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In this paper, we examine theoretically the system of periodically poled LiNbO₃ waveguide arrays that feature a quadratic nonlinearity and that have recently become accessible to experimental studies. Motivated by these earlier works, we provide a detailed analytical study of the existence, stability, and dynamics of plane wave delocalized solutions, as well as that of strongly localized modes consisting of a few sites. The linear stability of both classes of solutions is quantified as a function of the system parameters, such as the wave vector mismatch parameter or the interchannel coupling strength, using experimentally accessible ranges. Our findings are, in all the cases, corroborated by numerical bifurcation analysis results; furthermore, when the solutions are found to be unstable, typical examples of the instability evolution are shown.

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I. INTRODUCTION

The past few years have witnessed a tremendous growth of interest in spatially discrete systems, both in optics as well as in other areas of physics. One of the prototypical systems that has spurred this growth has been the setting of fabricated AlGaAs waveguide arrays [1]. Numerous interesting features have emerged in that context from the interplay between diffraction and nonlinearity; these include among others the Peierls-Nabarro potential barriers due to discreteness, diffraction, and diffraction management [2] and gap solitary waves [3]; see, e.g., Refs. [4,5] for some recent discussions of this field.

Another setting of quasidiscrete systems that has developed over the past few years is that of optically induced lattices in photorefractive crystals such as Strontium Barium Niobate (SBN). After the original theoretical proposition of Ref. [6], and its experimental realizations in Refs. [7,8], a wide and diverse array of phenomena have been predicted and experimentally observed in such crystals. Among them, one can highlight a few such as dipole [9], quadrupole [10] or necklace solitons [11], impurity modes [12], and discrete vortices [13] or rotary solitons [14]. For a recent review of this activity, we refer the interested reader to Ref. [15]. We should note in passing that similar dynamical lattices have also come to be of interest in an entirely different area of physics, namely, the atomic physics of Bose-Einstein condensates, when trapped in periodic potentials; see, e.g., the recent reviews [16,17].

All of the above settings share the principal feature that the nonlinearities of relevance are essentially cubic (i.e., of the χ^3 type); this is with the exception of the photorefractive media which feature a saturable nonlinearity that can be well approximated by a cubic one in the small intensity limit (through an appropriate Taylor expansion). More recently, however, another particularly interesting direction has emerged through the study of quadratic dynamical lattices,

motivated by the pioneering experiments of Refs. [18,19]. By fabricating waveguide arrays of periodically poled LiNbO₃, the authors of Refs. [18,19] were able to examine the prototypical formation of weakly, as well as strongly localized discrete, quadratic solitary waves, in a wide range of parameters (such as the wave vector mismatch parameter between the fundamental and the second harmonic or the coupling constant between adjacent waveguides); see also Ref. [20] for a recent review of these findings. We note in passing that a detailed review of the solitary waves and properties of continuum quadratic media can be found in Ref. [21], while a detailed discussion of the stability properties of one- and two-dimensional discrete quadratic solitons is given in Ref. [22]. More recently, additional features of the system were examined, including the formation of surface solitons at the interface between the waveguide array and a semi-infinite medium [23], and the modulational instability study of Ref. [24].

The modulational stability, i.e., the linear stability of plane waves, is a standard topic of interest widely in nonlinear wave systems [25]. However, to the best of our knowledge, it has not previously been theoretically addressed in the context of the experimental setting of the works of Refs. [18,19,24], although earlier studies of this topic have appeared in a slightly different context in Refs. [26,27]. Such an analysis should be particularly relevant in determining when a wide input beam is going to experience parametric amplification and subsequent filamentation.

Another aspect that has occasionally been studied, but was not systematically analyzed is the issue of stability of localized modes (i.e., solitary waves) in these quadratic media. Earlier works such as the first prediction of discrete quadratic solitons [28] and subsequent efforts in Refs. [22,27,29,30] involved setting up the model for waveguide arrays [29], proposing new modes (such as the so-called twisted modes [27]), examining the numerical stability of strongly localized states, or attempting an examination of stability using a “defect” type mode in the linearization equations [30]. However, in addition to the fact that most of these earlier works were not geared toward the recent experiments of Refs. [18,19], we have not found in the literature a systematic and detailed study of the different modes that are

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possible and, especially, of their stability features, similar, for instance, to the classification that exists for cubic waveguides [31,32].

In keeping with the experimental works of Refs. [18–20,24], we consider the following Hamiltonian theoretical model:

$$i\frac{\partial u_n}{\partial z} + i\delta\frac{\partial u_n}{\partial t} + c(u_{n+1} + u_{n-1}) + 2\gamma u_n^* v_n = 0, \quad (1)$$

$$i\frac{\partial v_n}{\partial z} - \Delta\beta v_n + \gamma u_n^2 = 0,$$

where u_n and v_n are the fundamental and harmonic fields at the center of the n th channel, c represents the linear coupling constant, δ is the inverse group velocity mismatch, $\Delta\beta$ is the linear wave vector mismatch, and γ is the effective quadratic nonlinear coefficient. Note that $*$ denotes the complex conjugate. The Hamiltonian of the system (1) is given by

$$H = - \sum_{n=-\infty}^{\infty} \int dt \left(i\frac{\delta}{2} u_n^* \frac{\partial}{\partial t} u_n + c u_n^* u_{n+1} + \gamma u_n^2 v_n^* - \frac{1}{2} \Delta\beta |v_n|^2 + \text{c.c.} \right). \quad (2)$$

Our aim in the present work is to consider plane wave solutions in the context of Eq. (1), and establish conditions for their existence and stability (Sec. II). We then focus on the range of parameters relevant to the experiments in the above works, and study the instability growth rate in the wide parameter space of the system at hand. The analytical results for the parametric gain are corroborated by direct numerical simulations, demonstrating the dynamical manifestation of the instability (Sec. III). We then turn to the examination of the existence and linear stability of localized modes with a few excited sites (Sec. IV). Our starting point in the latter setting will be the anticontinuum (AC) limit of $c=0$; for definiteness, in that limit we excite configurations containing one, two, or three sites. Since the former (single-site waveform) is always a stable configuration, as discussed earlier, e.g., in Ref. [22], we focus particularly on the two- and three-site structures as the prototypical examples of how to handle cases with arbitrary numbers of excited sites. Considering the interchannel coupling strength as a small parameter, we develop a systematic perturbative approach that allows us to identify the relevant linearization spectrum and hence the wave stability. Our analytical results are again compared to numerics, both through numerical linear stability, as well as through the dynamical evolution of unstable configurations (Sec. V). Finally, we briefly summarize our findings and discuss some directions of future interest in Sec. VI.

