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An Elliptic Partial Differential Equation With a Symmetrical Almost Periodic Term

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An Elliptic Partial Differential Equation with a Symmetrical Almost Periodic Term

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1. Introduction

In [STT], a Hamiltonian system of the form

$$(1.0) \quad -u'' + u = h(t)\nabla F(u)$$

was studied, where h is an almost periodic (defined in a moment) function, and $F : \mathbf{R}^N \rightarrow \mathbf{R}$ a “superquadratic” potential. That is, $F(q)$ behaves like q to a power greater than 2, with $F(q)/|q|^2 \rightarrow 0$ as $|q| \rightarrow 0$ and $F(q)/|q|^2 \rightarrow \infty$ as $|q| \rightarrow \infty$. For example, $F(q) = |q|^{p-1}q$ with $p > 1$ would qualify. The authors found that (1.0) must have a nonzero solution homoclinic to zero. Since this result, many papers (see [CMN], [R1], and [ACM], for example) have been written concerning Hamiltonian systems with almost periodic terms.

“Almost periodic” is defined for functions on $\mathbf{R}^N$ ($N > 1$) and even on more general topological groups. Thus one can write a PDE version of (1.0),

$$(1.1) \quad -\Delta u + u = h(x)f(u),$$

wherein h is almost periodic and the primitive F of f satisfies appropriate superquadraticity and critical growth conditions. Then a natural question is, does (1.1) have a nontrivial “homoclinic-type” solution, that is, a solution u with $|\nabla u(x)| + |u(x)| \rightarrow 0$ as $|x| \rightarrow \infty$? In [S1] we took a step toward answering in the affirmative. We considered an equation of the form $-\epsilon^2 \Delta u + V(x)u = f(u)$, with V almost periodic and f as above. It was shown that if V satisfied an additional condition, then for small enough ϵ , the equation has a homoclinic-type solution. That condition was automatically satisfied when $N = 2$. In this paper, we obtain a similar result for equation (1.1), that is, an equation without the ϵ . We will impose an extra symmetry condition on h , however.

Let us define an almost periodic function on $\mathbf{R}^N$ ($\mathbf{R}$ is a special case, and defining an a.p. function on other topological groups is an obvious generalization). First, a set $\mathcal{A} \subset \mathbf{R}^N$ is *relatively dense* if there exists $L > 0$ such that for every $x \in \mathbf{R}^N$, there exists $y \in \mathcal{A}$ with $|x - y| < L$. Next, for $\epsilon > 0$, $\vec{v} \in \mathbf{R}^N$, and $h : \mathbf{R}^N \rightarrow \mathbf{R}$, we say $\vec{v}$ is an ϵ -almost period of h if for all $x \in \mathbf{R}^N$, $|h(x + \vec{v}) - h(x)| < \epsilon$. Finally, h is defined to be *almost periodic* if h is continuous and for every $\epsilon > 0$, there exists a relatively dense set $\mathcal{A} \equiv \mathcal{A}(\epsilon) \subset \mathbf{R}^N$ such that for all $a \in \mathcal{A}$, a is an ϵ -almost period of h . For properties of almost periodic functions (many properties of a.p. functions on $\mathbf{R}$ extend to functions on $\mathbf{R}^N$), see [Be], [Bo], [C], [Z].

We will prove the following:

THEOREM 1.2 *Let h and f satisfy*

- (h_1) $h \in C(\mathbf{R}^N, \mathbf{R})$,
- (h_2) $\inf_{\mathbf{R}^N} h > 0$
- (h_3) h is almost periodic.
- (h_4) $h(x_1, \dots, -x_i, \dots, x_N) = h(x_1, \dots, x_i, \dots, x_N)$ for all $x = (x_1, \dots, x_N) \in \mathbf{R}^N$ and all $i = 1, \dots, N - 1$.
- (f_1) $f \in C^1(\mathbf{R}^+, \mathbf{R})$
- (f_2) $f'(0) = 0$.
- (f_3) There exist $A, s > 0$ such that $|f'(q)| \leq A(1 + |q|^{s-1})$ for all $q \geq 0$. If $n \geq 3$, then $s < 4/(n - 2)$.
- (f_4) For some $\mu > 2$, $0 < \mu F(q) \leq f(q)q$ for all $q > 0$, where $F(q) \equiv \int_0^q f(t) dt$.
- (f_5) The function $q \mapsto f(q)/q$ is increasing on $(0, \infty)$.

Then equation (1.1) has a positive homoclinic-type solution. More precisely, if c_0 is the mountain-pass value of the functional I associated with (1.1), then for each $\epsilon > 0$, there exists a positive solution u of (1.1) with $c_0 \leq I(u) < c_0 + \epsilon$.

In a moment we will give a precise definition of c_0 . An example of f satisfying (f_1) – (f_5) is $f(q) = q^s$, where s is as in (f_3). Condition (f_5) is an important convexity condition found in many papers, such as [R2], [WZ], and [FdP1-3]. Condition (h_4) states that h is even with respect to the hyperplane $\{x_i = 0\}$ for all $i = 0, \dots, N - 1$ (but not necessarily for $i = N$). An example of h satisfying (h_1) – (h_4) is $h(x_1, x_2) = 6 + \cos(x_1) + \cos(\sqrt{2}x_1) + \sin(x_2) + \sin(\sqrt{2}x_2) + \sin(\sqrt{6}x_2)$. This h is even with respect to the x_2 -axis, but not the x_1 -axis.

Although the symmetry assumption (h_4) is a strong one, proving Theorem 1.2 is not easy. The proof of [STT]'s $N = 1$ result cannot be directly generalized (see [S1] for explanation). Also, an almost periodic function of several variables need not have a local minimum (see [S2]), ruling out some promising variational approaches.

