rho \Lambda_\gamma^\sigma C_{\rho\sigma}^\alpha, \quad (2.3)$$

$$2P^\mu C_{\mu\nu}^\alpha \Lambda_\beta^\nu = \dot{\Lambda}_\beta^\alpha \quad (2.4)$$

the overdot denoting differentiation with respect to time. Due to (2.3), $\Lambda_\beta^\alpha(t)$ belongs to the automorphism group of the Bianchi type in question and transformations (2.2) describe the gauge freedom of the emanating system of Einstein's field equations in the case of vacuum. So, the three arbitrary functions of time contained in $\Lambda_\beta^\alpha(t)$ and $P^\alpha(t)$ can be used to simplify the line element (2.1) and, thus, also the aforementioned equations. Automorphisms induced by general coordinate transformations have also been considered in Ref. 26, while the rigid gauge symmetries have been analyzed in Ref. 27. Time-dependent automorphisms, seen as tangent space transformations, have also been considered previously,^{28,29} while the first (known to us) reference of the relevance of automorphisms to a systematic analysis of Bianchi cosmologies goes back to 1962.³⁰

The above result also holds true when the matter content is such that the linear constraints G_i^0 do not acquire an extra term T_i^0 . For orthogonal perfect fluids and scalar fields depending only on time, T_i^0 is indeed zero and therefore we can use the gauge freedom (2.2)–(2.4) to diagonalize $\gamma_{\alpha\beta}(t)$ without losing generality. Therefore, the scale factor matrix is taken to be

$$\gamma_{\alpha\beta}(t) = \text{diag}(a^2, b^2, c^2) \quad (2.5)$$

while for the shift vector we have $N_\alpha(t)=0$. Our choice of time is specified by the gauge condition $N = \sqrt{\det \gamma_{\mu\nu}}$, which is frequently used due to the simplification of the Einstein tensor and whose importance for the decoupling is essential.

The matter content is a minimally coupled scalar field with an arbitrary potential $V(\Phi)$, thus having an energy-momentum tensor

$$T_{\mu\nu}^{(1)} = \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} (g^{\kappa\lambda} \Phi_{,\kappa} \Phi_{,\lambda} + 2V(\Phi)) \quad (2.6)$$

and a perfect fluid part

$$T_{\mu\nu}^{(2)} = (p + \rho) u_\mu u_\nu + p g_{\mu\nu}, \quad (2.7)$$

where u^μ is the unit four-velocity vector and a general equation of state $p=p(\rho)$ is adopted. The governing Einstein's field equations are taken to be

$$R_\nu^\mu - \frac{1}{2} \delta_\nu^\mu R = T_\nu^\mu, \quad (2.8)$$

where $T_{\mu\nu} = T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)}$, while the scalar field equation (Klein-Gordon) is

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) - V'(\Phi) = 0 \quad (2.9)$$

the prime, from now onwards, denoting differentiation with respect to the argument.

The “equation of motion” for the perfect fluid is the conservation of its energy-momentum tensor:

$$T_{\nu;\mu}^{(2)\mu} = 0. \quad (2.10)$$

(The scalar field energy-momentum tensor is separately conserved by virtue of the Klein-Gordon equation).

III. DECOUPLING OF THE SCALAR DEGREE AND THE SOLUTION SPACE FOR BIANCHI TYPE I

The basis one-forms are $\sigma_i^\alpha(x) = \delta_i^\alpha$ and, thus, with (2.5), zero shift and the chosen time gauge, the initial form of the metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -a^2 b^2 c^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & c^2 \end{pmatrix}. \quad (3.1)$$