II. THEORETICAL STABILITY ANALYSIS OF PLANE WAVES

We perform a stability analysis for plane wave solutions of Eq. (1). We begin by taking:

$$u_n(z, t) = A e^{i(k_1 t + k_2 n + \omega z)} [1 + \epsilon \tilde{u}_n(z, t)], \quad (3)$$

$$v_n(z, t) = B e^{2i(k_1 t + k_2 n + \omega z)} [1 + \epsilon \tilde{v}_n(z, t)],$$

where A and B are taken to be real; ϵ is a small parameter; k_1 , k_2 , and ω are wave number and frequency parameters; and $\tilde{u}_n(z, t)$ and $\tilde{v}_n(z, t)$ are the small perturbations. Plugging Eq. (3) into Eq. (1) and equating orders of ϵ , one finds equations for A , B , $\tilde{u}_n(z, t)$, and $\tilde{v}_n(z, t)$. The leading order equations are given by

$$-\omega A - \delta k_1 A + 2cA \cos(k_2) + 2\gamma A^* B = 0 \quad (4)$$

$$-2\omega B - \Delta\beta B + \gamma A^2 = 0.$$

At order ϵ , we obtain

$$-\omega A \tilde{u}_n + iA \frac{\partial \tilde{u}_n}{\partial z} - \delta k_1 A \tilde{u}_n + i\delta A \frac{\partial \tilde{u}_n}{\partial t} + cA(e^{ik_2} \tilde{u}_{n+1} + e^{-ik_2} \tilde{u}_{n-1}) + 2\gamma A^* B(\tilde{u}_n^* + \tilde{v}_n) = 0 \quad (5)$$

$$-2\omega B \tilde{v}_n - (\Delta\beta) B \tilde{v}_n + iB \frac{\partial \tilde{v}_n}{\partial z} + 2\gamma A^2 \tilde{u}_n = 0.$$

Solving Eqs. (4) for A and B , one determines

$$B = \frac{\omega + \delta k_1 - 2c \cos(k_2)}{2\gamma}, \quad A^2 = \frac{2\omega + \Delta\beta}{\gamma} B, \quad (6)$$

where A and B are real valued. Taking $\gamma > 0$ and using the fact that $A^2 > 0$, we have that either both $(2\omega + \Delta\beta)$ and B be positive or negative. Now if we let the quantity $(2\omega + \Delta\beta) > 0$ or $(2\omega + \Delta\beta) < 0$, one finds, respectively, that

$$\omega > \max \left\{ -\frac{\Delta\beta}{2}, 2c \cos(k_2) - \delta k_1 \right\} \quad \text{or} \quad \omega < \min \left\{ -\frac{\Delta\beta}{2}, 2c \cos(k_2) - \delta k_1 \right\}. \quad (7)$$

We note that with Eqs. (6), we can rewrite Eqs. (5) as

$$-\omega A \tilde{u}_n + iA \frac{\partial \tilde{u}_n}{\partial z} - \delta k_1 A \tilde{u}_n + i\delta A \frac{\partial \tilde{u}_n}{\partial t} + cA(e^{ik_2} \tilde{u}_{n+1} + e^{-ik_2} \tilde{u}_{n-1}) + 2\gamma A^* B(\tilde{u}_n^* + \tilde{v}_n) = 0, \quad (8)$$

$$i\frac{\partial \tilde{v}_n}{\partial z} - r\tilde{v}_n + 2r\tilde{u}_n = 0,$$

where $r \equiv 2\omega + \Delta\beta$. At this point, we express $\tilde{u}_n(z, t)$ and $\tilde{v}_n(z, t)$ as follows:

$$\tilde{u}_n(z, t) = D e^{i(\kappa_1 t + \kappa_2 n + \Omega z)} + F e^{-i(\kappa_1 t + \kappa_2 n + \Omega z)},$$

$$\tilde{v}_n(z, t) = G e^{i(\kappa_1 t + \kappa_2 n + \Omega z)} + H e^{-i(\kappa_1 t + \kappa_2 n + \Omega z)}, \quad (9)$$

where D , F , G , and H are real coefficients. Taking the forms of $\tilde{u}_n$ and $\tilde{v}_n$ in Eq. (9) and plugging this into Eq. (8), we derive equations for D , F , G , and H of the form

$$-(\Omega + r)G + 2rD = 0,$$

$$(\Omega - r)H + 2rF = 0,$$

$$\begin{aligned}
& -[\omega + \Omega + \delta\kappa_1 + \delta\kappa_1 - 2c \cos(k_2 + \kappa_2)]D + (2\gamma B)F \\
& + (2\gamma B)G = 0, \\
& -[\omega - \Omega + \delta\kappa_1 - \delta\kappa_1 - 2c \cos(k_2 - \kappa_2)]F + (2\gamma B)D \\
& + (2\gamma B)H = 0,
\end{aligned} \tag{10}$$

where $r=2\omega+\Delta\beta$ and B is given by Eqs. (6). We now express the previous set of algebraic equations in terms of a matrix equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$, where the zero column matrix is on the right hand side of the equation and $\mathbf{A}$ and $\mathbf{x}$ are given by

$$\mathbf{A} = \begin{pmatrix} 2r & 0 & -(\Omega + r) & 0 \\ 0 & 2r & 0 & \Omega - r \\ \tilde{a}_{31} - \Omega & 2\gamma B & 2\gamma B & 0 \\ 2\gamma B & \tilde{a}_{42} + \Omega & 0 & 2\gamma B \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} D \\ F \\ G \\ H \end{pmatrix}. \tag{11}$$

Here, we have that $r \equiv 2\omega + \Delta\beta$, B is defined by Eqs. (6) and

$$\begin{aligned}
\tilde{a}_{31} & \equiv -\omega - \delta\kappa_1 - \delta\kappa_1 + 2c \cos(k_2 + \kappa_2), \\
\tilde{a}_{42} & \equiv -\omega - \delta\kappa_1 + \delta\kappa_1 + 2c \cos(k_2 - \kappa_2).
\end{aligned} \tag{12}$$

We note that the above equations have a nontrivial solution for $\mathbf{x}$ if $\det(\mathbf{A})=0$. For this case, we derive the following quartic polynomial in Ω :

$$\begin{aligned}
& \Omega^4 - (\tilde{a}_{31} - \tilde{a}_{42})\Omega^3 + (4\gamma^2 B^2 - 8r\gamma B - r^2 - \tilde{a}_{31}\tilde{a}_{42})\Omega^2 \\
& + (4r\gamma B + r^2)(\tilde{a}_{31} - \tilde{a}_{42})\Omega + 12r^2\gamma^2 B^2 \\
& + 4r^2\gamma B(\tilde{a}_{31} + \tilde{a}_{42}) + r^2\tilde{a}_{31}\tilde{a}_{42} = 0.
\end{aligned} \tag{13}$$

This quartic polynomial has four complex roots; we are interested in the one with the maximal imaginary part, as the imaginary part of the roots corresponds to the growth rate of the relevant perturbation. Therefore, our aim in the next section is to systematically examine the dependence of the growth rate on the system parameters and the plane wave features in the case of the experimentally accessible LiNbO₃ waveguide arrays.

