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A New Expenditure Function

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The usual expenditure function exploits the duality of expenditure and direct utility; a new expenditure function is proposed which exploits the duality of expenditure and indirect utility. The derivative of the new expenditure function is the inverse demand or willingness-to-pay. The new expenditure function offers the easiest way to value quantity changes.

In applied welfare economics, the value of a price change is measured by the equivalent change in income. The most direct way to convert price changes into income equivalents is to use the expenditure function, which is the minimum expenditure achieving a given utility level,

$$e(1, q, u) \equiv \min_{x_0, x} x_0 + q \cdot x \mid u(x_0, x) \geq u, \quad (1)$$

where

x_0 = gross demand for numeraire goods, whose price is unity,

x = vector of net demands for non-numeraire goods,

q = vector of relative prices of non-numeraire goods, and

u = utility.

We shall propose an equally compact expression for the income equivalent of a quantity change. First we offer a formal definition of the new expenditure function, and then we shall explain its interpretation and use. Consider the problem of finding the prices and income at which the quantity weighted sum of these variables is minimized, subject to the constraint that the indirect utility function assumes a particular value,

$$\min_{q, I} -I + q \cdot x \mid v(1, q, I) \leq u.$$

In fig. 1 we find the minimum by moving along an indifference curve in the space of

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Fig. 1.

prices, holding quantities x constant, searching for the lowest budget line. The new expenditure function is found by adding the constant x_0 to this expression,

$$E(x_0, x, u) \equiv \min_{q, I} x_0 - I + q \cdot x \mid v(1, q, I) \leq u. \quad (2)$$

We shall call the familiar form of the expenditure function 'indirect' because its arguments are prices, and we shall call the new form 'direct' because its arguments are quantities.

There is a simple interpretation of the direct form of the expenditure function. Let us define χ_0 to be the quantity of the numeraire which satisfies the budget identity $-\chi_0 = q \cdot x - I$. Finding the values of q and I which minimize $q \cdot x - I$, subject to a constraint upon indirect utility, is equivalent to minimizing χ_0 subject to a constraint upon direct utility,

$$\begin{aligned} E(x_0, x, u) &= x_0 - \min \chi_0 \mid u(x_0, x) \geq u \\ &= x_0 - \chi_0^*. \end{aligned} \quad (3)$$

In fig. 2 we find the minimum by searching down the line of constant consumption $x = \bar{x}$ for its intersection with the indifference curve $u(x_0, x)$. In words, the direct expenditure function is the difference between actual numeraire consumption and the hypothetical level of numeraire consumption necessary to achieve a given level of utility for given non-numeraire consumption.

The interpretation becomes clearer when we consider how the direct expenditure function values a quantity change. Suppose that the quantity change $(x_0^a, x^a) \rightarrow (x_0^b, x^b)$ causes utility to change from $u^a = u(x_0^a, x^a)$ to $u^b = u(x_0^b, x^b)$. Notice that

Fig. 2.

the actual change in numeraire consumption $x_0^b - x_0^a$ is already in dollars, but the actual change in non-numeraire consumption $x^b - x^a$ is not. The hypothetical change in numeraire consumption which compensates for the actual change in non-numeraire consumption is found by inverting

$$u(x_0^*, x^b) = u(x_0^a, x^a),$$

and solving for x_0^* ,

$$x_0^* = u^{-1}(x^b, u^a).$$

The dollar value which the direct expenditure function assigns to the change in the consumption bundle is minus the sum of the actual and compensating changes in numeraire consumption,

$$-[(x_0^b - x_0^a) + (x_0^a - x_0^*)] = -[x_0^b - x_0^*].$$

In other words, the direct expenditure function values a quantity change according to the difference between the actual numeraire consumption and the hypothetical change in numeraire consumption which holds utility constant, or according to *minus the compensating variation in numeraire consumption*.

This point can be seen from a slightly different angle by using some properties of the direct expenditure function. We convert from utils to dollars when we value the change $u^b - u^a$ according to the direct expenditure function,

$$E(x_0^b, x^b, u^b) - E(x_0^b, x^b, u^a) = x_0^* - x_0^b.$$

Equality follows from eq. (3) and $0 = E(x_0^b, x^b, u^b)$; the function is nil at any inter-

section of a Hicksian and Marshallian demand curve because at that point it equals net expenditure. The direct expenditure function evaluates the utility change as equal to minus the compensating variation in numeraire consumption.