Spatial homogeneity implies that $\Phi = \Phi(t)$ and $\rho = \rho(t)$. The nonzero components of the Einstein tensor G_ν^μ , and the energy-momentum tensors $T_\nu^{(1)\mu}$, $T_\nu^{(2)\mu}$ (all multiplied by $a^2 b^2 c^2$) are given by

$$G_0^0 = -\frac{\dot{a}\dot{b}}{ab} - \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc}, \quad (3.2)$$

$$G_1^1 = \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}^2}{c^2} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}, \quad (3.3)$$

$$G_2^2 = \frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}^2}{c^2} - \frac{\ddot{a}}{a} - \frac{\ddot{c}}{c}, \quad (3.4)$$

$$G_3^3 = \frac{\dot{a}^2}{a^2} + \frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} + \frac{\dot{a}\dot{c}}{ac} + \frac{\dot{b}\dot{c}}{bc} - \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b}, \quad (3.5)$$

$$T_0^{(1)0} = -a^2 b^2 c^2 V(\Phi) - \frac{1}{2} \dot{\Phi}^2, \quad (3.6)$$

$$T_1^{(1)1} = T_2^{(1)2} = T_3^{(1)3} = -a^2 b^2 c^2 V(\Phi) + \frac{1}{2} \dot{\Phi}^2, \quad (3.7)$$

$$T_0^{(2)0} = -a^2 b^2 c^2 \rho(t), \quad (3.8)$$

$$T_1^{(2)1} = T_2^{(2)2} = T_3^{(2)3} = a^2 b^2 c^2 p(\rho(t)). \quad (3.9)$$

Due to the equalities (3.7) and (3.9), we can subtract the corresponding Einstein's equations $G_\nu^\mu = T_\nu^{(1)\mu} + T_\nu^{(2)\mu}$, i.e., form the differences (all multiplied by $a^2 b^2 c^2$) $G_2^2 - G_1^1 = T_2^{(1)2} + T_2^{(2)2} - T_1^{(1)1} - T_1^{(2)1}$ and $G_3^3 - G_1^1 = T_3^{(1)3} + T_3^{(2)3} - T_1^{(1)1} - T_1^{(2)1}$ and get the following equations involving only the scale factors:

$$\frac{\dot{a}^2}{a^2} - \frac{\dot{b}^2}{b^2} - \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} = 0, \quad (3.10)$$

$$\frac{\dot{a}^2}{a^2} - \frac{\dot{c}^2}{c^2} - \frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} = 0. \quad (3.11)$$

These equations provide the following integrals of motion:

$$b(t) = e^{\lambda t} a(t), \quad (3.12)$$

$$c(t) = e^{\mu t} a(t), \quad (3.13)$$

leaving us with just one undetermined scale factor. At this stage the Klein-Gordon equation (2.9) becomes

$$-V'(\Phi) - \frac{e^{-2(\lambda+\mu)t} \ddot{\Phi}(t)}{a^6(t)} = 0. \quad (3.14)$$

Now, the conservation of the fluid energy-momentum tensor (continuity equation) gives

$$3\frac{\dot{a}}{a} + \frac{\dot{\rho}}{p(\rho) + \rho} + \lambda + \mu = 0. \quad (3.15)$$

It is straightforward to check that the time derivative of the quadratic constraint equation $G_0^0 = T_0^{(1)0} + T_0^{(2)0}$ is identically satisfied by virtue of the remaining spatial equation $G_1^1 = T_1^{(1)1} + T_1^{(2)1}$, the Klein-Gordon equation and the continuity equation [solved for $\ddot{a}(t)$, $\ddot{\Phi}(t)$, $\dot{\rho}(t)$], as expected from the consistency between the aforementioned constraint with the spatial Einstein and the matter equations. Therefore, the equations to be solved are the Klein-Gordon equation (3.14), the continuity equation (3.15) and the constraint equation which, upon multiplication by $-2a^6 e^{2(\lambda+\mu)t}$ reads

$$6\left(\frac{\dot{a}}{a}\right)^2 + 4(\lambda + \mu)\frac{\dot{a}}{a} + 2\lambda\mu - \dot{\Phi}^2 - 2e^{2(\lambda+\mu)t} a^6 [V(\Phi) + \rho] = 0. \quad (3.16)$$

In order to have a closed form for the integral of the continuity Eq. (3.15), it is convenient to use the parametrization

$$p(\rho) = \frac{g(\rho)}{g'(\rho)} - \rho \quad (3.17)$$

(the prime denoting differentiation with respect to the argument) through the use of which one obtains the integral

$$\rho = h[\rho_0 a^{-3} e^{-(\lambda+\mu)t}] \quad (3.18)$$

with h being the inverse function to g , i.e., satisfying $h(g(x)) = x$.