III. NUMERICAL STABILITY ANALYSIS OF PLANE WAVES

From the works of Refs. [18–20,23,24], we infer the following parameter values for Eq. (1). The wave vector mismatch parameter $\Delta\beta = \pi s/L$, where $L=7$ cm is the typical propagation distance of interest and $s \in [-50, 150]$ is a dimensionless mismatch parameter range. On the other hand, the coupling strength $c = \pi/(2L_c)$, where L_c is the interchannel coupling length; for the four configurations used in the experiment, $L_c = 25.5, 15.74, 12.16$, and 9.53 mm. Notice that we will not consider the effects of the group velocity mismatch in what follows (i.e., $\delta=0$), similarly to the experiments and analysis of Refs. [23,24]. Finally, the quadratic nonlinear coefficient $\gamma = 40 \text{ m}^{-1}$ [33]. In what follows, all the above relevant system parameters have been rendered adimensional by rescaling, using a typical propagation distance of $z_0 = 25$ mm (motivated by rendering $\tilde{\gamma} = \gamma z_0 = 1$). Dimen-

sionless parameters (rescaled by z_0) will be denoted by a tilde.

In Fig. 1(a), a contour plot of the stability region is shown for the case when $\tilde{c} = 1.5399$, $\omega = 1$, and $k_2 = 0$. Equation (13) is solved and the maximum of the imaginary part of the four roots is plotted for every $(\kappa_2, \tilde{\Delta}\beta)$. The region of instability is a small thin band in the part of the plane where $\tilde{\Delta}\beta$ is mostly negative. Figure 1(b) gives the same contour plot but with $\tilde{c} = 4.12066$. The size of the region of instability increases with $\tilde{c}$. Figure 1(c) gives slices of Figs. 1(a) and 1(b) for $\kappa_2 = \frac{\pi}{2}$ and Fig. 1(d) gives slices of Figs. 1(a) and 1(b) for $\kappa_2 = \pi$. Figure 1(e) gives slices of Figs. 1(a) and 1(b) for $\tilde{\Delta}\beta = -6$ and Fig. 1(f) gives slices of Figs. 1(a) and 1(b) for $\tilde{\Delta}\beta = -14$. In Fig. 1(g) a time evolution of a perturbed plane wave according to Eq. (3) with $\epsilon = 0.00001$ is presented. This evolution corresponds to the point $(\kappa_2, \tilde{\Delta}\beta) = (\pi, -6)$ in Fig. 1(a). In Fig. 1(h) the maximum norms of u and v [the solutions of which the contour plot of the square modulus is shown in Fig. 1(g)] are plotted at each step.

When $\tilde{c} = 1.5399$ is fixed and ω increases and approaches $2\tilde{c} \cos(k_2) = 3.08$ from below, then the region of instability vanishes (not shown here). Figure 2(a) shows the contour plot for $\omega = 4 > 3.08$ with fixed $\tilde{c} = 1.5399$. The region of instability reappears mostly for $\tilde{\Delta}\beta > 0$, concentrated only at the lower and higher wave numbers of κ_2 . Figure 2(b) gives the stability region resulting from increasing ω in Fig. 1(b) from 1 to 4. The band of instability in Fig. 1(b) becomes thinner in Fig. 2(b) but remains topologically the same. Despite the fact that $\omega = 4$, ω is still less than $2\tilde{c} \cos(k_2) = 8.24$ for $\tilde{c} = 4.12066$. It is observed more generally that the stability diagrams resemble Figs. 1(a), 1(b), and 2(b) when ω is less than $2\tilde{c} \cos(k_2)$ and $\tilde{c}$ not too large. Figure 2(c) gives slices of Figs. 2(a) and 2(b) for $\kappa_2 = \frac{\pi}{4}$ and Fig. 2(d) gives slices of Figs. 2(a) and 2(b) for $\kappa_2 = \pi$. Figure 2(e) gives slices of Fig. 2(a) for $\tilde{\Delta}\beta = 0$ and $\tilde{c} = 1.5399$ and of Fig. 2(b) for $\tilde{\Delta}\beta = -8$ and $\tilde{c} = 4.12066$. Figure 2(f) gives slices of Figs. 2(a) for $\tilde{\Delta}\beta = -20$ and $\tilde{c} = 1.5399$ and of Figs. 2(b) for $\tilde{\Delta}\beta = 50$ and $\tilde{c} = 4.12066$. In Fig. 2(g) a case example of the evolution of the perturbed plane waves is given [similarly to Fig. 1(g)]. This evolution corresponds to the point $(\kappa_2, \tilde{\Delta}\beta) = (\frac{\pi}{4}, 50)$ in Fig. 2(a). In Fig. 2(h), the maximum norms of u and v are plotted [for the solutions in Fig. 2(g)].

In Fig. 3 the results of increasing ω further are shown. In Figs. 3(a) and 3(b), contour plots for $\omega = 10$ are given for $\tilde{c} = 1.5399$ and $\tilde{c} = 4.12066$, respectively. ω is greater than $2\tilde{c} \cos(k_2)$ for both values of $\tilde{c}$ and therefore the region of instability lies now mostly where $\tilde{\Delta}\beta > 0$. In Fig. 3(a), because of the larger value of ω , the region of instability moves away from the boundary toward the center. Figure 3(c) gives slices of Figs. 3(a) and 3(b) for $\kappa_2 = \frac{\pi}{4}$ and Fig. 3(d) gives slices of Figs. 3(a) and 3(b) for $\kappa_2 = \pi$. Figure 3(e) gives slices of Figs. 3(a) and 3(b) for $\tilde{\Delta}\beta = 10$ and Fig. 3(f) gives slices of Figs. 3(a) and 3(b) for $\tilde{\Delta}\beta = 50$. In Fig. 3(g) the perturbed plane wave evolution is given for the case of $(\kappa_2, \tilde{\Delta}\beta) = (\pi, 50)$ in Fig. 3(a). In Fig. 3(h), the maximum norms of u and v are plotted at each step for the solutions shown in Fig. 3(g).

