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In the framework of a one-dimensional model with a tightly localized self-attractive nonlinearity, we study the formation and transfer (dragging) of a trapped mode by “nonlinear tweezers,” as well as the scattering of coherent linear wave packets on the stationary localized nonlinearity. The use of a nonlinear trap for dragging allows one to pick up and transfer the relevant structures without grabbing surrounding “radiation.” A stability border for the dragged modes is identified by means of analytical estimates and systematic simulations. In the framework of the scattering problem, the shares of trapped, reflected, and transmitted wave fields are found. Quasi-Airy stationary modes with a divergent norm, which may be dragged by a nonlinear trap moving at a constant acceleration, are briefly considered too.

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I. INTRODUCTION

Solitons, in the form of robust localized pulses propagating in nonlinear dispersive media, have been studied extensively in various physical contexts [1]. Many works have dealt with soliton dynamics in media featuring uniform nonlinearities, modeled by wave equations with constant nonlinearity coefficients. More recently, much attention has been drawn to the study of solitons in structured media, which feature spatial modulations of the local nonlinearity strength (see [2] for a recent review), including the case when stable *bright* solitons may be supported by *self-defocusing* nonlinearity whose strength grows toward the periphery [3]. In solid-state physics, effective potentials induced by nonlinearity modulation are often called *pseudopotentials* [4]. Recently developed experimental techniques make it possible to realize such structures in photonics and Bose-Einstein condensates (BECs). Particularly, in photonic media, the modulation profiles can be induced by nonuniform distributions of resonant dopants affecting local nonlinearity, which may be created by means of in-diffusion technology [5]. Another possibility is to use photonic crystals with voids infiltrated by liquids, which are matched to the host material in terms of the refractive index but feature a mismatch in the Kerr coefficient [6]. On the other hand, in the context of BECs, the local nonlinearity can be spatially modulated by means of the Feshbach resonance, induced by nonuniform magnetic [7] or electric [8] fields, as well as by an appropriate optical field [9].

One-dimensional (1D) solitons were studied theoretically in the context of pseudopotential structures in BECs [10,11] and optics [3,12]. The possible use of 1D nonlinearity-modulation profiles was also elaborated in models of matter-wave lasers, designed so as to release periodic arrays of coherent pulses [13,14] (although the first experimentally realized prototypes of atom lasers were built in a different way [15]). Self-focusing pseudopotentials were theoretically studied in the

two-dimensional (2D) geometry too, demonstrating that it is much more difficult (but possible) to stabilize 2D solitons in such settings than using familiar linear potentials or lattices [16–18], while the above-mentioned self-defocusing structures solve this problem easily [3].

The simplest example of a pseudopotential corresponds to nonlinearity concentrated at a single point, which is accounted for by the δ function [19]. In optics, it may represent a planar waveguide with a narrow nonlinear stripe embedded in it (stable spatial solitons supported by narrow stripes carrying quadratic, rather than cubic, nonlinearity were found in Refs. [20] and [21]). In BECs, localized nonlinearity may be imposed by a tightly focused laser beam inducing the Feshbach resonance. A relevant extension of such a nonlinear spot is the model of the *double-well* pseudopotential; in the simplest form, it is based on a symmetric set of two δ functions multiplying the cubic term in the respective nonlinear Schrödinger (NLS) equation [22] or the quadratic nonlinearity in the model of the second-harmonic generation [21]. In either case, full analytical solutions have been found for symmetric, asymmetric, and antisymmetric localized modes trapped by the set of two δ functions. In addition to that, full symmetric and asymmetric solutions can be found [23] for a similar discrete model, with two cubic-nonlinear sites embedded in a linear host chain, which was introduced (without consideration of spontaneous symmetry breaking) in Refs. [24]. The symmetry-breaking problem was also recently analyzed in the 2D setting, for a symmetric pair of nonlinear circles embedded in a linear host medium [18].