Note that other than symmetry, there are few extra restrictions on h ; for example, no assumptions on the variation of h are necessary (as in [S3]), nor on the magnitude of $|\nabla h|$ (if defined, or on a modulus of continuity for h if h is not C^1).

Variational Framework and Plan of Proof

Define $E = W^{1,2}(\mathbf{R}^N)$. Extend f and F to $\mathbf{R}$ by defining $f(-q) = f(q)$. Define a C^2 functional I on E by

$$(1.3) \quad I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} h(x)F(u) \, dx.$$

Then, by elliptic regularity theory, the set of critical points of I equals the set of homoclinic-type solutions of (1.1). We will find a nonzero critical point of I , then show it is a positive function. Define

$$(1.4) \quad \Gamma_0 = \{\gamma \in C([0, 1], E) \mid \gamma(0) = 0, I(\gamma(1)) < 0\}$$

and

$$(1.5) \quad c_0 = \inf_{\gamma \in \Gamma} \max_{\theta \in [0, 1]} I(\gamma(\theta)).$$

This is the c_0 from the statement of Theorem 1.2. I satisfies most of the hypotheses of the Mountain Pass Theorem ([AR]), so $c_0 > 0$ (see [CR1] for a similar example). For $i = 1, \dots, N - 1$, define the reflection operator $R_i : \mathbf{R}^N \rightarrow \mathbf{R}^N$ by

$$(1.6) \quad R_i(x_1, \dots, x_i, \dots, x_N) = (x_1, \dots, -x_i, \dots, x_N),$$

Also define $R_i : E \rightarrow E$ by

$$(1.7) \quad (R_i u)(x) = u(R_i x).$$

For $\{i_1, i_2, \dots, i_k\} \subset \{1, \dots, N - 1\}$, define $E[i_1, i_2, \dots, i_k] \subset E$ by

$$(1.8) \quad E[i_1, i_2, \dots, i_k] = \{u \in E \mid R_i u = u \text{ for } i = i_1, i_2, \dots, i_k\},$$

$\Gamma[i_1, i_2, \dots, i_k] \subset \Gamma$ by

$$(1.9) \quad \Gamma[i_1, i_2, \dots, i_k] = \{\gamma \in \Gamma \mid \gamma(\theta) \in E[i_1, i_2, \dots, i_k] \text{ for all } \theta \in [0, 1]\},$$

and

$$(1.10) \quad c[i_1, i_2, \dots, i_k] = \inf_{\gamma \in \Gamma[i_1, i_2, \dots, i_k]} \max_{\theta \in [0, 1]} I(\gamma(\theta)).$$

Finally, for $k = 1, 2, \dots, N-1$, define

$$(1.11) \quad c_k = \min \{c[i_1, i_2, \dots, i_k] \mid \{i_1, i_2, \dots, i_k\} \subset \{1, \dots, N-1\}\}$$

and

$$(1.12) \quad E_k = \bigcup \{E[i_1, i_2, \dots, i_k] \mid i_1, i_2, \dots, i_k \subset \{1, \dots, N-1\}\}$$

Then $c_0 \leq c_1 \leq \dots \leq c_{N-1}$. We will prove Theorem 1.2 by proving the following two propositions:

PROPOSITION 1.13 *If $c_{N-1} = c_0$, then for any $\epsilon > 0$, I has a critical point u with $c_0 \leq I(u) < c_0 + \epsilon$.*

PROPOSITION 1.14 *If there exists $d > 0$ such that the interval $[c_0, c_0 + d)$ contains no critical values of I , then for $i = 0, 1, \dots, N-2$, $c_i = c_0$ implies $c_{i+1} = c_0$.*

The proof of Theorem 1.2 then runs as follows: Let $\epsilon > 0$. If there does not exist a critical point u of I with $c_0 \leq I(u) < c_0 + \epsilon$, then by Proposition 1.14, $c_0 = c_1 = \dots = c_{N-1}$. Thus by Proposition 1.13, I has a critical level in the interval $[c_0, c_0 + \epsilon)$. This is a contradiction. Therefore (1.1) has a homoclinic-type solution u_ϵ with $c_0 \leq I(u_\epsilon) < c_0 + \epsilon$. We then show that if ϵ is small enough, u_ϵ does not change sign.

Organization of Paper

Section 2 contains some technical results. Section 3 contains the proofs of Propositions 1.13 and 1.14.

2. Technical Results

There is another definition of almost periodic that is equivalent ([Be]) to that given in the Introduction. For $h : \mathbf{R}^N \rightarrow \mathbf{R}$ and $x \in \mathbf{R}^N$, define $\tau_x h$ by $\tau_x h(y) = h(y - x)$. That is, $\tau_x h$ is h translated by x . Then, a continuous function $h : \mathbf{R}^N \rightarrow \mathbf{R}$ is almost periodic if the set of its translates is precompact under the uniform norm. That is, h is almost periodic if, for any sequence $(x_m) \subset \mathbf{R}^N$, there is a subsequence $(y_m) \subset (x_m)$ and a continuous function $\bar{h} : \mathbf{R}^N \rightarrow \mathbf{R}$ with $\tau_{y_m} h \rightarrow \bar{h}$ uniformly on $\mathbf{R}^N$ as $m \rightarrow \infty$. For almost periodic h , define the *hull* of h , $Hull(h)$, to be the set of all limit points of $\{\tau_x h \mid x \in \mathbf{R}^N\}$ under the topology of uniform convergence.