FIG. 1. (Color online) (a) Contour plot of the absolute value of the largest imaginary part of Ω (most unstable eigenmode) as a function of $(\kappa_2, \tilde{\Delta}\beta)$. Here $\tilde{c}=1.5399$, $\omega=1$, and $k_2=0$. (b) Same as (a) but for $\tilde{c}=4.12066$, $\omega=1$, and $k_2=0$. (c) Slices of (a) and (b) for $\kappa_2=\frac{\pi}{2}$. (d) Slices of (a) and (b) for $\kappa_2=\pi$. (e) Slices of (a) and (b) for $\tilde{\Delta}\beta=-6$. (f) Slices of (a) and (b) for $\tilde{\Delta}\beta=-14$. (g) The evolution of the square modulus of the perturbed plane wave solutions for u and v as a function of n and z with $\kappa_2=\pi$ and $\tilde{\Delta}\beta=-6$ and for $\tilde{c}=1.5399$. Notice the oscillatory nature of the evolution upon the manifestation of the instability. (h) The z evolution of the maximal amplitude for the solutions given in (g).

The effects of changing k_2 are now examined. For the value $k_2=\frac{\pi}{2}$ all regions appear to be stable. For the case $k_2=\pi$ and $\omega=1$ the stability diagram is presented in Fig. 4(a) for $\tilde{c}=1.5399$ and in Fig. 4(b) for $\tilde{c}=4.12066$. As ω increases this region gets smaller (results not shown here). In Fig. 4(c), the evolution of the magnitudes of a perturbed plane wave solution for u and v with $\kappa_2=\pi$ and $\tilde{\Delta}\beta=0.5$ is presented in Fig. 4(a). In Fig. 4(d), the time evolution of the maximum norms of u and v [from Fig. 4(c)] for the solutions are also shown. We note in passing how the above results highlight the potential nonlinear dynamical outcomes of the instability and how these may vary from a more [as, e.g., in Fig. 3(g)] to a less [as, e.g., in Fig. 1(g)] oscillatory evolution and from the formation of localized filamentary solitary structures [as, e.g., in Fig. 2(g)] to the chaotic dynamics [of, e.g., Fig. 4(c)].

IV. THEORETICAL STABILITY ANALYSIS OF LOCALIZED MODES

We next consider a perturbation analysis for the existence and the stability of discrete solitons of Eq. (1). As indicated above, we focus on solitons with two and three excited sites. We will start the discussion of our analytical results by considering the two excited site case.

By taking $u_n(z, t) = e^{i\Lambda z} U_n$ and $v_n(z, t) = e^{2i\Lambda z} v_n$ with U_n and v_n independent of variable t , Eq. (1) (setting $\delta=0$) becomes

$$i \frac{\partial U_n}{\partial z} - \Lambda U_n + c(U_{n+1} + U_{n-1}) + 2\gamma U_n^* v_n = 0,$$

FIG. 2. (Color online) (a) Contour plot of the absolute value of the largest imaginary part of Ω as a function of $(\kappa_2, \tilde{\Delta}\beta)$. Here $\tilde{c} = 1.5399$, $\omega = 4$, and $k_2 = 0$. (b) Same as (a) but for $\tilde{c} = 4.12066$, $\omega = 4$, and $k_2 = 0$. (c) Slices of (a) and (b) for $\kappa_2 = \frac{\pi}{4}$. (d) Slices of (a) and (b) for $\kappa_2 = \pi$. (e) Slice of (a) for $\tilde{\Delta}\beta = 0$ and $\tilde{c} = 1.5399$ and of (b) for $\tilde{\Delta}\beta = -8$ and $\tilde{c} = 4.12066$. (f) Slice of (a) for $\tilde{\Delta}\beta = -20$ and $\tilde{c} = 1.5399$ and of (b) for $\tilde{\Delta}\beta = 50$ and $\tilde{c} = 4.12066$. (g) The evolution of the square modulus of a perturbed plane wave in u and v as a function of n and z with $\kappa_2 = \frac{\pi}{4}$ and $\tilde{\Delta}\beta = 50$; $\tilde{c} = 1.5399$. Notice how, in the present case, the instability leads to filamentation. (h) The z evolution of the maximum amplitudes of the two fields for the solutions given in (g).

$$i \frac{\partial v_n}{\partial z} - 2\Lambda v_n - (\Delta\beta) v_n + \gamma U_n^2 = 0, \quad (14)$$

where Λ denotes the propagation constant. Looking, as is customary, for real, z -independent (i.e., standing wave) solutions, we obtain the following steady state equations:

$$-\Lambda U_n + c(U_{n+1} + U_{n-1}) + 2\gamma U_n v_n = 0,$$

$$-2\Lambda v_n - (\Delta\beta) v_n + \gamma U_n^2 = 0.$$

We will also be concerned with the linear stability of the ensuing solutions which can be studied using the linearization ansatz

$$u_n(z, t) = e^{i\Lambda z} [U_n + \epsilon(e^{-i\lambda z} p_n + e^{i\lambda^* z} p_n^*)],$$

$$v_n(z, t) = e^{2i\Lambda z} [v_n + \epsilon(e^{-i\lambda z} q_n + e^{i\lambda^* z} q_n^*)],$$

where ϵ is a formal linearization parameter and λ is the stability parameter (eigenfrequency). The resulting linear stability problem can then be written in the following compact form:

$$\lambda f_n = \mathbf{C} f_{n-1} + \mathbf{M}_n f_n + \mathbf{C} f_{n+1}, \quad (16)$$

where $f_n = (p_n, P_n^*, q_n, Q_n^*)^T$ and $\mathbf{C}$ and $\mathbf{M}_n$ are the following 4×4 matrices:

$$\mathbf{C} = \begin{pmatrix} -c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

FIG. 3. (Color online) (a) Contour plot of the absolute value of the imaginary part of Ω as a function of $(\kappa_2, \tilde{\Delta}\beta)$. Here $\tilde{c}=1.5399$, $\omega=10$, and $k_2=0$. (b) Same as (a) but for $\tilde{c}=4.12066$, $\omega=10$, and $k_2=0$. (c) Slices of (a) and (b) for $\kappa_2=\pi/4$. (d) Slices of (a) and (b) for $\kappa_2=\pi$. (e) Slice of (a) for $\tilde{\Delta}\beta=10$ and $\tilde{c}=1.5399$ and of (b) for $\tilde{\Delta}\beta=10$ and $\tilde{c}=4.12066$. (f) Slice of (a) for $\tilde{\Delta}\beta=50$ and $\tilde{c}=1.5399$ and of (b) for $\tilde{\Delta}\beta=50$ and $\tilde{c}=4.12066$. (g) The n - z evolution of the fields' square modulus for $\kappa_2=\pi$ and $\tilde{\Delta}\beta=50$, while $\tilde{c}=1.5399$. Notice the regular oscillatory evolution in this case. (h) The z evolution of the solutions' maximum amplitude for the case of (g).

$$\mathbf{M}_n = \begin{pmatrix} \Lambda & -2\gamma\nu_n & -2\gamma U_n & 0 \\ 2\gamma\nu_n & -\Lambda & 0 & 2\gamma U_n \\ -2\gamma U_n & 0 & 2\Lambda + \Delta\beta & 0 \\ 0 & 2\gamma U_n & 0 & -(2\Lambda + \Delta\beta) \end{pmatrix}.$$

We note that superscript $*$ denotes complex conjugation and superscript T denotes the transpose of a matrix. The eigenfrequency λ and the eigenvector f_n determine the stability of the configuration as follows: a configuration will be (neutrally) stable for this Hamiltonian system if $\forall \lambda$, the imaginary part λ_i of the eigenfrequency ($\lambda = \lambda_r + i\lambda_i$) is such that $\lambda_i = 0$. This is due to the fact that the Hamiltonian nature of the system enforces that if λ is an eigenfrequency, so are λ^* , $-\lambda$, and $-\lambda^*$.