From the Klein-Gordon equation (3.14) the scale factor and its logarithmic derivative is expressed in terms of the scalar field $\Phi(t)$ and its derivatives as well as $V(\Phi)$,

$$a = \left[-\frac{e^{-2(\lambda+\mu)t} \ddot{\Phi}}{V'(\Phi)} \right]^{1/6}, \quad (3.19)$$

$$\frac{\dot{a}}{a} = \frac{1}{6} \left(\frac{\ddot{\Phi}}{\ddot{\Phi}} - \frac{V''(\Phi)}{V'(\Phi)} \dot{\Phi} - 2(\lambda + \mu) \right). \quad (3.20)$$

The promised decoupling of the scalar field dynamics from the geometry occurs upon inserting (3.18)–(3.20) into (3.16). The result is the following nonlinear, third order ODE for $\Phi(t)$:

$$\left(\frac{\ddot{\Phi}}{\ddot{\Phi}} - \frac{V''(\Phi)}{V'(\Phi)} \dot{\Phi} \right)^2 - 4(\lambda + \mu)^2 + 12\lambda\mu - 6\dot{\Phi}^2 + 12\frac{\ddot{\Phi}}{V'(\Phi)} V(\Phi) + 12\frac{\ddot{\Phi}}{V'(\Phi)} h \left[\rho_0 \left(-\frac{\ddot{\Phi}}{V'(\Phi)} \right)^{-1/2} \right] = 0. \quad (3.21)$$

At this stage, any solution to this equation determines, through (3.18) and (3.19) a corresponding solution to the full Einstein plus matter system whose entire space of solution is therefore attained

from the solution space of (3.21). Of course, the price paid, for the moment, is the nonlinearity in the highest time derivative, which has also been raised to third order. Furthermore, normally one would expect that the arbitrary functions h , V need first to be specified before hoping to actually get a solution. Nevertheless, it is quite interesting that further reduction of the order of (3.21), and subsequent complete integration of the whole system, is possible. To this end, we first observe that many terms in (3.21), namely all the nontrivial except the fourth and fifth, are functions of the combination $-\ddot{\Phi}/V'(\Phi)$. We thus define

$$\chi = -\frac{\ddot{\Phi}}{V'(\Phi)} \quad (3.22)$$

and write (3.21) as

$$\left(\frac{\dot{\chi}}{\chi}\right)^2 - 4(\lambda + \mu)^2 + 12\lambda\mu - 6f(\chi) - 12\chi h(\rho_0\chi^{-1/2}) = 0, \quad (3.23)$$

where

$$f(\chi) \equiv \dot{\Phi}^2 + 2\chi V(\Phi) \quad (3.24)$$

implicitly reflects the arbitrariness in choosing $V(\Phi)$. Now (3.23) can be integrated and, by judicious choices for f , h , even give $\chi(t)$ in closed form. Suppose $\chi(t)$ does indeed solve (3.23). Then, multiplying (3.22) by $2\dot{\Phi}V'(\Phi)$ and using (3.24) we get the first order linear differential equation for $f(t)$:

$$\frac{df(t)}{dt} = 2\dot{\chi}(t)V(t) \quad (3.25)$$

and consequently $\Phi(t)$ is given by

$$\frac{d\Phi(t)}{dt} = \pm \sqrt{f(\chi(t)) - 2\chi(t)V(t)}. \quad (3.26)$$

Finally, a change of time variable from t to χ [defined in (3.22)] permits the presentation of the entire space of solutions to the system under consideration in closed form. Indeed, considering V , f , Φ as functions of χ one gets from (3.25) and (3.26), respectively,

$$\frac{df}{d\chi} = 2V(\chi), \quad (3.27)$$

$$\frac{d\Phi(\chi)}{d\chi} \dot{\chi}(t) = \pm \sqrt{f(\chi) - 2\chi V(\chi)}, \quad (3.28)$$

where the time derivative of χ is given by the constraint (3.23). The integration of these two equations is trivial, yielding f , Φ as

$$f(\chi) = \sigma + 2 \int V(\chi) d\chi, \quad (3.29)$$

$$\Phi(\chi) = \kappa \pm \int \frac{1}{\chi} \sqrt{\frac{f(\chi) - 2\chi V(\chi)}{4(\lambda + \mu)^2 - 12\lambda\mu + 6f(\chi) + 12\chi h(\rho_0\chi^{-1/2})}} d\chi. \quad (3.30)$$