I fails the Palais-Smale condition. That is, a *Palais-Smale sequence*, or sequence (u_m) with $I'(u_m) \rightarrow 0$ and $(I(u_m))$ convergent, need not be precompact. The following concentration compactness result, however, describes Palais-Smale sequences of I . For $\bar{h} : \mathbf{R}^N \rightarrow \mathbf{R}$, define $I[\bar{h}]$ by $I[\bar{h}](u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} \bar{h}(x)F(u(x)) dx$. In particular, $I = I[h]$.

PROPOSITION 2.0 *Let h and f satisfy $(h_1) - (h_3)$ and $(f_1) - (f_4)$. Let $(u_m) \subset E$ with $I'(u_m) \rightarrow 0$ and $I(u_m) \rightarrow b > 0$ as $m \rightarrow \infty$. Then there exists a subsequence of (u_m) (also denoted (u_m)), $k \in \mathbf{N}$, sequences $(x_m^i)_{m \geq 1}^{i=1, \dots, k}$, functions $(\bar{h}_i)_{i=1, \dots, k} \subset Hull(h)$, and $(v_i)_{i=1, \dots, k} \subset E \setminus \{0\}$ satisfying*

- (i) $|x_m^i - x_m^j| \rightarrow \infty$ for $i \neq j$ as $m \rightarrow \infty$
- (ii) $\tau_{-x_m^i} h \rightarrow \bar{h}_i$ uniformly as $m \rightarrow \infty$
- (iii) $I[\bar{h}_i]'(v_i) = 0$ for all i
- (iv) $\|u_m - \sum_{i=1}^k \tau_{x_m^i} v_i\| \rightarrow 0$ as $m \rightarrow \infty$
- (v) $\sum_{i=1}^k I[\bar{h}_i](v_i) = b$.

This is essentially a special case of a result in [S1]. The proof follows that of a similar result in [CR1-2], using principles found in [L].

The above result does not require (f_5) . If we assume (f_5) , then for any $u \neq 0$, the mapping $t \mapsto I(tu)$ is increasing for small positive t , achieves a maximum at some $t' > 0$, then decreases to infinity on (t', ∞) (see [CR1]). Therefore, if $u \neq 0$ with $I'(u) = 0$, or

even just $I'(u)u = 0$, then $I(u) \geq c_0$, where c_0 is the minimax value defined in (1.5). If we define $c(\bar{h})$ to be the minimax value associated with the functional $I[\bar{h}]$, then $c(\bar{h}) = c_0$ for all $\bar{h} \in \text{Hull}(h)$ (see [CMN]). Therefore, Proposition 2.0, in particular Proposition 2.0(v), gives the following strong result:

PROPOSITION 2.1 *Let h and f satisfy $(h_1) - (h_3)$ and $(f_1) - (f_5)$. Let $(u_m) \subset E$ with $I'(u_m) \rightarrow 0$ and $I(u_m) \rightarrow b \in (0, 2c_0)$ as $m \rightarrow \infty$. Then there exists a subsequence of (u_m) (also denoted (u_m)), a sequence $(x_m) \subset \mathbf{R}^N$, a function $\bar{h} \in \text{Hull}(h)$, and $v \in E$ satisfying*

- (i) $\tau_{-x_m} h \rightarrow \bar{h}$ uniformly as $m \rightarrow \infty$
- (ii) $I[\bar{h}](v) = 0$
- (iii) $\|u_m - \tau_{x_m} v\| \rightarrow 0$ as $m \rightarrow \infty$
- (iv) $I[\bar{h}](v) = b$

Define the “solution manifold” or Nehari manifold $\mathcal{S} \subset E$ by $\mathcal{S} = \{u \in E \setminus \{0\} \mid I'(u)u = 0\}$. Because of (f_5) , $\mathcal{S}$ is homeomorphic to the unit sphere in E via a radial map, and

$$(2.2) \quad c_0 = \inf_{\mathcal{S}} I.$$

(see [R2]). We can strengthen this result to include sequences which are “close to $\mathcal{S}$.” That is, the following is true:

PROPOSITION 2.3

$$(2.4) \quad c_0 = \inf \left\{ \liminf_{m \rightarrow \infty} I(u_m) \mid (u_m) \subset E, 0 < \liminf_{m \geq 1} I(u_m) \leq \limsup_{m \geq 1} I(u_m) < \infty, \right. \\ \left. I'(u_m)u_m \rightarrow 0 \right\}.$$

This is essentially the same result as Proposition 3 in [CT], extended to a PDE setting. In that paper, the authors worked with a Palais-Smale sequence (u_m) , with $I'(u_m) \rightarrow 0$. However, their proof did not use the full strength of that assumption, which could have been replaced by $I'(u_m)u_m \rightarrow 0$. The distinction is important to us, for we will be working with sequences (u_m) which satisfy $I'(u_m)u_m \rightarrow 0$ but not, apparently, $I'(u_m) \rightarrow 0$.

Proof of Proposition 2.3: the proof is similar to that in [CT]. The direction $c_0 \geq \dots$ is obvious by (2.2). To prove the other inequality, let $(u_m) \subset E$ be as in (2.4). By arguments of [CMN], for example, (u_m) is bounded in E . By arguments of [CR2], Lemma 2.25 (these arguments assume $I'(u_m) \rightarrow 0$ but require only $I'(u_m)u_m \rightarrow 0$), there exist $\rho > 0$ and a sequence $(x_m) \subset \mathbf{R}^N$ such that $\|u_m\|_{L^2(B_1(x_m))} > \rho$ for large m . Since (u_m) is bounded, there exists $v \in E$ such that, along a subsequence, $\tau_{-x_m}u_m \rightharpoonup v$ weakly in E as $m \rightarrow \infty$ and also in L^p_{loc} for $1 \leq p < 2n/(n-2)$. $\|\tau_{-x_m}u_m\|_{L^2(B_1(0))} > \rho$ for large m , so $v \neq 0$.