In the next section, we examine the steady-state solutions U_n and ν_n as a function of the coupling parameter c . We use Eqs. (16) to determine the linear stability of our multisite discrete solitons (as a function of parameter c).

V. ANALYSIS OF DISCRETE SOLITONS AT MULTIPLE SITES

A. Perturbation equations in powers of the coupling parameter c

We start our perturbative approach by expanding the stationary solutions U_n and ν_n in powers of c as

$$U_n = u_n^{(0)} + cu_n^{(1)} + c^2 u_n^{(2)} + \dots, \quad (18)$$

FIG. 4. (Color online) (a) Contour plot of the absolute value of the imaginary part of Ω as a function of $(\kappa_2, \Delta\beta)$. Here $\tilde{c}=1.5399$, $\omega=1$, and $k_2=\pi$. (b) Same as (a) but for $\tilde{c}=4.12066$, $\omega=1$, and $k_2=0$. (c) The n - z evolution of the magnitudes of the perturbed plane waves for $\kappa_2=\pi$ and $\tilde{\Delta}\beta=0.5$; $\tilde{c}=1.5399$. Notice the chaotic nature of the dynamics in this case. (d). The z evolution of the field amplitudes for the solutions given in (c).

$$\nu_n = \nu_n^{(0)} + c\nu_n^{(1)} + c^2\nu_n^{(2)} + \dots \quad (19)$$

The perturbed equations are given at the appropriate order of the coupling parameter c as follows. $O(1)$ equations (anti-continuum limit):

$$-\Lambda u_n^{(0)} + 2\gamma u_n^{(0)}\nu_n^{(0)} = 0, \quad -2\Lambda\nu_n^{(0)} - (\Delta\beta)\nu_n^{(0)} + \gamma(u_n^{(0)})^2 = 0. \quad (20)$$

$O(c)$ equations:

$$-\Lambda u_n^{(1)} + (u_{n+1}^{(0)} + u_{n-1}^{(0)}) + 2\gamma u_n^{(0)}\nu_n^{(1)} + 2\gamma u_n^{(1)}\nu_n^{(0)} = 0, \\ -2\Lambda\nu_n^{(1)} - (\Delta\beta)\nu_n^{(1)} + 2\gamma u_n^{(0)}u_n^{(1)} = 0. \quad (21)$$

$O(c^2)$ equations:

$$-\Lambda u_n^{(2)} + (u_{n+1}^{(1)} + u_{n-1}^{(1)}) + 2\gamma u_n^{(0)}\nu_n^{(2)} + 2\gamma u_n^{(1)}\nu_n^{(1)} \\ + 2\gamma(u_n^{(2)})^2\nu_n^{(0)} = 0, \\ -2\Lambda\nu_n^{(2)} - (\Delta\beta)\nu_n^{(2)} + \gamma(u_n^{(1)})^2 + 2\gamma u_n^{(0)}u_n^{(2)} = 0. \quad (22)$$

Equations (20)–(22) yield discrete soliton solutions in terms of the coupling parameter c . We initiate our examination of two- and three-site structures at the anticontinuum limit of $c=0$.

B. Anticontinuum limit

Solving Eq. (20) yields the zero and nonzero steady state solutions

$$u_n^{(0)} = \left\{ \pm \sqrt{\frac{\Lambda(2\Lambda + \Delta\beta)}{2\gamma^2}}, 0 \right\}, \quad \nu_n^{(0)} = \frac{\gamma(u_n^{(0)})^2}{2\Lambda + \Delta\beta}. \quad (23)$$

Notice that our assumption of considering real solutions implicitly requires that $\Lambda(2\Lambda + \Delta\beta) > 0$.

The two-site discrete soliton solution may assume an in-phase or an out-of-phase form (depending on the relative phase of the adjacent sites). At the AC limit, the soliton is of the form

$$u_n^{(0)} = \begin{cases} s_n \sqrt{\frac{\Lambda(2\Lambda + \Delta\beta)}{2\gamma^2}}, & n = 1, 2, \\ 0, & n \neq 1, 2, \end{cases} \quad (24)$$

where $s_i = \pm 1$, $i=1, 2$ determines the type of the soliton, i.e., in phase when $s_1 s_2 = 1$ and out of phase otherwise. Without loss of generality, we may set $s_1 = 1$.

In the AC limit, Eq. (16) becomes

$$\lambda f_n = \mathbf{M}_n^{(0)} f_n, \quad (25)$$

where f_n is defined as before and

$$\mathbf{M}_n^{(0)} = \begin{pmatrix} \Lambda & -2\gamma\nu_n^{(0)} & -2\gamma u_n^{(0)} & 0 \\ 2\gamma\nu_n^{(0)} & -\Lambda & 0 & 2\gamma u_n^{(0)} \\ -2\gamma u_n^{(0)} & 0 & 2\Lambda + \Delta\beta & 0 \\ 0 & 2\gamma u_n^{(0)} & 0 & -(2\Lambda + \Delta\beta) \end{pmatrix}. \quad (26)$$

One can subsequently examine the linear stability of these prototypical configurations, as a starting point for the finite c

FIG. 5. (Color online) The top panels depict an in-phase discrete soliton when $c=0.2$ and the distribution of its eigenfrequencies. The (connected) circles and stars show the first and the second component, i.e., u_n and v_n , of the soliton, respectively. The middle panels show the dependence of the eigenfrequencies on the coupling constant c . As a comparison with our perturbation analysis, we also plot in dashed-line the approximate eigenfrequencies (36)–(38). The bottom panel shows the dynamics of an unstable discrete in-phase soliton for $c=0.2$.

case. In the AC limit, the linear stability simplifies greatly due to the fact that the sites become uncoupled. One can then easily see that for all zero sites [from Eq. (23)], the relevant stability matrix of the AC limit is identical and has the form

$$\begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & 2\Lambda + \Delta\beta & 0 \\ 0 & 0 & 0 & -(2\Lambda + \Delta\beta) \end{pmatrix}. \quad (27)$$

This yields the following eigenfrequencies

$$\lambda = \pm \Lambda, \quad \pm (2\Lambda + \Delta\beta). \quad (28)$$

The first pair of eigenfrequencies will expand for nonzero coupling and will constitute the continuous spectrum of a discrete soliton, while the second pair will constitute a “point” spectrum with infinite multiplicity.