The line element in the new time χ reads

$$ds^2 = - \frac{d\chi^2}{\chi[4(\lambda + \mu)^2 - 12\lambda\mu + 6f(\chi) + 12\chi h(\rho_0\chi^{-1/2})]} + [e^{-2(\lambda+\mu)t(\chi)}\chi]^{1/3} dx^2 + [e^{(4\lambda-2\mu)t(\chi)}\chi]^{1/3} dy^2 + [e^{(-2\lambda+4\mu)t(\chi)}\chi]^{1/3} dz^2 \quad (3.31)$$

with $t(\chi)$ given by the integral form of (3.23),

$$t(\chi) = \pm \int \frac{1}{\chi \sqrt{4(\lambda + \mu)^2 - 12\lambda\mu + 6f(\chi) + 12\chi h(\rho_0\chi^{-1/2})}} d\chi \quad (3.32)$$

while the density and pressure are given as

$$\rho(\chi) = h(\rho_0\chi^{-1/2}), \quad p(\chi) = \rho_0\chi^{-1/2}h'(\rho_0\chi^{-1/2}) - h(\rho_0\chi^{-1/2}) \quad (3.33)$$

the prime denoting differentiation with respect to the argument. Quite independently of the way these solutions were obtained, one can straightforwardly check (through, say, a symbolic computing facility) that they do satisfy all 10 Einstein's equations, the generalized KG and the continuity equation. Furthermore, since no extra ansatz has been involved in the process of integration of the system (2.8)–(2.10), equations (3.29)–(3.33) represent the full space of solutions to the Einstein plus matter system considered. The functions V , h can be freely specified to obtain special case solutions.

IV. DECOUPLING OF THE SCALAR DEGREE AND THE SOLUTION SPACE FOR THE BIANCHI TYPE V

In this case the basis one-forms are $\sigma^1 = e^{-x} dy$, $\sigma^2 = e^{-x} dz$, $\sigma^3 = dx$. The $G_1^0 = 0$ Einstein equation [due to spatial homogeneity $\Phi = \Phi(t)$ and $\rho = \rho(t)$ and therefore there is no corresponding component of the matter tensor] implies $c = ab$. Thus the initial metric is taken as

$$g_{\mu\nu} = \begin{pmatrix} -a^3b^3 & 0 & 0 & 0 \\ 0 & ab & 0 & 0 \\ 0 & 0 & a^2e^{-2x} & 0 \\ 0 & 0 & 0 & b^2e^{-2x} \end{pmatrix}. \quad (4.1)$$

The nonzero components of the Einstein tensor G_ν^μ , and the energy-momentum tensors $T_\nu^{(1)\mu}$, $T_\nu^{(2)\mu}$, all multiplied by a^3b^3 , are given by

$$G_0^0 = 3a^2b^2 - \frac{\dot{a}^2}{2a^2} - \frac{2\dot{a}\dot{b}}{ab} - \frac{\dot{b}^2}{2b^2}, \quad (4.2)$$

$$G_1^1 = a^2b^2 + \frac{3\dot{a}^2}{2a^2} + \frac{2\dot{a}\dot{b}}{ab} + \frac{3\dot{b}^2}{2b^2} - \frac{\ddot{a}}{a} - \frac{\ddot{b}}{b}, \quad (4.3)$$

$$G_2^2 = a^2b^2 + \frac{\dot{a}^2}{a^2} + \frac{2\dot{a}\dot{b}}{ab} + \frac{2\dot{b}^2}{b^2} - \frac{\ddot{a}}{2a} - \frac{3\ddot{b}}{2b}, \quad (4.4)$$

$$G_3^3 = a^2b^2 + \frac{2\dot{a}^2}{a^2} + \frac{2\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} - \frac{3\ddot{a}}{2a} - \frac{\ddot{b}}{2b}, \quad (4.5)$$