Let $\Omega \subset \mathbf{R}^N$ be bounded and have positive Lebesgue measure, with $|v| > 2\delta > 0$ a.e. on Ω . Since $\tau_{-x_m}u_m \rightarrow v$ in $L^2(\Omega)$, there exists $a > 0$ with

$$(2.5) \quad \lambda(\{x \mid |u_m(x)| > \delta\}) \geq \lambda(\{x \mid |u_m(x)| > \delta\} \cap \Omega) > a$$

for large enough m , where λ denotes Lebesgue measure.

Define ρ_m to be the unique positive number satisfying $\rho_m u_m \in \mathcal{S}$. Let $h_- = \inf_{\mathbf{R}^N} h > 0$. Then for large m , $I'(\rho_m u_m)(\rho_m u_m) = 0 = I'(\rho_m u_m)(u_m)$, so

$$(2.6) \quad \begin{aligned} \|u_m\|^2 &= \frac{1}{\rho_m} \int_{\mathbf{R}^N} h f(\rho_m u_m) u_m \geq \frac{h_-}{\rho_m} \int_{\{|u_m| > \delta\}} f(\rho_m u_m) u_m = \\ &= h_- \int_{\{|u_m| > \delta\}} \frac{f(\rho_m u_m)}{\rho_m u_m} u_m^2 \geq h_- \delta^2 \int_{\{|u_m| > \delta\}} \frac{f(\delta \rho_m)}{\delta \rho_m} = \\ &\quad (\text{by } (f_5)) \\ &= h_- \delta^2 \frac{f(\delta \rho_m)}{\delta \rho_m} \lambda(\{|u_m| > \delta\}) > (h_- \delta^2 a) \left(\frac{f(\delta \rho_m)}{\delta \rho_m} \right). \end{aligned}$$

Since $f(q)/q \rightarrow \infty$ as $|q| \rightarrow \infty$, and $(\|u_m\|)$ is bounded, it follows that (ρ_m) is bounded. Define $g_m, h_m : \mathbf{R}^+ \rightarrow \mathbf{R}$ by

$$(2.7) \quad g_m(\rho) = I(\rho u_m) = \frac{1}{2} \rho^2 \|u_m\|^2 - \int_{\mathbf{R}^N} h F(\rho u_m)$$

and

$$(2.8) \quad h_m(\rho) = \frac{1}{2} \rho^2 \int_{\mathbf{R}^N} h f(u_m) u_m - \int_{\mathbf{R}^N} h F(\rho u_m).$$

(f_5) gives $h'_m > 0$ on $(0, 1)$ and $h'_m < 0$ on $(1, \infty)$. Now

$$(2.9) \quad g_m(\rho_m) - h_m(\rho_m) = \frac{1}{2} \rho_m^2 (\|u_m\|^2 - \int_{\mathbf{R}^N} h f(u_m) u_m) = \frac{1}{2} \rho_m^2 I'(u_m) u_m.$$

Since (ρ_m) is bounded and $I'(u_m)u_m \rightarrow 0$, $g_m(\rho_m) - h_m(\rho_m) \rightarrow 0$. The same argument shows that $g_m(1) - h_m(1) \rightarrow 0$. Therefore

$$(2.10) \quad \begin{aligned} c_0 &\leq \liminf_{m \rightarrow \infty} I(\rho_m u_m) \equiv \liminf_{m \rightarrow \infty} g_m(\rho_m) = \liminf_{m \rightarrow \infty} h_m(\rho_m) \leq \\ &\leq \liminf_{m \rightarrow \infty} h_m(1) = \liminf_{m \rightarrow \infty} g_m(1) \equiv \liminf_{m \rightarrow \infty} I(u_m). \end{aligned}$$

The direction $c_0 \leq \dots$ in (2.4) is proven.

The proof of (2.2) can be adapted easily to show

$$(2.11) \quad c_k = \inf_{S \cap E_k} I,$$

where E_k is defined in (1.12). Therefore, the proof of Proposition 2.3 shows that for $k = 0, 1, \dots, N-1$,

$$(2.12) \quad \begin{aligned} c_k &= \inf \{ \liminf_{m \rightarrow \infty} I(u_m) \mid (u_m) \subset E_k, \\ &0 < \liminf_{m \rightarrow \infty} I(u_m) \leq \limsup_{m \rightarrow \infty} I(u_m) < \infty, \\ &I'(u_m)u_m \rightarrow 0 \}. \end{aligned}$$

3. Proofs of Propositions 1.13 and 1.14

Proof of Proposition 1.13

Suppose $c_{N-1} = c_0$. Let $\epsilon > 0$ and let $\gamma \in \Gamma[1, 2, \dots, N-1]$ with

$$(3.0) \quad \max_{\theta \in [0,1]} I(\gamma(\theta)) < c_0 + \epsilon.$$

Let V denote the gradient of I . That is, $(V(u), w) = I'(u)w$ for all $u, w \in E$. Let η be the solution of the initial value problem $\frac{d\eta}{dt} = -V(\eta)$; $\eta(0, u) = u$. By arguments in [CMN] or [S1], for all $u \in E$, either $\eta(t, u)$ is well-defined and $I(\eta(t, u)) \geq 0$ for all $t > 0$, or $I(\eta(t, u)) < 0$ for some $t > 0$. Because of the mountain-pass structure of I , it is apparent that for some $\theta \in [0, 1]$, $\lim_{t \rightarrow \infty} I(\eta(t, \gamma(\theta))) \in [c_0, c_0 + \epsilon)$. By Lemma 3.1 of [CMN], $I'(\eta(t, \gamma(\theta))) \rightarrow 0$ as $t \rightarrow \infty$. From here on, denote $\eta \equiv \eta(t) \equiv \eta(t, \gamma(\theta))$.