As for the nonzero sites, one can easily recover that the stability matrix is given by

$$\begin{pmatrix} \Lambda & -\Lambda & -K & 0 \\ \Lambda & -\Lambda & 0 & K \\ -K & 0 & 2\Lambda + \Delta\beta & 0 \\ 0 & K & 0 & -(2\Lambda + \Delta\beta) \end{pmatrix} \quad (29)$$

with $K = \sqrt{2\Lambda(2\Lambda + \Delta\beta)}$. This gives us the following eigenfrequencies:

$$\lambda_0^2 = 0, \quad (2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta). \quad (30)$$

These are the eigenfrequencies that are of particular interest to us since the instability must arise from these ones, when the coupling c becomes finite. More specifically, the zero eigenfrequency with algebraic multiplicity $2N$ and geometric multiplicity N , will become an eigenfrequency of algebraic multiplicity 2 (and geometric multiplicity 1), when $c \neq 0$.

FIG. 6. (Color online) The same as Fig. 5, but for out-of-phase configuration. The presence of an imaginary eigenfrequency shown in the bottom left panel when the coupling c is above the critical coupling constant $c_{cr} \approx 0.17$ implies that this mode is stable only when c is relatively small. The bottom right panel shows the instability evolution of this solution for supercritical couplings; the wave deforms toward a single site profile which hops between adjacent sites.

This entails that $N-1$ eigenfrequency pairs will move away from the spectral plane origin and may become (purely) imaginary (and, in principle, even complex) when $c \neq 0$, therefore potentially inducing instabilities. Therefore, it is especially relevant to track down these eigenfrequencies when $c \neq 0$. We also note in passing that our consideration of solutions that are stable in the anticontinuum limit necessitates that $(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) > 0$ herein. The above discussion fully characterizes the $c=0$ spectrum, hence we now move on to the finite coupling case.

C. Finite coupling case: Two-site discrete soliton

Let us first consider the solution profile. It is clear that for finite c the solutions will be deformed from their AC-limit

profile of Eqs. (23) and (24). The leading-order correction at $O(c)$ is found by solving Eqs. (21) which gives

$$u_n^{(1)} = \frac{(u_{n+1}^{(0)} + u_{n-1}^{(0)})}{\Lambda - 2\gamma \left[\nu_n^{(0)} + \frac{2\gamma(u_n^{(0)})^2}{(2\Lambda + \Delta\beta)} \right]}, \quad \nu_n^{(1)} = \frac{2\gamma u_n^{(0)} u_n^{(1)}}{2\Lambda + \Delta\beta} \quad (31)$$

for all sites. Upon evaluating the above forms, one finds

$$u_1 = K/(2\gamma) - cs_2(2\Lambda + \Delta\beta)/(2\gamma K) + O(c^2),$$

$$u_2 = s_2 K/(2\gamma) - c(2\Lambda + \Delta\beta)/(2\gamma K) + O(c^2),$$

$$\nu_{1,2} = \Lambda/(2\gamma) - cs_2/(2\gamma) + O(c^2),$$

where $K = \sqrt{2\Lambda(2\Lambda + \Delta\beta)}$ as previously defined.

FIG. 7. (Color online) The same as Fig. 6, but for +++ discrete solitons.

The next step is to consider the stability problem when the coupling is turned on. It is clear that one excited site will contribute two pairs of eigenfrequencies [cf. Eq. (30)]. Hence, in our two-excited sites case, there will be four pair of eigenfrequencies. Since we have expanded u_n and v_n in a power series of c , then it is natural that we also expand the eigenfrequencies in c . Therefore, we write

$$\lambda = \sqrt{\lambda_0^2 + c\lambda_1 + O(c^2)},$$

where λ_0 is given by Eq. (30).

For $0 < c \ll 1$, the eigenvalue problem is given by

$$\lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \underbrace{\begin{pmatrix} M_1 & C \\ C & M_2 \end{pmatrix}}_M \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \quad (33)$$

The submatrix M_j for $j=1,2$ in Eq. (33) is given by

$$M_j = \begin{pmatrix} \Lambda & -2\gamma v_j^{(0)} & -2\gamma u_j^{(0)} & 0 \\ 2\gamma v_j^{(0)} & -\Lambda & 0 & 2\gamma u_j^{(0)} \\ -2\gamma u_j^{(0)} & 0 & 2\Lambda + \Delta\beta & 0 \\ 0 & 2\gamma u_j^{(0)} & 0 & -(2\Lambda + \Delta\beta) \end{pmatrix}. \quad (34)$$

To find the eigenfrequencies, we then need to calculate the following determinant:

$$|M - \lambda \text{Id}_{8 \times 8}| = 0, \quad (35)$$

where $\text{Id}_{n \times n}$ is the identity matrix of size $n \times n$. The above equation will then give us the characteristic polynomial solutions λ as a power series in c . From this polynomial, one can extract the following expressions for the eigenfrequencies:

$$\lambda = \sqrt{(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) - 4cs_2(2\Lambda + \Delta\beta) + O(c^2)}, \quad (36)$$

 FIG. 8. (Color online) The same as Fig. 6, but for $++-$ discrete solitons.

$$\lambda = \sqrt{(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) - 4cs_2 \frac{4\Lambda^2 + 6\Lambda\Delta\beta + \Delta\beta^2}{6\Lambda + \Delta\beta}} + O(c^2), \quad (37)$$

$$\lambda = \sqrt{-4cs_2\Lambda \frac{2\Lambda + \Delta\beta}{6\Lambda + \Delta\beta}} + O(c^2), \quad (38)$$

$$\lambda = 0. \quad (39)$$

In this way, one can see that, for $\Lambda > 0$, in-phase discrete solitons are unstable due to the split of a pair of zero eigenfrequencies along the imaginary axis. On the other hand, for out-of-phase solitons, we found this type of waveforms to be stable close to the anticontinuum limit. This prediction will be compared with our numerical results in the next section. We note in passing that in the case of $\Lambda < 0$, the above stability conclusions are reversed (numerically, we will focus on the $\Lambda > 0$ case in what follows). Next, we consider the case of three excited sites.