A model with a single nonlinear δ function readily gives rise to a family of exact soliton solutions [see Eq. (5) below] that are completely unstable but may be stabilized if the point-like nonlinearity is combined with a periodic linear potential (in the same setting, localized nonlinearity with a repulsive sign may support stable gap solitons) [11]. In the present work, we demonstrate that nonlinear self-trapped states may also be stabilized, without adding any linear potential, by regularized localized nonlinearity, if the ideal δ function is replaced by a Gaussian profile.

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In addition to supporting stationary trapped modes, localized potentials may be used as *tweezers*, for transfer of trapped modes, which is a topic with many potential applications [25]. In particular, the use of δ -functional linear trapping potentials makes it possible to find exact solutions for various transfer problems [26,27]. The dynamical process of extracting matter-wave pulses from a BEC reservoir with nonlinear tweezers, induced by an appropriate laser beam, was simulated in detail in Ref. [14].

The main objective of the present work is to analyze possibilities of controlled transfer of localized wave packets by a moving nonlinear trapping pseudopotential (*nonlinear tweezers*), represented by a narrow Gaussian profile multiplying the self-focusing cubic nonlinearity. An advantage of using nonlinear tweezers is that they would not grab and drag small-amplitude “garbage” (radiation) surrounding the target object in a realistic setting, and one may use the amplitude of the object to control its transfer (this resembles the well-known advantage of nonlinear optical amplifiers in comparison with their linear counterparts; see Ref. [28] and references therein). The model is introduced in Sec. II, where we also produce simple analytical results for the transfer problem, obtained by means of the adiabatic approximation, which is appropriate for a slowly moving trap. Basic numerical results are reported in Sec. III. We also briefly consider the related problem of dragging wave patterns with a divergent norm by a nonlinear trap moving at a constant acceleration, which is relevant in connection with the recently considered transmission regimes for Airy beams, exact [29] or approximate [30], including their nonlinear generalizations [31], and the transfer of linear trapped wave packets by a potential well moving at a constant acceleration [32]. In Sec. IV, we report results for another natural problem related to the present setting, namely, the scattering of linear wave packets on a localized stationary attractive nonlinearity. Section V concludes the paper.

II. THE MODEL AND ADIABATIC APPROXIMATION

A. The formulation

The model with a tightly localized nonlinear trap, which represents moving tweezers, is based on the following normalized version of the NLS equation for wave function $\psi(x, t)$,

$$i\psi_t = -\frac{1}{2}\psi_{xx} - \tilde{\delta}(x - \xi(t))|\psi|^2\psi, \quad (1)$$

where subscripts denote partial derivatives, while $\tilde{\delta}(x)$ and $\xi(t)$ define the shape of the nonlinear trap and its law of motion. In the context of BECs, Eq. (1) is a normalized version of the Gross-Pitaevskii equation, with time t and coordinate x [note that in an early work [19], Eq. (1) was introduced as a model for tunneling of attractively interacting particles through a junction]. In the optical model, t is replaced by the propagation distance (z), while x is the transverse coordinate in the respective planar waveguide.

In the first version of the model, the nonlinear trap was taken in the form of the ideal δ function, $\tilde{\delta}(x) = \delta(x)$ [19]. In the present work, numerical results are reported for its practically relevant Gaussian regularization, namely,

$$\tilde{\delta}(x) = (2\sqrt{\pi\epsilon})^{-1} \exp(-x^2/(4\epsilon)), \quad (2)$$

with a sufficiently small width ϵ . The law of motion for the moving trap will be taken as

$$\xi(t) = \begin{cases} 0 & \text{at } t \leq 0, \\ \frac{1}{2}\Xi[1 + \sin(\pi \frac{t-T/2}{T})] & \text{at } 0 < t < T, \\ \Xi & \text{at } t \geq T, \end{cases} \quad (3)$$

which implies that the trap is smoothly transferred from the initial position, $\xi = 0$, starting at $t = 0$, to the final position, $\xi = \Xi$, at $t = T$. Both the initial and final velocities corresponding to this law of motion are 0, i.e., $d\xi/dt(t = 0) = d\xi/dt(t = T) = 0$.