Let $u_m = \eta(m)$. Since $I'(\eta(t)) \rightarrow 0$ as $t \rightarrow \infty$, $\|u_{m+1} - u_m\| \rightarrow 0$ as $m \rightarrow \infty$. Using Proposition 2.0, together with an indirect argument, it is easy to show that there

exists $\rho > 0$ and a sequence $(y_m) \subset \mathbf{R}^N$ with $\|u_m\|_{W^{1,2}(B_1(y_m))} > \rho$ for large m . Let $\mathbf{e}_i = (0, 0, \dots, 1, \dots, 0)$ denoted the i th standard basis vector for $\mathbf{R}^N$. We claim that for $i \leq N - 1$, $(y_m \cdot \mathbf{e}_i)$ is bounded in m . Proof: suppose not. Then $(y_m \cdot \mathbf{e}_i)$ has an unbounded subsequence for, say, $i = 1$. Let $(u_{m_i}) \subset (u_m)$ be a subsequence along which $|y_{m_i} \cdot \mathbf{e}_1| \rightarrow \infty$. Without loss of generality assume $\epsilon < c_0$. Apply Proposition 2.1 to (u_{m_i}) to obtain a subsequence (also denoted (u_{m_i})), a sequence $(x_i) \subset \mathbf{R}^N$, and $v \in E$ with $\|u_{m_i} - \tau_{x_i} v\| \rightarrow 0$ as $i \rightarrow \infty$. Then

$$(3.1) \quad \|\tau_{x_i} v\|_{W^{1,2}(B_1(y_{m_i}))} > \rho/2$$

for large i . The flow η preserves symmetry with respect to the x_1 -, x_2 -, $\dots$, x_{N-1} -hyperplanes. Therefore $u_m \in E[1, 2, \dots, N - 1]$ for all m , using the notation of (1.8). So for large i , $\|u_{m_i}\|_{W^{1,2}(B_1(R_1(y_{m_i})))} > \rho$, and

$$(3.2) \quad \|\tau_{x_i} v\|_{W^{1,2}(B_1(R_1(y_{m_i})))} > \rho/2.$$

By (3.1)-(3.2),

$$(3.3) \quad \|v\|_{W^{1,2}(B_1(y_{m_i}-x_i))} > \rho/2 \text{ and } \|v\|_{W^{1,2}(B_1(R_1(y_{m_i})-x_i))} > \rho/2.$$

This is impossible because $|y_{m_i} \cdot \mathbf{e}_1| \rightarrow \infty$ as $i \rightarrow \infty$, implying $|y_{m_i} - R_1(y_{m_i})| \rightarrow \infty$ and $|y_{m_i} - x_i| + |R_1(y_{m_i}) - x_i| \rightarrow \infty$. The claim is proven, and for all $i \leq N - 1$, $(y_m \cdot \mathbf{e}_i)$ is bounded in m .

Next, we claim that $y_m - y_{m+1}$ is bounded in m . To prove, assume otherwise. Then along a subsequence $(y_{m_i}) \subset (y_m)$, $|y_{m_i+1} - y_{m_i}| \rightarrow \infty$. Apply Proposition 2.1 to (y_{m_i}) . By a similar argument to that above, we obtain a contradiction.

$(y_m \cdot \mathbf{e}_N)$ is either bounded or unbounded in m . If bounded, then (y_m) is bounded. Taking a subsequence, we may assume $y_m \rightarrow \bar{y}$ as $m \rightarrow \infty$. Applying Proposition 2.1 to (u_m) , we obtain a subsequence (also called (u_m)), $(x_m) \subset \mathbf{R}^N$, $\bar{h} \in \text{Hull}(h)$ with $\tau_{-x_m} h \rightarrow \bar{h}$ uniformly, and v with $\|u_m - \tau_{x_m} v\| \rightarrow 0$, $I[\bar{h}](v) = 0$, and $I[\bar{h}](v) \in [c_0, c_0 + \epsilon)$. It suffices to show that $\bar{h}$ is merely a translate of h . By arguments similar to those above, it is easy to show that $(x_m - y_m)$ is bounded. Therefore, we assume that (x_m) is convergent. Then obviously $\bar{h}$ is a translate of h .

The other alternative is that $(y_m \cdot \mathbf{e}_N)$ is unbounded in m . By [S1], for any $\delta > 0$, there exists a relatively dense subset $\mathcal{A} \subset \mathbf{R}$ such that for all $a \in \mathcal{A}$, $a\mathbf{e}_N$ is a δ -almost

period of h . Since $(y_{m+1} - y_m)$ is bounded, and $(y_m \cdot \mathbf{e}_i)$ is bounded for all $i \leq N - 1$, there exist a sequence $\delta_m \rightarrow 0$, a sequence $(a_m) \subset \mathbf{R}$ with $a_m \mathbf{e}_N$ a δ_m -almost period of h for all m , and a subsequence of (u_m) (also denoted (u_m)) with $\sup_{m \geq 1} |y_m - a_m \mathbf{e}_N| < \infty$. Apply Proposition 2.1 to this latest (u_m) to obtain yet another subsequence (u_m) , and $(x_m) \subset \mathbf{R}^N$, $\bar{h} \in \text{Hull}(h)$, and v satisfying (2.1)(i)-(iv). As before, it suffices to show $\bar{h}$ is a translate of h . By similar arguments to before, $(x_m - y_m)$ is bounded, so by taking a subsequence, we may assume $(x_m - a_m \mathbf{e}_N)$ is convergent, with $x_m - a_m \mathbf{e}_N \rightarrow z$. Since $a_m \mathbf{e}_N$ is a δ_m -almost period of h , $a_m \mathbf{e}_N$ is a δ_m -almost period of any translate of h as well. $\tau_{-x_m} h \rightarrow \bar{h}$ uniformly, so $\tau_{-z-a_m \mathbf{e}_N} h \rightarrow \bar{h}$ uniformly, and $\bar{h} = \tau_{-z} h$. Proposition 1.13 is proven.