$$T_0^{(1)0} = -a^3b^3V(\Phi) - \frac{1}{2}\dot{\Phi}^2, \quad (4.6)$$

$$T_1^{(1)1} = T_2^{(1)2} = T_3^{(1)3} = -a^3 b^3 V(\Phi) + \frac{1}{2} \dot{\Phi}^2, \quad (4.7)$$

$$T_0^{(2)0} = -a^3 b^3 \rho(t), \quad (4.8)$$

$$T_1^{(2)1} = T_2^{(2)2} = T_3^{(2)3} = a^3 b^3 p(\rho(t)). \quad (4.9)$$

The situation is similar to the type I case, and thus forming the difference (multiplied by $a^3 b^3$) $G_2 - G_1 = T_2^{(1)2} + T_2^{(2)2} - T_1^{(1)1} - T_1^{(2)1}$ we get the following equation involving only the scale factors:

$$-\frac{\dot{a}^2}{2a^2} + \frac{\dot{b}^2}{2b^2} + \frac{\ddot{a}}{2a} - \frac{\ddot{b}}{2b} = 0, \quad (4.10)$$

which can be integrated yielding

$$b(t) = e^{\lambda t} a(t). \quad (4.11)$$

The Klein-Gordon equation (2.9) becomes

$$-V'(\Phi) - \frac{e^{-3\lambda t} \ddot{\Phi}(t)}{a^6(t)} = 0 \quad (4.12)$$

while the conservation of the fluid energy-momentum tensor (continuity equation) gives

$$3\frac{\dot{a}}{a} + \frac{\dot{\rho}}{p(\rho) + \rho} + \frac{3}{2}\lambda = 0. \quad (4.13)$$

Again, the only other equation to be solved is the constraint equation which reads

$$6\left(\frac{\dot{a}}{a}\right)^2 + 6\lambda\frac{\dot{a}}{a} + \lambda^2 - 6e^{2\lambda t} a^4 - \dot{\Phi}^2 - 2e^{3\lambda t} a^6 (V(\Phi) + \rho) = 0, \quad (4.14)$$

where the fourth term is the only nontrivial difference from the corresponding equation (3.16) and its presence is due to the nonvanishing curvature of the spatial slice.

Integrating (4.13) [in the parametrization (3.17)] and solving (4.12) for $a(t)$, we obtain the following results for the matter density $\rho(t)$, the scale factor $a(t)$ and its logarithmic derivative

$$\rho = h(\rho_0 a^{-3} e^{-(3/2)\lambda t}), \quad (4.15)$$

$$a = \left(-\frac{e^{-3\lambda t} \ddot{\Phi}}{V'(\Phi)} \right)^{1/6}, \quad (4.16)$$

$$\frac{\dot{a}}{a} = \frac{1}{6} \left(\frac{\ddot{\Phi}}{\ddot{\Phi}} - \frac{V''(\Phi)}{V'(\Phi)} \dot{\Phi} - 3\lambda \right). \quad (4.17)$$

Use of these equations in the quadratic constraint equation (4.14) (multiplied by $-2e^{3\lambda t} a^6$) yields

$$\begin{aligned} & \left(\frac{\ddot{\Phi}}{\ddot{\Phi}} - \frac{V''(\Phi)}{V'(\Phi)} \dot{\Phi} \right)^2 - 3\lambda^2 - 36 \left(-\frac{\ddot{\Phi}}{V'(\Phi)} \right)^{2/3} - 6\dot{\Phi}^2 + 12 \frac{\ddot{\Phi}}{V'(\Phi)} V(\Phi) \\ & + 12 \frac{\ddot{\Phi}}{V'(\Phi)} h \left(\rho_0 \left(-\frac{\ddot{\Phi}}{V'(\Phi)} \right)^{-1/2} \right) = 0 \end{aligned} \quad (4.18)$$

which, with the same definition of χ , translates into

$$\left(\frac{\dot{\chi}}{\chi}\right)^2 - 3\lambda^2 - 36\chi^{2/3} - 6f(\chi) - 12\chi h(\rho_0\chi^{-1/2}) = 0. \quad (4.19)$$