B. Pulse solutions

The model with a quiescent trap ($\xi = 0$), described by the ideal δ function, allows one to construct solutions to Eq. (1) as a combination of two solutions of the linear Schrödinger equation in free space at $x < 0$ and $x > 0$, coupled by the derivative-jump condition at $x = 0$ (while the wave function itself must be continuous at this point):

$$\psi_x(x = +0) - \psi_x(x = -0) = -2|\psi(x = 0)|^2\psi(x = 0). \quad (4)$$

This condition gives rise to a family of obvious solutions to Eq. (1) with $\tilde{\delta}(x) = \delta(x)$ and $\xi = 0$:

$$\psi = (-2\mu)^{1/4} e^{-i\mu t} e^{-\sqrt{-2\mu}|x|}, \quad (5)$$

where $\mu < 0$ is an arbitrary chemical potential. Such soliton-like solutions, with a discontinuous first derivative, are usually called *peakons* (see, e.g., Refs. [33]). Note that the norm of the peakon family *does not* depend on the chemical potential, namely,

$$N \equiv \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1. \quad (6)$$

This degeneracy resembles the well-known feature of *Townes solitons* in the 2D NLS equation with uniform nonlinearity [34]. Accordingly, application of the Vakhitov-Kolokolov (VK) stability criterion, $dN/d\mu < 0$ [34,35], formally predicts the neutral stability of the peakon family; nevertheless, a numerical study demonstrates that the entire family is unstable [11], the instability being similar to that of Townes solitons in two dimensions [34]. As mentioned above, peakons can be stabilized by the addition of a linear periodic potential (with the nonlinear δ function placed at an arbitrary position with respect to the potential [11]). In the next section, we demonstrate that stabilization may also be induced by the regularization of the δ function, as per Eq. (2). Generally speaking, it is easy to stabilize the family of Townes-like solitons, because linearization of the underlying equation with respect to small perturbations around the soliton does not give rise to any unstable eigenvalue; in fact, the instability is nonlinear, i.e., it grows not exponentially but, rather, as a power of time, and is represented by a respective pair of zero eigenvalues. Therefore, any modification of the equation that shifts the zero eigenvalues in the direction of real frequencies may stabilize the entire family. This will be the case in our considerations in Sec. III, with a finite-width Gaussian (2) replacing the δ function.

Assuming that a stabilization mechanism is in action, but the shape of the soliton does not substantially deviate from Eq. (5), it is natural to expect that the trap, slowly moving according to given $\xi(t)$ [see, e.g., Eq. (3)], may *drag* the trapped soliton, which will then be described by the following modification of solution (5), in the adiabatic approximation:

$$\psi = (-2\mu)^{1/4} \exp \left[-i\mu t + \frac{i}{2} \int \left(\frac{d\xi}{dt} \right)^2 dt + i \frac{d\xi}{dt} x \right] \times \exp[-\sqrt{-2\mu}|x - \xi(t)|]. \quad (7)$$

In fact, expression (7) is the usually defined *Galilean boost* of the peakon, (5), moving at instantaneous velocity $d\xi/dt$.

C. Airy waves

For consideration of a nonlinear trap moving with constant acceleration W , we set $\xi(t) = (W/2)t^2$. Then Eq. (1) can be rewritten in an accelerating reference frame moving along with the trap, i.e., in terms of the following variables:

$$\psi(x, t) \equiv (W/4)^{1/6} \phi(z, \tau) \exp[iWt(x - Wt^2/3)], \quad (8)$$

$$z \equiv (2W)^{1/3}(x - Wt^2/2), \quad \tau \equiv (W^2/2)^{1/3}t. \quad (9)$$

The accordingly transformed Eq. (1) reads

$$i\phi_\tau + \phi_{zz} - z\phi + \delta(z - z_0)|\phi|^2\phi = 0, \quad (10)$$

where z_0 is the location of the nonlinear trap with respect to the accelerating reference frame. Stationary solutions to Eq. (10) are given by $\phi(z, \tau) \equiv a(z)$, with the real function $a(z)$ obeying the following equation:

$$\frac{d^2a}{dz^2} - za + \delta(z - z_0)a^3 = 0. \quad (11)$$

Without the nonlinear term, Eq. (11) is the classical Airy equation, whose relevant solutions at $z < z_0$ and $z > z_0$ are, respectively, given by

$$a_-(z) = C_1 \text{Ai}(z) + C_2 \text{Bi}(z), a_+(z) = C_3 \text{Ai}(z). \quad (12)$$