This expression for the change is the most meaningful, but not the most useful. If we use the fact that the derivative of the direct expenditure function with respect to quantity is willingness-to-pay (inverse compensated demand), then a little manipulation yields the expression

$$E(x_0^b, x^b, u^b) - E(x_0^b, x^b, u^a) = (x_0^a - x_0^b) + \int_{x_0^b}^{x_0^a} q(x, u) dx,$$

where $q(\cdot)$ is the inverse compensated demand. These two terms represent the sum of the actual change in numeraire consumption and the hypothetical change in numeraire consumption which compensates for the actual change in non-numeraire consumption. Notice that the sum of the vertical areas under the inverse demand curves for the non-numeraire goods is the hypothetical change in numeraire consumption $x_0^* - x_0^a$.

The direct and indirect expenditure functions order changes in the opposite direction; any change in income and prices $(I^a, q^a) \rightarrow (I^b, q^b)$, with corresponding change in quantities $(x_0^a, x^a) \rightarrow (x_0^b, x^b)$, has the opposite sign when measured by the direct and indirect expenditure functions,

$$\text{sign}[\epsilon(1, q^b, u^b) - \epsilon(1, q^b, u^a)] = -\text{sign}[E(x_0^b, x^b, u^b) - E(x_0^b, x^b, u^a)]$$

$$\Leftrightarrow \text{sign}[(I^b - I^a) + \int_{q^b}^{q^a} x dq] = \text{sign}[(x_0^b - x_0^a) + \int_{x_0^b}^{x_0^a} q dx].$$

In other words, the actual income change plus the consumer's surplus summed over all markets, has the same sign as the actual change in numeraire consumption plus the willingness-to-pay summed over all markets. These observations about signs follow from the fact that ϵ is monotonic increasing in u , and E is monotonic decreasing in u .

The direct expenditure function is the easiest and simplest way to evaluate quantity changes, just as the indirect expenditure function is the best way to value price changes. The chief advantage is the ability to reason with confidence when several markets are influenced by the change. For example, it is not uncommon in cost-benefit analysis to value inputs by their opportunity cost (area under demand or supply curve) and then to value price changes in outputs by the consumer's surplus (area to left of demand curve). This is an example of the kind of error which is eliminated by our dual approach.

As another example, consider the effect upon welfare of taxing complements to leisure and subsidizing substitutes. We can construct the simplest proof that this policy increases welfare when there is a pre-existing tax on labor income. We adopt

the following notation:

x_0 = leisure,
 x_1 = complement to leisure,
 x_2 = substitute for leisure,
 R = tax revenues, and
 t_i = tax on commodity i .

The value of a change in E is

$$dE = dx_0 + q_1 dx_1 + q_2 dx_2.$$

Suppose this change is caused by a tax on x_1 , with an offsetting subsidy on x_2 , when there is a pre-existing tax t_0 on labor income. By differentiating the individual's budget constraint and government's revenue constraint, the preceding equation can be written ¹

$$\left. \frac{dE}{dt_1} \right|_R = - \left(\frac{\partial x_0}{\partial q_1} + \frac{\partial x_0}{\partial q_2} \frac{dt_2}{dt_1} \right) \Big|_R t_0,$$

which is positive by our assumptions.

The direct expenditure function also offers the simplest way to construct shadow prices. For example, a government project which exactly substitutes public for private consumption of good i should use the shadow price

$$s_i = \int \frac{\partial E}{\partial x_i} dx_i \Big/ \int dx_i = \int q_i(x, u) dx_i \Big/ \int dx_i.$$

This definition of i 's shadow price is consistent with measuring value by compensating variations, as proved by the Fundamental Equality of Value. The development of a general theory of shadow pricing requires the introduction of new concepts, namely the direct profit function for firms and government. These new concepts are not difficult, because the usual profit function bears the same relationship to the direct profit function as exists between the indirect and direct expenditure functions. However, the account of such uses must be postponed for a longer paper.

¹ The budget constraint for the individual is

$$0 = x_0 + \frac{\bar{p}_1}{1 - t_0} x_1 + \frac{\bar{p}_2}{1 - t_0} x_2$$

$$\Rightarrow 0 = dx_0 + \frac{\bar{p}_1}{1 - t_0} dx_1 + \frac{\bar{p}_2}{1 - t_0} dx_2 + x_1 dt_1 + x_2 dt_2,$$

where $\bar{p}$ is the constant producer's price. The differential of the revenue constraint is
 $0 = -t_0 dx_0 + x_1 dt_1 + x_2 dt_2.$

Appendix

Properties of direct expenditure function

Assume that the indirect utility function is continuous and quasi-convex in normalized prices and income. The direct expenditure function, defined by

$$E(x_0, x, u) \equiv \min_{q, I} x_0 - I + q \cdot x \mid v(1, q, I) \leq u,$$

has the following properties:

- (i) *Non-decreasing in x_0, x* ; specifically, $E_{x_0} = 1$ and $E_x = q(x, u)$ where q is inverse compensated demand, assuming the derivatives exist.
- (ii) *Decreasing in u* ; E_u may be interpreted as minus the reciprocal of the marginal utility of income.
- (iii) *Concave in x_0, x* .
- (iv) *Continuous in x_0, x* .