D. Finite coupling case: Three-site discrete soliton

For this case in the AC limit, the configuration is given by

FIG. 9. (Color online) The same as Fig. 6, but for $+-+$ discrete solitons. The dynamics of an unstable $+-+$ discrete soliton presented in the bottom right panel is for $c=0.25$

$$u_n^{(0)} = \begin{cases} s_n \sqrt{\frac{\Lambda(2\Lambda + \Delta\beta)}{2\gamma^2}}, & n = 1, 2, 3, \\ 0, & n \neq 1, 2, 3, \end{cases} \quad (40)$$

where $s_i = \pm 1$, $i = 1, 2, 3$. Again, we set $s_1 = 1$. The corresponding expression of Eq. (32) for this case is then

$$u_n = \begin{cases} K/(2\gamma) - cs_2(2\Lambda + \Delta\beta)/(2\gamma K) + O(c^2), & n = 1, \\ s_2 K/(2\gamma) - c(1 + s_3)(2\Lambda + \Delta\beta)/(2\gamma K) + O(c^2), & n = 2, \\ s_3 K/(2\gamma) - cs_2(2\Lambda + \Delta\beta)/(2\gamma K) + O(c^2), & n = 3, \\ O(c^2), & \text{otherwise,} \end{cases} \quad (41)$$

$$\nu_n = \begin{cases} \Lambda/(2\gamma) - cs_2/(2\gamma) + O(c^2), & n=1, \\ \Lambda/(2\gamma) - cs_2(1+s_3)/(2\gamma) + O(c^2), & n=2, \\ \Lambda/(2\gamma) - cs_2s_3/(2\gamma) + O(c^3), & n=3, \\ O(c^2), & \text{otherwise.} \end{cases} \quad (42)$$

Following the same procedures as described for the two excited sites above, one obtains a stability matrix of size 12×12

$$M = \begin{pmatrix} M_1 & C & 0 \\ C & M_2 & C \\ 0 & C & M_3 \end{pmatrix}, \quad (43)$$

where the submatrices M_j for $j=1,2,3$ are determined by Eq. (34).

Again from calculating $|M - \lambda \text{Id}_{12 \times 12}| = 0$, one obtains, in addition to a pair of $\lambda=0$, (i) $s_3=1$

$$\lambda = \sqrt{(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) - 2s_2c \frac{3\Delta\beta^2 + 21\Delta\beta\Lambda + 24\Lambda^2 \pm \kappa_1}{6\Lambda + \Delta\beta} + O(c^2)}, \quad (44)$$

$$\lambda = \sqrt{(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) - 4s_2c \frac{\Delta\beta^2 + 7\Delta\beta\Lambda + 8\Lambda^2}{6\Lambda + \Delta\beta} + O(c^2)}, \quad (45)$$

$$\lambda = \sqrt{-6s_2c\Lambda \frac{2\Lambda + \Delta\beta}{6\Lambda + \Delta\beta} + O(c^2)}, \quad (46)$$

$$\lambda = \sqrt{-2s_2c\Lambda \frac{2\Lambda + \Delta\beta}{6\Lambda + \Delta\beta} + O(c^2)}, \quad (47)$$

(ii) $s_3=-1$

$$\lambda = \sqrt{(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) \pm 4c \frac{\kappa_2}{\kappa_4} + O(c^2)}, \quad (48)$$

$$\lambda = \sqrt{(2\Lambda + \Delta\beta)(6\Lambda + \Delta\beta) + O(c^2)}, \quad (49)$$

$$\lambda = \sqrt{\pm \frac{\kappa_3}{\kappa_4}c + O(c^2)}, \quad (50)$$

where

$$\begin{aligned} \kappa_1 &= \sqrt{\Delta\beta^4 + 14\Delta\beta^3\Lambda + 73\Delta\beta^2\Lambda^2 + 176\Delta\beta\Lambda^3 + 192\Lambda^4}, \quad \kappa_2 = \sqrt{\Delta\beta^4 + 14\Delta\beta^3\Lambda + 67\Delta\beta^2\Lambda^2 + 128\Delta\beta\Lambda^3 + 96\Lambda^4}, \quad \kappa_3 \\ &= 2\sqrt{3}\sqrt{\Delta\beta^2\Lambda^2 + 4\Delta\beta\Lambda^3 + 4\Lambda^4}, \quad \kappa_4 = \sqrt{\Delta\beta^2 + 12\Delta\beta\Lambda + 36\Lambda^2}. \end{aligned}$$

From the above expressions, it can be observed that for $\Lambda > 0$, the case with $s_3=-1$ is necessarily unstable, while the case with $s_3=1$ is only stable if $s_2=-1$, leading to the configuration $+-+$ as the only possible stable configuration in this setting.

VI. NUMERICAL ANALYSIS OF LOCALIZED MODES

In the following, we present our numerical results for the existence and stability of discrete solitons. Throughout the section, we use the parameter values $\Lambda=1$, $\tilde{\Delta}\beta=1$, and $\tilde{\gamma}=1$ [34] for concreteness; however, note that our results in

the previous section are entirely general and can be applied for any set of system parameters. To identify the standing wave solutions of Eqs. (15), we apply a Newton-Raphson fixed point method together with continuation to obtain the soliton for a given c from the AC limit. Afterwards, we use MATLAB's eigenvalue algorithm to study the state's linear stability. Regarding the dynamics when it is unstable, we use the same numerical integrator as the one used in Sec. III. The only difference here is that we perturb the initial state by a random perturbation with a maximum amplitude of the perturbation equal to 0.01. The perturbation is relatively high so that we can see an early manifestation of the instability. We

have found this to be especially helpful for configurations that are stable in the vicinity of the AC limit, but unstable for couplings above a certain threshold due to an oscillatory instability. Those configurations can be quite robust against small perturbations (due to their typically small instability growth rate).

A. Two excited sites case

It has been mentioned that for two excited sites, there are four pairs of eigenfrequencies. One $\lambda=0$ pair does not change because it is related to the gauge invariance property (which is also connected with the conservation of the total power). The other three pairs move as one turns on the coupling.

Let us first consider the in-phase discrete mode, i.e., $++$ case. As is predicted by our perturbation analysis, this mode is unstable. As soon as we turn on the coupling, there is a pair of eigenfrequencies bifurcating from $\lambda=0$ onto the imaginary axis. This results in the mode's instability throughout its entire existence region. In Fig. 5 we depict the eigenfrequencies as a function of c . In the same figure, we also show the other point spectrum pairs as a function of c . We notice that the nonzero eigenfrequencies decrease as c increases and finally enter the continuous spectrum. We should underscore here the very good agreement of our theoretical predictions (shown by the dashed line) with the full numerical computations for $0 \leq c \leq 0.1$, where the leading order correction obtained analytically should be expected to be dominant. For higher coupling strengths, higher order perturbative corrections should be expected to come into play. This type of agreement will be ubiquitous throughout our numerical results.