Here, C_1 and C_2 are constants, and $\text{Ai}(z)$ and $\text{Bi}(z)$ are the standard Airy functions, which are defined by their asymptotic forms at $z \rightarrow -\infty$ and $z \rightarrow +\infty$ [36]:

$$\{\text{Ai}, \text{Bi}\}(z) \approx \frac{1}{\sqrt{\pi}(-z)^{1/4}} \{\sin, \cos\} \left(\frac{\pi}{4} + \frac{2}{3}(-z)^{3/2} \right), \quad (13)$$

$$\{\text{Ai}, \text{Bi}\}(z) \approx \frac{1}{2\sqrt{\pi}z^{1/4}} \exp \left(\mp \frac{2}{3}z^{3/2} \right). \quad (14)$$

Using the Wronskian of the Airy functions, which is equal to $1/\pi$ [36], the continuity condition for $\phi(z)$ and jump relation (4) at $z = z_0$ give coefficients $C_{1,2,3}$ [see Eq. (12)] in terms of the amplitude of the solution, $A \equiv a(z = z_0)$, and z_0 itself, which may be considered two free parameters of the solution family:

$$\begin{aligned} C_1 &= [(\text{Ai}(z_0))^{-1} - \pi \text{Bi}(z_0)A^2]A, \\ C_2 &= \pi \text{Ai}(z_0)A^3, \\ C_3 &= A(\text{Ai}(z_0))^{-1}. \end{aligned} \quad (15)$$

The known peculiarity of solutions based on Airy functions is the divergence of the norm in the tail at $z \rightarrow -\infty$, as the corresponding average density produced by Eqs. (12) and (13),

$$\langle a^2(z) \rangle \approx \frac{C_1^2 + C_2^2}{2\pi\sqrt{-z}}, \quad (16)$$

decays too slowly. In time-dependent settings, the divergence of the norm implies a gradual decay of the initial pulse, which loses its power sucked into the tail growing toward $z \rightarrow -\infty$ [30].

III. NUMERICAL RESULTS FOR THE TRANSFER AND DRAGGING PROBLEMS

A. Stability conditions for the transfer

First, we have constructed a family of stationary solutions to Eq. (1) with a quiescent ($\xi = 0$) regularized nonlinearity profile, taken per Eq. (2), in the form of $\psi(x, t) = e^{-i\mu t} w(x)$, where the real function $w(x)$ satisfies the equation

$$\mu w = -\frac{1}{2} \frac{d^2w}{dx^2} - \tilde{\delta}(x)w^3. \quad (17)$$

A typical example of the solution is displayed in Fig. 1(a), and the entire family of solutions is represented in Fig. 1(b) by the corresponding $N(\mu)$ curve; compare the degenerate dependence given by Eq. (6) for the ideal δ function. It is obvious that the entire family satisfies the VK criterion, $dN/d\mu < 0$, hence the solitons may be stable. This conjecture has been verified through the computation of the stability eigenvalues for perturbations around the steady-state solution of Eq. (17) and, also, by means of direct simulations of the perturbed evolution (not shown here). All the eigenvalues were found to be purely imaginary (they are defined so that imaginary values correspond to the real frequencies, i.e., neutral stability), therefore only the edge of the continuous spectrum¹ lying on the imaginary axis is shown in Fig. 1(c). This stabilization is a direct result of the regularization of the δ function in Eq. (17). We have checked that, as smaller values of the regularization parameter ϵ are taken, the corresponding stationary wave function w approaches the peakon profile of Eq. (5). It is noteworthy that, as the limit of the ideal δ function is approached, we observe a pair of eigenvalues that bifurcate from the edge of the continuous spectrum and approach the origin, where they formally arrive in the limit of $\epsilon \rightarrow 0$, at which point the scaling invariance associated with Eq. (6) is restored, and the nonlinear instability emerges.