By arguments completely analogous to the previous type I case, the final form of the solution is, in this case

$$f(\chi) = \sigma + 2 \int V(\chi) d\chi, \quad (4.20)$$

$$\Phi(\chi) = \kappa \pm \int \frac{1}{\chi} \sqrt{\frac{f(\chi) - 2\chi V(\chi)}{3\lambda^2 + 36\chi^{2/3} + 6f(\chi) + 12\chi h(\rho_0\chi^{-1/2})}} d\chi, \quad (4.21)$$

$$ds^2 = - \frac{d\chi^2}{\chi[3\lambda^2 + 36\chi^{2/3} + 6f(\chi) + 12\chi h(\rho_0\chi^{-1/2})]} + \chi^{1/3} dx^2 + e^{-\lambda t(\chi)-2x} \chi^{1/3} dy^2 + e^{\lambda t(\chi)-2x} \chi^{1/3} dz^2 \quad (4.22)$$

with $t(\chi)$ given by the integral form of (4.19),

$$t(\chi) = \pm \int \frac{1}{\chi \sqrt{3\lambda^2 + 36\chi^{2/3} + 6f(\chi) + 12\chi h(\rho_0\chi^{-1/2})}} d\chi, \quad (4.23)$$

while the density and pressure are given as

$$\rho(\chi) = h(\rho_0\chi^{-1/2}), \quad p(\chi) = \rho_0\chi^{-1/2}h'(\rho_0\chi^{-1/2}) - h(\rho_0\chi^{-1/2}). \quad (4.24)$$

The remarks at the end of the preceding section apply also here.

V. DISCUSSION AND CONCLUSIONS

We have discussed the dynamics of a scalar field with an arbitrary potential, minimally coupled to a general (anisotropic) Bianchi type I and V geometry, in the presence of a perfect fluid source obeying a general equation of state. In the case of vacuum, the rich structure of outer automorphisms for these two symmetry groups entails the existence of some integrals of motion: Indeed, consider the generators of the rigid symmetries (2.3), i.e., the vector fields in the space of dependent variables $X_I = \lambda_{I\alpha}^\rho \gamma_{\rho\beta} (\partial / \partial \gamma_{\alpha\beta})$ where λ_β^α satisfy $\lambda_{I\rho}^\alpha C_{\beta\gamma}^\rho = \lambda_{I\beta}^\rho C_{\rho\gamma}^\alpha + \lambda_{I\gamma}^\rho C_{\beta\rho}^\alpha$. If one performs the appropriate change of variables $\gamma_{\alpha\beta} \rightarrow z_{\alpha\beta}$ which brings one (or more, if possible) generator into each canonical form, say $\partial / \partial z_{11}$, then Einstein's field equations written in the new variables do not explicitly depend on z_{11} . Thus the system becomes of first order in the variable $\dot{z}_{11}$ and therefore an integral of motion arises.

The initial choice of the time gauge in which the lapse is equal to the determinant of the scale factor matrix has a twofold advantage: First, it enables the corresponding integrals of motion (in the presence of the matter content chosen) to be revealed. Second, it makes the Klein-Gordon equation purely algebraic in the scale factor variables. As a result, when the utilization of the integrals of motion has reduced the number of these variables to one, this equation gives this last scale factor as a function of the second derivative of the scalar field, the derivative of the potential with respect to the scalar field, and of time. Substitution of this form of the scale factor into the only equation remaining to be solved, i.e., the quadratic constraint, results in a single ODE for the scalar field (without any explicit time dependence). The utilization of a final time gauge adapted to the scalar field enables the reduction of this equation to first order and subsequently leads to the complete integration of the entire system of Einstein plus matter field equations. The description of the space of solutions contains, in an integral form, the arbitrary functions of the final time χ , $V(\chi)$, and $h(\rho_0\chi^{-1/2})$. The presence of this twofold arbitrariness corresponds to the fact that we have not specified either the form of the potential (as a function of the scalar field) or the equation of state. It is evident that prescribing $V(\chi)$ and $h(\rho_0\chi^{-1/2})$ implicitly corresponds to a choice of