In the bottom panel of Fig. 5, we also present the dynamics of this soliton. As a particular case, we take the corresponding soliton of $c=0.2$. One can see that the soliton is unstable with respect to a single localized state (which is always stable, as mentioned previously).

Next, we consider the case of out-of-phase discrete solitons. We have shown using perturbation analysis, i.e., Eqs. (36)–(38), that this mode is stable for small coupling. This prediction is in agreement with our numerical results that are shown in Fig. 6. As we increase the coupling, there is a critical coupling c_{cr} above which the soliton is unstable. The critical coupling is approximately given by $c_{cr} \approx 0.17$. This complex eigenfrequency is caused by the collision of a pair bifurcating from zero and one bifurcating from the edge of the continuous spectrum. As we increase the coupling further, there is another critical coupling for the existence of an out-of-phase mode which approximately is given by $c \approx 0.3034$. In the vicinity of this point, the solution disappears through a sequence of a pitchfork bifurcation with $++-$ and $+-$, rendering the configuration unstable and a saddle node bifurcation with a $++-$ discrete soliton, leading to the disappearance of the branch. The details of the bifurcation diagram observed here are essentially identical the corresponding result of Ref. [31], hence are not discussed further.

We have also analyzed the dynamics of this mode when it is unstable. As a particular example we consider an out-of-

phase soliton again with $c=0.2$. The dynamics of the discrete soliton is presented in the bottom right panel of Fig. 6. Similar to the typical dynamics of an in-phase unstable discrete soliton, we also observed that the manifestation of the instability leads to a single site localized soliton. Yet, there are some differences that we can note in the present case, namely, the two excited sites oscillate in an out-of-phase way in z before the solution breaks down into a single excited site; also, the single soliton hops from one site to its neighbor after it is formed, rather than staying immobile as in the former case.

B. Three excited sites case

Next, we consider the case of three excited sites, where we expect to have six pairs of point spectrum eigenfrequencies. Again, one of the pairs stays at $\lambda=0$ due to the gauge invariance. The other five pairs move as one turns on the coupling.

Among this group, we start by considering $+++$ configuration. According to our perturbation analysis, this configuration is unstable throughout its existence region. This is due to two pairs of eigenfrequencies bifurcating from zero along the imaginary axis for nonzero coupling. If one increases the coupling further, there is a critical coupling at which the structure disappears in a saddle node bifurcation with a $+0+$ soliton, i.e., a two excited sites configuration with one nonexcited site in between. We present our numerical results for the existence and the stability of this configuration in Fig. 7. Regarding the dynamics of the instability, we present the numerical results of our time integration in the bottom panel of Fig. 7. The manifestation of the instability is quite similar to the case of the two excited site in-phase discrete soliton in that it results into a single-site configuration.

Among the group of three excited site configurations, there is another mode that becomes unstable as soon as we turn on the coupling, namely, the $++-$ soliton. This wave is the same as $---$ soliton because of the gauge invariance property. In this case, the instability is due to the bifurcation of a single pair of eigenfrequencies from $\lambda=0$. Another pair of zero eigenfrequencies (of the AC limit) bifurcates along the real axis. This pair also gives another unstable eigendirection for higher values of the coupling. The complete figure from our numerical calculations is presented in Fig. 8. We have also examined the result of the dynamical instability for this type of discrete solitons. We depict our simulation in the bottom right panel of Fig. 8, where one can see that this configuration also dynamically evolves toward a single excited site configuration. Interestingly, however, before it becomes a single site soliton, the structure “visits” a two-excited-site, out-of-phase configuration. Eventually, however, in this case as well, the dynamics chiefly involves a single site.

The last configuration that we consider is the $+-+$. In accordance with our prediction, this soliton is stable for relatively small coupling. We present our numerical results in Fig. 9. We also find numerically the stability boundary of this soliton, which is approximately given by $c \approx 0.15$. For larger values of c , this mode also disappears in a saddle-node bi-

furcation similarly to the case of $+-$. Therefore, one might guess as well that the collisions of this branch occur with the branches $++-$ (pitchfork that initially destabilizes it) and eventually with $++-++$ (saddle node that terminates it). In the bottom right panel of Fig. 9, we present the dynamics of a discrete soliton of this type for $c=0.25$. One can see clearly that the soliton is unstable, eventually leading to a single excited soliton.

VII. CONCLUSIONS AND FUTURE CHALLENGES

In this manuscript, we have considered qualitatively as well as quantitatively the existence, linear stability, and dynamics of solutions to the quadratic nonlinearity model for optical waveguide arrays, emulating directly the experimental setup of Refs. [18–20,24]. Our analysis was twofold. On the one hand, we considered the extended plane wave solutions of the model and examined their modulational stability analysis, finding the corresponding instability bands and evolving the plane waves dynamically when unstable to observe the ensuing formation of localized pulses (or of oscillatory or of even chaotic behavior). On the other hand, we also considered such localized wave profiles systematically starting from the anticontinuum limit, where the interchannel coupling is absent. Upon a perturbative evaluation of the leading order corrections to the solutions for finite coupling, we have substituted those solutions to the eigenvalue problem, identifying which of the localized waveforms may be

stable. Interestingly, although the relevant technical details and the nature of the spectrum are significantly different in this case, i.e., the number of eigenvalues of a localized mode of the considered system in general will be twice the number of eigenvalues of the corresponding localized mode in the cubic nonlinearity, the principal conclusion for the stability of such modes is similar to the conclusion of the cubic case, i.e., only alternating phase solutions (in the first component) may be stable in the one-dimensional case [32]. Quantitative comparisons between the (in)stability of discrete modes of the cubic [32] and the quadratic nonlinearity would not be quite straightforward as the parameters involved do not immediately correspond to each other.

It would be of particular interest to extend the present considerations to higher spatial dimensions, and especially to two-dimensional systems, which are also amenable to optical experiments with waveguide arrays. It is well-known [21] that the two-dimensional quadratic medium supports a stable solitary wave (i.e., not subject to collapse, in contrast to the cubic nonlinearity case). It would therefore be particularly relevant to examine systematically (similarly to Ref. [35]) the localized solutions that emerge in that setting, especially the discrete vortices [36] and soliton necklaces [11] among other interesting and relevant patterns. It would be especially interesting in that context to determine whether the stability of the structures has the same characteristics as in the cubic case or not. Such studies are currently in progress and will be reported in future publications.

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 - [33] We are particularly grateful to G. Stegeman and D. Christodoulides for relaying to us relevant values of the parameters and other useful resources regarding their work on the topic.
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