We now turn to characteristic examples (cf. Fig. 2) of the transfer of originally stable quiescent solitons by a nonlinear trap moving according to Eq. (3). As might be expected, the soliton survives the transfer, provided that it is not dragged too hard, i.e., the transfer time, T , is not too short. We define the soliton as “surviving the transfer” if, at time $t = T$, its amplitude is no lower than a sufficiently high fraction of the initial soliton amplitude—say, 90% or 80%. For instance, in the case shown in Fig. 2, the soliton dragged by a trap

¹The spectrum is purely continuous, aside from a pair of eigenvalues at the origin, which exist due to the U(1) invariance of the model.

FIG. 1. (Color online) (a) An example of a numerically found stationary solution of Eq. (17) with $\mu = -1$ and $\bar{\delta}(x)$ taken per Eq. (2). (b) The norm of the solutions, defined according to Eq. (6), vs the chemical potential μ . (c) The edge of the continuous spectrum lying on the imaginary axis [the full eigenvalue is defined as $i\lambda_i$, the continuous spectrum occupying the area above the border shown in (c)]. These results are obtained for the width of the moving trap $\epsilon = 0.05$.

of width $\epsilon = 0.05$ keeps 90% of the initial amplitude for $T \geq 20$. It can be checked that, in all cases of “survival,” the motion of the dragged soliton accurately follows the prediction of the adiabatic approximation [Eq. (7)], as clearly shown in the example displayed in Fig. 2(c), where the actual trajectory of the dragged mode closely follows the adiabatic trajectory $\xi(t)$. The surviving solitons essentially preserve their original shape of the smoothed peakon [cf. Figs. 1(a) and 2(d)–2(f)]. The latter three figures demonstrate that the eventual shape of the transferred mode is nearly the same for three different values of the trap’s width, $\epsilon = 0.05$, 0.1, and 0.025; i.e., the eventual results are robust, withstanding the

variation of essential parameters of the scheme within wide limits.

Results of the systematic simulations are summarized in Fig. 3, where the survival border is shown in the plane of the transfer parameters, Ξ and T [see Eq. (3)], for the fixed initial soliton and two survival criteria, based on the demand of keeping the amplitude at the 90% or 80% level. It may again be concluded that the results are robust, as they only weakly depend on the particular choice of the criterion for the soliton’s survival. The roughly parabolic shape of the border may be explained by the fact that (as discussed in the previous section) the gradual destruction of the dragged soliton is determined

FIG. 2. (Color online) (a)–(c) Density contour plots illustrating the dragging of the soliton, shown in Fig. 1(a), by a nonlinear trap with width $\epsilon = 0.05$, moving per Eq. (3), for three values of the dragging time T . Solid black lines show the corresponding trajectory $\xi(t)$. Adopting the criterion that the surviving soliton must keep $\geq 90\%$ of its initial amplitude, we conclude that it survives the transfer for $T \geq 20$. (d) The shape of the soliton in the case of $T = 20$, i.e., at the border of the survival, at time $t = 30$ (some time after the completion of the transfer). (e), (f) The same as (d), but for a moving trap with widths $\epsilon = 0.1$ and $\epsilon = 0.025$, respectively.

FIG. 3. (Color online) The top and bottom solid curves represent the boundary above which the initial soliton with chemical potential $\mu = -1$, shown in Fig. 1(a), survives, keeping more than 90% or 80%, respectively, of the initial amplitude by $t = 30$. The dashed curve depicts the analytical estimate (see text).

by the absolute value of the acceleration of the moving trap. In the case of the motion law (3), the acceleration is

$$\frac{d^2\xi}{dt^2} = -\frac{\pi^2\Xi}{2T^2} \sin\left(\pi \frac{t - T/2}{T}\right), \quad (18)$$

hence its average absolute value in the process of dragging is $\langle |d^2\xi/dt^2| \rangle = \pi\Xi/T^2$. Assuming that the soliton survives when the absolute value of the acceleration does not exceed a certain critical level, the latter result predicts a parabolic shape of the survival border, $T \sim \sqrt{\Xi}$. To confirm the reasonable accuracy of the prediction, we have drawn a parabola with an empirically fitted coefficient, $T = \sqrt{7\pi\Xi}$ (the dashed line), to be compared to the numerically determined survival borders displayed in Fig. 3.