Proof

(i) It is obvious from fig. 1 that E is non-decreasing in x_0 and x . Let q^*, I^* solve $\min_{q, I} x_0 - I + q \cdot x \mid v(1, q, I) \leq u$. Because $E(\cdot)$ is a minimum we have $(\bar{x}_0 - I^* + q^* \cdot \bar{x}) - E(\bar{x}_0, \bar{x}, u) \geq 0$. This expression goes to zero as $(\bar{x}_0, \bar{x}) \rightarrow (x_0, x)$, so the derivatives with respect to the x 's must be nil at that point: $1 - E_{x_0} = 0$ and $q - E_x = 0$.

(ii) E must decrease in u because the indirect utility function is decreasing in q and increasing in I . In the body of the text we saw that the budget constraint $-x_0 = I + q \cdot x$ can be used to write the direct expenditure function in the form

$$\bar{E} = x_0 - \min_{x_0} x_0 \mid u(x_0, x) \geq u.$$

So we may write $E_u = -1/(\partial u / \partial x_0)$, assuming that this derivative exists.

(iii) We require specific values for some variables:

Let $\bar{q}, \bar{I}$ solve $\min_{q, I} \bar{x}_0 - I + q \cdot \bar{x} \mid v(1, q, I) \leq u$.

Let q', I' solve $\min_{q, I} x'_0 - I + q \cdot x' \mid v(1, q, I) \leq u$.

Let $(x''_0, x'') = t(\bar{x}_0, \bar{x}) + (1-t)(x'_0, x')$ where $0 \leq t \leq 1$.

Let q'', I'' solve $\min_{q, I} x''_0 - I + q \cdot x'' \mid v(1, q, I) \leq u$.

By definition of the direct expenditure function, we have

$$\begin{aligned} E(x''_0, x'', u) &= (1, q'') \cdot (x''_0, x'') - I'' \\ &= t(1, q'') \cdot (\bar{x}_0, \bar{x}) + (1-t)(1, q'') \cdot (x'_0, x') - I''. \end{aligned}$$

But minimization ensures that

$$(1, q'') \cdot (\bar{x}_0, \bar{x}) - I'' \geq E(\bar{x}_0, \bar{x}, u),$$

$$(1, q'') \cdot (x'_0, x') - I'' \geq E(x'_0, x', u).$$

It follows that E is concave:

$$E(x''_0, x'', u) \geq tE(\bar{x}_0, \bar{x}, u) + (1-t)E(x'_0, x', u).$$

(iv) Continuity follows from concavity.

Comment

The properties of the direct expenditure function parallel those of the indirect expenditure function, except the indirect expenditure function is homogeneous of degree 1 in prices, whereas the direct expenditure function is not homogeneous in quantities. This difference is a consequence of the fact that doubling prices and income does not influence utility, but doubling consumption increases utility. The choice of the numeraire good affects the value of either expenditure function, but not the ranking of changes.

Proposition about direct expenditure function

$$E(x_0^b, x^b, u^b) - E(x_0^b, x^b, u^a) = x_0^a - x^b + \int_{x^b}^{x^a} q(x, u) dx.$$

Proof

$$(1) \quad E(x_0^b, x^b, u^b) = x_0^b - I(x^b, u^b) + q(x^b, u^b) \cdot x^b$$

$$= 0 \quad \text{by budget constraint.}$$

(2) From the definition it is obvious that

$$E(x_0^b, x^b, u^a) = -x_0^a + x_0^b + E(x_0^a, x^b, u^a).$$

(3) It follows from the preceding steps that

$$E(x_0^b, x^b, u^b) - E(x_0^b, x^b, u^a) = x_0^a - x_0^b - E(x_0^a, x^b, u^a).$$

(4) From step 1 it is obvious that $0 = E(x_0^a, x^a, u^a)$; hence

$$E(x_0^b, x^b, u^b) - E(x_0^b, x^b, u^a) = x_0^a - x_0^b + \int_{E(x_0^a, x^b, u^a)}^{E(x_0^a, x^a, u^a)} dE.$$

(5) We already proved that $E_x = q$. It is well known that the compensated demands are symmetrical in prices and twice continuously differentiable; the same properties characterize the inverse compensated demands, so we can expand $\int dE$ into an exact line integral,

$$= x_0^a - x_0^b + \int_{x^b}^{x^a} q(x, u) dx.$$