Proof of Proposition 1.14

Let $d > 0$ such that $[c_0, c_0 + d)$ contains no critical values of I . Fix $l \in 0, 1, \dots, N - 2$ and assume $c_l = c_0$. Let $\epsilon \in (0, d)$. Let $\{i_1, \dots, i_l\} \subset \{1, \dots, N - 1\}$ and $\gamma \in \Gamma[i_1, \dots, i_l]$ with

$$(3.4) \quad \max_{\theta \in [0, 1]} I(\gamma(\theta)) < c_0 + \epsilon.$$

For convenience, assume without loss of generality that $\gamma \in \Gamma[1, 2, \dots, l]$. Let η be as in the proof of Proposition 1.13. As before, there exists $\theta \in [0, 1]$ with

$$(3.5) \quad \lim_{t \rightarrow \infty} I(\eta(t, \gamma(\theta))) \in [c_0, c_0 + \epsilon).$$

Let $h^+ = \sup_{\mathbf{R}^N} h$ and let s be as in (f_3) . Let $P > 0$ be large enough so that if $\Omega = \mathbf{R}^N$ or the half-space $\{x \in \mathbf{R}^N \mid x_1 > 0\}$, then $\|u\|_{L^{s+1}(\Omega)} \leq P\|u\|_{W^{1,2}(\Omega)}$. By $(f_1) - (f_3)$, there exists $A > 0$ such that for all $q \in \mathbf{R}$, $|f(q)| \leq |q|/(2h^+) + A|q|^s$. For $\Omega \subset \mathbf{R}^N$, define $\|u\|_\Omega = \|u\|_{W^{1,2}(\Omega)}$ and J_Ω on $W^{1,2}(\Omega)$ by $J_\Omega(u) = \|u\|_\Omega^2 - \int_\Omega h f(u)u$. Then $J_{\mathbf{R}^N}(u) = I'(u)u$. Define

$$(3.6) \quad r_0 = \frac{1}{3} \left(\frac{1}{4h^+ A P^{s+1}} \right)^{\frac{1}{s-1}}.$$

If $\Omega = \mathbf{R}^N$ or $\{x_1 > 0\}$, and $\|u\|_\Omega \leq 3r_0$, then

$$(3.7) \quad J_\Omega(u) = \|u\|_\Omega^2 - \int_\Omega h f(u)u \geq \|u\|_\Omega^2 - h^+ \int_\Omega f(u)u \geq$$

$$\begin{aligned}
&\geq \|u\|_\Omega^2 - h^+ \int_\Omega \frac{u^2}{2h^+} + A|u|^{s+1} \geq \frac{1}{2}\|u\|_\Omega^2 - Ah^+ \int_\Omega |u|^{s+1} = \\
&= \frac{1}{2}\|u\|_\Omega^2 - Ah^+ \|u\|_{L^{s+1}(\Omega)}^{s+1} \geq \frac{1}{2}\|u\|_\Omega^2 - Ah^+ P^{s+1} \|u\|_\Omega^{s+1} = \\
&= \|u\|_\Omega^2 \left(\frac{1}{2} - h^+ AP^{s+1} \|u\|_\Omega^{s-1} \right) \geq \\
&\geq \|u\|_\Omega^2 \left(\frac{1}{2} - h^+ AP^{s+1} (3r_0)^{s-1} \right) \geq \|u\|_\Omega^2 / 4.
\end{aligned}$$

The following estimates will also be useful. For any $w \in E$,

$$\begin{aligned}
\|w\|^2 &= 2I(w) + 2 \int_{\mathbf{R}^N} hF(u) \leq 2I(w) + \frac{2}{\mu} \int_{\mathbf{R}^N} hf(w)w = \\
&= 2I(w) + \frac{2}{\mu} (\|w\|^2 - I'(w)w),
\end{aligned}$$

so

$$(3.8) \quad \|w\|^2 \leq \frac{2\mu I(w) - 2I'(w)w}{\mu - 2},$$

and with $h_- = \inf_{\mathbf{R}^N} h > 0$,

$$\begin{aligned}
(3.9) \quad \int_{\mathbf{R}^N} wf(w) &\leq \frac{1}{h_-} \int_{\mathbf{R}^N} hwf(w) = \frac{1}{h_-} (\|w\|^2 - I'(w)w) \leq \\
&\leq \left(\frac{\mu}{h_-} \right) \frac{2I(w) - I'(w)w}{\mu - 2}
\end{aligned}$$

and

$$(3.10) \quad \int_{\mathbf{R}^N} F(w) \leq \frac{1}{\mu} \int_{\mathbf{R}^N} wf(w) \leq \left(\frac{1}{h_-} \right) \frac{2I(w) - I'(w)w}{\mu - 2}.$$

Let $\eta \equiv \eta(t) \equiv \eta(t, \gamma(\theta))$. Since $I'(\eta(t)) \rightarrow 0$, by (3.8) and (3.7), we may pick $t_0 > 0$ to be large enough so that for all $t \geq t_0$,

$$(3.11) \quad |I'(\eta(t))\eta(t)| < \min(1, \epsilon, r_0^2/8) \text{ and } \|\eta(t)\| > 3r_0.$$