B. Quasi-Airy profiles dragged at a constant acceleration

We have also briefly studied profiles of waves dragged at a constant acceleration, i.e., those built per Eqs. (12) and (15), and their stability within the framework of Eq. (10). A typical example of the stationary wave attached to the accelerating nonlinear δ function, i.e., a solution to Eq. (11), is displayed in Fig. 4. Direct simulations of Eq. (10) with this stationary wave used as the initial configuration were performed in order to examine its robustness. In particular, $a(z)$ was constructed and an approximate local minimum of the quantity $C_1^2 + C_2^2$ [which determines the tail's amplitude, $\langle a^2(z) \rangle$, as per Eq. (16)]

was identified for appropriate values of A and z_0 . The temporal evolution of the resulting configuration is shown in Fig. 4.

A more detailed analysis of quasi-Airy waves in the present nonlinear model will be the subject of a separate work. We note here that if we use small A (such as $A = 0.00057$ in the left column in Fig. 4), the apparent discontinuity of the beam at z_0 is barely visible; in this low-amplitude case, the solution appears to be more robust, and moves to the left, preserving its shape. If A is larger ($A = 1.14$ in the right panel in Fig. 4), a significant fraction of the solution gets trapped by the δ function at z_0 , while the larger remaining fraction still moves to the left, and a smaller fraction of the initial power disperses to the right of z_0 .

IV. SCATTERING OF COHERENT WAVE PACKETS ON A LOCALIZED NONLINEARITY

The scattering problem in the framework of Eq. (1) with the ideal quiescent δ function was first studied in Ref. [19], where the scattering of an incident plane wave was considered. It was found that a localized self-attractive nonlinearity gave rise to an accordingly localized modulational instability (MI) of the incident wave, provided that its amplitude exceeded a certain threshold (minimum) value. Unlike the commonly known MI in the NLS equation with uniform nonlinearity, the localized MI features a complex growth rate (i.e., it corresponds to the oscillatory instability).

Here, we aim to study scattering from the regularized Gaussian nonlinear potential of incident wave packets in the form of coherent states. The latter represent the fundamental solutions of the linear Schrödinger equation in free space,

$$\psi_{\text{in}}(x, t) = \frac{B}{\sqrt{1 + ibt}} \exp\left[-\frac{b(x - ct)^2}{1 + ibt} + icx + \frac{ic^2}{2}t\right]. \quad (19)$$

Here, B and b are real constants that determine, respectively, the amplitude and inverse width of the initial wave packet; $A = (1 + b^2t^2)^{-1/4}B$ and $W = \sqrt{b^{-2} + t^2}$ are the same characteristics of the spreading packet; the norm of the packet is $N = \sqrt{\pi/(2b)}B^2$; and c is its velocity. In particular, in the case of small B^2 , the scattering problem may be considered in the Born's approximation (see, e.g., Ref. [37]) by setting

FIG. 4. Temporal evolution of $a(z)$ with $z_0 = -3.86235$, for quasi-Airy waves dragged by the accelerating nonlinear trap with width $\epsilon = 0.05$. Left: The case of $A = 0.00057$. Right: Unstable solution with $A = 1.14$.

FIG. 5. Examples of the interaction of an incident coherent pulse, (19), with a localized nonlinear potential corresponding to a regularized profile, (2), for $\epsilon = 0.05$, at different values of pulse amplitude and velocity.