potential form and equation of state. For example, the choice $h=A(\rho_0\chi^{-1/2})^{1+\gamma}$ gives through (4.24) $p=\gamma\rho$, i.e., the barotropic equation of state. For the scalar field, if we take the particular case $\lambda=\mu=\sigma=h(\rho_0\chi^{-1/2})=0$ (i.e., flat Robertson-Walker with no fluid) the choice $V(\chi)=C/\sqrt{\chi}$ corresponds to the functional form $V=\pm Ce^{\sqrt{3}(\kappa-\Phi)}$. If, on the other hand, someone insists in prescribing $V(\Phi)$ and $p(\rho):=p(h)$ then the situation must be dealt with in the following manner: As far as the density is concerned, the equation giving the pressure becomes the holonomic, first order differential equation $p(h)=wh'(w)-h(w)$ with w standing for $\rho_0\chi^{-1/2}$ which can be straightforwardly integrated. For example, a barotropic equation $p:=\gamma\rho$ gives $\rho=A(\rho_0\chi^{-1/2})^{1+\gamma}$. As far as the matter field is concerned, the situation is somewhat more complicated as the results of choosing a particular form $V(\Phi)$ are influenced by the choice of the density h . As an example consider again the case $\lambda=\mu=\sigma=h(\rho_0\chi^{-1/2})=0$ and an arbitrary potential form $V(\Phi)$. Relaxing for a moment (3.30), we get the following Klein-Gordon equation

$$-\frac{V'(\Phi)}{V(\Phi)} + \frac{6\chi}{6\chi^2\Phi'^2(\chi)}[2\chi\Phi''(\chi) - 6\chi^2\Phi'^3(\chi) + 3\Phi'(\chi)] = 0,$$

which may be difficult to solve depending on the choice of $V(\Phi)$. This is the price paid for insisting on prescribing $V(\Phi)$ and not $V(\chi)$ in which case the solution would be given by (3.29)–(3.33) or (4.20)–(4.24) correspondingly. The particular choice $V=V_0e^{-\lambda\Phi}$ corresponds to the case considered in Ref. 11 and the equation above can be dealt with by choosing a new time $\chi=e^{3\pi/2}$, in which the equation becomes the following first order in the derivative $\omega\equiv d\Phi/d\tau$:

$$48\frac{d\omega}{d\tau} - 96\omega^3 + 24\lambda\omega^2 - 9\lambda = 0.$$

Finally, there are two cases which, at first sight, need to be separately examined. The first concerns the case of a constant potential $V(\Phi)\equiv V_0$, i.e., a cosmological constant term. Then, $V'(\Phi)=0$ and the definition of the final time χ in Eq. (3.32) seems to be precarious. However, the Klein-Gordon Eq. (3.14) then implies that also $\ddot{\Phi}=0$ and, surprisingly enough, χ does exist. Indeed, substituting $V(\Phi)\equiv V_0$ in the solutions (3.29)–(3.33) or (4.20)–(4.24) one can see that the solution is still valid. This holds true even for $V_0=0$. The second case arises when the ratio $\ddot{\Phi}/V'(\Phi)=-1/c^6$, for, then, the change of time from t to χ is not valid. In this case, the remaining scale factor is given by $a(t)=ce^{-(\lambda+\mu)t/3}$ in the type I case or $a(t)=ce^{-\lambda t/2}$ in the type V case. The continuity equation [(3.15) or (4.12), respectively] implies that the pressure and density are constants, say, p_0 and ρ_0 . The remaining equations dictate $\Phi(t)=At+B$ and $V(\Phi)=V_0$ and there exists a relation between c , p_0 , ρ_0 , A , and B due to the quadratic constraint.

The general properties of the space of solutions found for both type I and type V cases, e.g., isotropization, attractors and self-similarity, and quintessence, will be examined in a forthcoming paper, in which also particular cases of high interest are to be explicitly elaborated. Possible applications of the method exhibited here can be the cases of $D+1$ spatially homogeneous space-times, in which the outer automorphisms are rich enough to provide sufficient integrals of motion. Then the method will be applicable if the matter content is such that the integrals of motion persist.

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