For $i = l+1, l+2, \dots, N-1$, let $T_i^- < 0 < T_i^+$, with $T_i^- \mathbf{e}_i$ and $T_i^+ \mathbf{e}_i$ both ϵ -almost periods of h for all i (this is possible by [S1]), and $|T_i^-|$ and T_i^+ large enough to satisfy

$$(3.12) \quad \|\eta(t_0)\|_{\{x_i > T_i^+\}} < r_0$$

and

$$(3.13) \quad \|\eta(t_0)\|_{\{x_i < T_i^-\}} < r_0$$

Two cases are possible:

(3.14) I. For all $t \geq t_0$ and $i = l+1, l+2, \dots, N-1$,

$$\|\eta(t)\|_{\{x_i < T_i^+\}} > r_0 \text{ and } \|\eta(t)\|_{\{x_i > T_i^-\}} > r_0.$$

II. Case I does not hold.

In Case I, let $u_m = \eta(m)$. As in the proof of Proposition 1.13, $I'(u_m) \rightarrow 0$ as $m \rightarrow \infty$ and $\|u_{m+1} - u_m\| \rightarrow 0$. As before, there exist a sequence $(x_m) \subset \mathbf{R}^N$ and $\rho > 0$ such that $(x_{m+1} - x_m)$ is bounded and $\|u_m\|_{W^{1,2}(B_1(x_m))} > \rho$ for large m . Assume without loss of generality that $c_0 > d > \epsilon$. By Proposition 2.1, and the fact that $u_m \in E[1, 2, \dots, l]$, $(x_m \cdot \mathbf{e}_i)$ is bounded in m for $i = 1, 2, \dots, N-1$. The argument from the proof of Proposition 1.13 shows that I has a critical point with critical level in the interval $[c_0, c_0 + \epsilon)$, contradicting our assumption that $[c_0, c_0 + d)$ contains no critical values of I . Thus Case II holds.

For Case II, assume without loss of generality that Case I fails for $\{x_{l+1} < T_{l+1}^+\}$. By (3.11), $\|\eta(t)\| > 3r_0$ for all $t \geq t_0$, so (3.12) implies $\|\eta(t_0)\|_{\{x_{l+1} < T_{l+1}^+\}} \geq 2r_0$, and we can define

$$(3.15) \quad t_2 = \min\{t > t_0 \mid \|\eta(t)\|_{\{x_{l+1} < T_{l+1}^+\}} = r_0\} > t_0.$$

Now $\|\eta(t_2)\|_{\{x_{l+1} > T_{l+1}^+\}} \geq 2r_0$, so by (3.12), we can define

$$(3.16) \quad t_1 = \max\{t < t_2 \mid \|\eta(t)\|_{\{x_{l+1} > T_{l+1}^+\}} = r_0\} \in (t_0, t_2).$$

For all $t \in (t_1, t_2)$,

$$(3.17) \quad \|\eta(t)\|_{\{x_{l+1} < T_{l+1}^+\}} > r_0 \text{ and } \|\eta(t)\|_{\{x_{l+1} > T_{l+1}^+\}} > r_0.$$

By (3.7) and (3.16),

$$(3.18) \quad J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t_1)) \geq \|\eta(t_1)\|_{\{x_{l+1} > T_{l+1}^+\}}^2 / 4 = r_0^2 / 4 > 0.$$

Likewise, by (3.7) and (3.15),

$$(3.19) \quad J_{\{x_{l+1} < T_{l+1}^+\}}(\eta(t_2)) \geq r_0^2 / 4,$$

so by (3.19) and (3.11),

$$(3.20) \quad \begin{aligned} J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t_2)) &= J_{\mathbf{R}^N}(\eta(t_2)) - J_{\{x_{l+1} < T_{l+1}^+\}}(\eta(t_2)) \leq \\ &\leq I'(u)u - r_0^2/4 < r_0^2/8 - r_0^2/4 < 0. \end{aligned}$$

By (3.18) and (3.20), there exists $t^* \in (t_1, t_2)$ with $J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t^*)) = 0$, so by (3.11),

$$(3.21) \quad \max(|J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t^*))|, |J_{\{x_{l+1} < T_{l+1}^+\}}(\eta(t^*))|) < \epsilon.$$

Define

$$(3.22) \quad z = \tau_{-T_{l+1}^+ \mathbf{e}_{l+1}} \eta(t^*).$$

Since $d < c_0$, (3.8)-(3.10) imply that for all $t \geq t_0$,

$$(3.23) \quad \begin{aligned} \max(|I'(\eta(t))\eta(t)|, \int_{\mathbf{R}^N} F(\eta(t)), \int_{\mathbf{R}^N} f(\eta(t))\eta(t)) &\leq \\ &\leq 1 + \frac{\mu}{h_-} \left(\frac{4c_0 + 1}{\mu - 2} \right) \equiv K(f, h). \end{aligned}$$

Thus

$$(3.24) \quad \begin{aligned} |I(z) - I(\eta(t^*))| &= \left| \int_{\mathbf{R}^N} hF(z) - \int_{\mathbf{R}^N} hF(\eta(t^*)) \right| = \\ &= \left| \int_{\mathbf{R}^N} h(x)F((\eta(t^*)(x - T_{l+1}^+ \mathbf{e}_{l+1}))) dx - \int_{\mathbf{R}^N} h(x)F(\eta(t^*)(x)) dx \right| = \\ &= \left| \int_{\mathbf{R}^N} (h(y + T_{l+1}^+ \mathbf{e}_{l+1}) - h(y))F(\eta(t^*)(y)) dy \right| \leq \\ &\leq \epsilon \int_{\mathbf{R}^N} F(\eta(t^*)) \leq \epsilon K(f, h). \end{aligned}$$