$\psi(x, t) = \psi_{\text{in}}(x, t) + \psi_{\text{Born}}(x, t)$, where the first correction is to be found from the linear inhomogeneous equation,

$$i(\psi_{\text{Born}})_t + \frac{1}{2}(\psi_{\text{Born}})_{xx} = -\delta(x)|\psi_{\text{in}}|^2\psi_{\text{in}}. \quad (20)$$

Equation (20) can be solved by means of the Fourier transform. Here, we instead focus on direct simulations of the scattering problem in a wide parametric region in the space of (B, b) , with the δ function regularized according to Eq. (2). As illustrated

by the examples displayed in Fig. 5, simulations demonstrate splitting of the incident wave packet into three parts: transmitted, trapped, and reflected ones. Naturally, the share of the trapped norm increases with B , i.e., with enhancement of the nonlinearity; on the other hand, the dependence of the observed phenomenology on the inverse-width parameter b is relatively weak. It is also noteworthy that, as shown in Fig. 5(d), the trapped pulse performs oscillations around the

FIG. 6. (a)–(c) Gray-scale plots of the trapped, reflected, and transmitted shares of the norm of the incident wave packet, as functions of its amplitude (B) and inverse width (b), for a fixed velocity $c = 20$ and width $\epsilon = 0.05$ of the nonlinear scatterer, at time $t = 0.5$. The trapped share is defined by the norm confined in the interval $-1 < x < +1$.

FIG. 7. The same as Fig. 6, but for a broader scatterer, with $\epsilon = 0.10$.

localized nonlinearity, while the transmitted beam splits into jets [see Figs. 5(c) and 5(d)].

Results of the simulations of the scattering are summarized in Figs. 6 and 7, where we show the shares of the trapped, reflected, and transmitted norm (or power, in terms of the optics model) as functions of the amplitude (B) and inverse width (b) of the incident pulse for a fixed velocity, $c = 20$, and two values of the scatterer's width, $\epsilon = 0.05$ and 0.1 , respectively. The plots correspond to the moment of time $t = 0.5$, taken after the incident pulse has passed the scatterer. Note that the sum of the three shares is, by construction, identically equal to 1. Analysis of the numerical data demonstrates that, in the limit of high amplitude (B), the transmission decreases while the reflection of the wave packet from the nonlinear barrier is enhanced.

Further, comparison of Figs. 6 and 7 shows that an increase in width leads to a gradual change in the scattering picture. It is quite natural that the wider scatterer (Fig. 7) provides weaker reflection but stronger trapping of the incident waves. Generally, the solution of the scattering problem is more sensitive to the width ϵ than the results reported above for the transfer problem, as in the latter case the wave packet had a chance to adjust itself to the particular shape of the nonlinear scatterer.

V. CONCLUSION

In this work, we have considered a 1D Schrödinger model with tightly localized nonlinearity, to study the following dynamical problems: the existence and stability of standing waves trapped by this nonlinear potential, the controllable transfer of the localized mode by a moving nonlinear trap (nonlinear tweezers), and the scattering of coherent wave packets on a stationary localized nonlinearity. By means of systematic simulations, the border of effective stability has

been identified for the transfer problem. Relative shares of the norm (power) of trapped, reflected, and transmitted waves were found, as functions of the amplitude and width of the incident pulse, for the scattering problem. The outcome of the transfer is less sensitive to particular characteristics of the nonlinear potential trap (such as its width) than the results of the solution of the scattering problem. Quasi-Airy stationary modes, dragged by a nonlinear trap moving at a constant acceleration, were briefly considered too and were found to be progressively less robust for beams with the increasing amplitude. Some results were obtained in an approximate analytical form, such as the adiabatic approximation for the slowly dragged localized mode, and a qualitative explanation of the quasiparabolic shape of the stability border.

This work may be naturally extended in other directions. It is especially interesting to consider the 2D version of the transfer problem, using the corresponding localized nonlinear trap in the form of a circle [16,18] and setting it into motion. A challenging issue in this case is the competition between the trapping of the 2D wave packet and its propensity to the intrinsic collapse. It may also be interesting to consider the scattering problem in one dimension in the case of a localized *repulsive* nonlinearity. Such studies will be reported in future publications.

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