Similarly,

$$(3.25) \quad \begin{aligned} |J_{\{x_{l+1} > 0\}}(z) - J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t^*))| &= \\ &= \left| \int_{x_{l+1} > 0} hf(z)z - \int_{x_{l+1} > T_{l+1}^+} hf(\eta(t^*))\eta(t^*) \right| = \\ &= \left| \int_{x_{l+1} > T_{l+1}^+} h(x - T_{l+1}^+ \mathbf{e}_{l+1})f(z(x - T_{l+1}^+ \mathbf{e}_{l+1}))z(x - T_{l+1}^+ \mathbf{e}_{l+1}) - \right. \\ &\quad \left. h(x)f(\eta(t^*)(x))\eta(t^*)(x) dx \right| = \\ &= \left| \int_{x_{l+1} > T_{l+1}^+} (h(x - T_{l+1}^+ \mathbf{e}_{l+1}) - h(x))f(\eta(t^*)(x))\eta(t^*)(x) dx \right| \leq \\ &\leq \epsilon \int_{x_{l+1} > T_{l+1}^+} f(\eta(t^*)(x))\eta(t^*)(x) dx \leq \epsilon \int_{\mathbf{R}^N} f(\eta(t^*))\eta(t^*) \leq \epsilon K(f, h) \end{aligned}$$

and

$$(3.26) \quad |J_{\{x_{l+1} < 0\}}(z) - J_{\{x_{l+1} < T_{l+1}^+\}}(\eta(t^*))| \leq K(f, h)\epsilon.$$

For $\Omega \subset \mathbf{R}^N$, define I_Ω by $I_\Omega(w) = \frac{1}{2}\|w\|_\Omega^2 - \int_\Omega hF(w)$. By (3.24),

$$\begin{aligned} I_{\{x_{l+1} < 0\}}(z) + I_{\{x_{l+1} > 0\}}(z) &= I(z) < I(\eta(t^*)) + \epsilon K(f, h) < \\ &< c_0 + \epsilon(1 + K(f, h)), \end{aligned}$$

so

$$(3.27) \quad \min(I_{\{x_{l+1} < 0\}}(z), I_{\{x_{l+1} > 0\}}(z)) < \frac{1}{2}(c_0 + \epsilon(1 + K(f, h))).$$

Assume without loss of generality that $I_{\{x_{l+1} > 0\}}(z) < \frac{1}{2}(c_0 + \epsilon(1 + K(f, h)))$, and define $\hat{z}$ by reflecting the $\{x_{l+1} > 0\}$ -half of z over the x_{l+1} -axis. That is,

$$(3.28) \quad \hat{z}(x) = \begin{cases} z(x); & x_{l+1} \geq 0 \\ z(R_{l+1}x); & x_{l+1} < 0, \end{cases}$$

where the reflection operator R_i is as in (1.6). Then $\hat{z} \in E[1, 2, \dots, l+1]$.

Now $\|\hat{z}\| \leq 2\|\eta(t^*)\| < 2K(f, h)$ by (3.23). Also,

$$(3.29) \quad \|\hat{z}\| \geq \|\hat{z}\|_{\{x_{l+1} > 0\}} = \|\eta(t^*)\|_{\{x_{l+1} > T_{l+1}^+\}} > r_0$$

by (3.17),

$$\begin{aligned} (3.30) \quad |I'(\hat{z})\hat{z}| &= |J_{\mathbf{R}^N}(\hat{z})| = 2|J_{\{x_{l+1} > 0\}}(\hat{z})| = 2|J_{\{x_{l+1} > 0\}}(z)| \leq \\ &\leq 2(|J_{\{x_{l+1} > T_{l+1}^+\}}(\eta(t^*))| + \epsilon K(f, h)) < 2\epsilon(1 + K(f, h)) \end{aligned}$$

by (3.21) and (3.25). Also, $I(\hat{z}) = 2I_{\{x_{l+1} > 0\}}(z) < c_0 + \epsilon(1 + K(f, h))$ by (3.27).

Letting $\epsilon_m \rightarrow 0$ and constructing $\hat{z}_m$ in this manner, we obtain a bounded sequence $(\hat{z}_m) \subset E_{l+1}$ as in (2.12) with $\|\hat{z}_m\|$ bounded away from zero, $I'(\hat{z}_m)\hat{z}_m \rightarrow 0$, and $\limsup_{m \rightarrow \infty} I(z) \leq c_0$. Therefore, by (2.12), $c_{l+1} \leq c_0$. Thus $c_{l+1} = c_0$, and Proposition 1.14 is proven.

Positive Solutions

By the argument in the Introduction, for any $\epsilon > 0$, there is a critical value of I in the interval $[c_0, c_0 + \epsilon)$. To complete the proof of Theorem 1.2, let $\epsilon_m \rightarrow 0$, and let u_m

be a critical point of I with $I(u_m) \in [c_0, c_0 + \epsilon_m)$. By Proposition 2.1, there exists $\bar{h}$ in the hull of h , a sequence $(x_m) \subset \mathbf{R}^N$, and $v \in E$ such that $I[\bar{h}](v) = 0$, $I[\bar{h}](v) = c_0$, and $\|u_m - \tau_{x_m} v\| \rightarrow 0$ as $m \rightarrow \infty$. Since c_0 is the minimax value associated with $I[\bar{h}]$, arguments of [CR1], employing (f_5) , show that v does not change sign. Without loss of generality, v is nonnegative. Since $\tau_{-x_m} u_m \rightarrow v$ in $W^{1,2}(\mathbf{R}^N)$, $\tau_{-x_m} u_m \rightarrow v$ uniformly. Then a standard maximum principle argument shows that for large m , u_m is a positive function.

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