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EXPONENTIAL GROWTH RATE OF PATHS AND ITS CONNECTION WITH DYNAMICS

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Let $L = (l_{ij})_{1 \leq i, j \leq p}$ be a square matrix with $l_{ij} > 0$. We present a method to calculate the exponential growth rate of the number of paths in an associated directed graph G with length information L , via classifying the paths by their types of primitive cycles. After computing several examples, we show that the exponential growth rate equals to the topological entropy of special suspension flows associated to L , and this entropy is equal to the unique number λ such that the principal eigenvalue of $(e^{-l_{ij}\lambda})_{1 \leq i, j \leq p}$ is 1.

1. Introduction

In this paper we consider a directed graph whose edges can have different length. Let $p \geq 2$ be an integer. Consider a matrix of length information $L = (l_{ij})_{1 \leq i, j \leq p}$ with $l_{ij} > 0$. We can associate a directed graph G to L as

- (1) $V(G) = \{v_1, \dots, v_p\}$ and there exists an directed edge e_{ij} from i to j for all $i, j = 1, \dots, p$.
- (2) The length of e_{ij} is l_{ij} for all $i, j = 1, \dots, p$.

For simplicity, we also allow the case $l_{ij} = +\infty$. When $l_{ij} = +\infty$, it simply means that there is no edge from v_i to v_j .

Let $T > 0$ be a positive real number. We consider the collection $\mathcal{P}(G, T)$ of piecewisely smooth paths $\gamma : [0, T] \rightarrow G$, where G has length information L , satisfying

- $\gamma(t_k) = v_{i(t_k)}$ for $0 = t_0 < \cdots < t_n < T$ with $t_{k+1} - t_k = l_{i(t_k), i(t_{k+1})}$ for $k = 0, \dots, n-1$.
- $\gamma|_{[t_k, t_{k+1}]}$ is exactly the direct edge $e_{i(t_k), i(t_{k+1})}$ with unit speed for $k = 0, \dots, n-1$.
- $\gamma|_{[t_n, T]}$ lies totally in one edge $e_{i(t_n), j}$ and $T - t_n \leq l_{i(t_n), j}$ for some unique $1 \leq j \leq p$.

Now we define the *exponential growth rate* of the directed graph G with length information L to be

$$\lambda(L) = \lambda(G) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \# \mathcal{P}(G, T). \quad (1)$$

We could replace the limit by $\limsup$ or $\liminf$ in (1) if the limit does not exist. However, we shall see that the limit always exists. To estimate $\lambda(G)$ we classify the paths in $\mathcal{P}(G, T)$ by different types of primitive paths of G . Thus we reduce the estimate of $\# \mathcal{P}(G, T)$ to the estimate the number of paths having each possible types. This procedure is performed in Section 2 for several simple cases, showing that in these cases the limit (1) does exist. The calculations there also suggest there might be some relation between $\lambda(L)$ and the *principal eigenvalue* of the matrix $e^{-L\lambda} = (e^{-l_{ij}\lambda})_{1 \leq i, j \leq p}$. Note that the exponential matrix we have here is different from the normal one. Here we just simply raise each entry of the matrix by e .

We prove in Section 3 that the exponential growth rate equals to the topological entropy of special suspension flows associated to L (hence the limit (1) always exist), and also equals to the unique real number λ such that the principal eigenvalue of $e^{-L\lambda}$ is 1. (We note that above λ is unique since the principal eigenvalue of $e^{-L\lambda}$ is strictly decreasing with respect to $\lambda \in \mathbb{R}$.)

The *Perron–Frobenius Theorem* states that for each irreducible nonnegative matrix $A = (a_{ij})$, i.e., $a_{ij} \geq 0$ and $(A^m)_{ij} > 0$ for some $m \geq 1$ for all $i, j = 1, \dots, p$, there is a unique simple and positive eigenvalue of A with maximal norm. This eigenvalue is called the principal eigenvalue of A . For general nonnegative matrix A there also exists an eigenvalue of maximal norm among all eigenvalues of A . In this case it may not be simple and there may exist other eigenvalue of same norm.

To conclude this section, we explain why directed graph with varying lengths of edges is interesting and useful. In a normal subshift of finite type, any entry of the transition matrix is either zero or one. That is either there is no path from one vertex to another, or there is one with fixed length. There are natural situations where transition to different states may

require different time. There is an abundance of such example in population dynamics. A simple example would be a species consisting two groups, with each group having different reproduction rate.

Another situation where we have directed graph with varying edge lengths is the suspension of Anosov diffeomorphism, or a suspension of hyperbolic invariant set. In general, the topological entropy will depend on the length of the transition at each point, therefore it is very difficult to calculate. However, if the transition time between any two Markov partitions is a constant, then calculation is possible and this is exactly what we used to derive our formula.

Finally as an example, let h the topological entropy for the directly graph with the following length matrix

$$A = \begin{pmatrix} 1 & 1 \\ \tau & \infty \end{pmatrix}$$

for some $\tau > 0$. Then $h = \ln \lambda_1$, where λ_1 is the unique positive root of $\lambda^{1+\tau} - \lambda^\tau - 1 = 0$.

2. Exponential Growth Rate of Directed Graph with Length Information

Let L be a matrix of length information and G the directed graph associated to L whose edge e_{ij} is of length l_{ij} for $1 \leq i, j \leq p$. Set $l^* = \max\{l_{ij} | l_{ij} < \infty\}$. We will use $f(T) \sim g(T)$ to denote the relation $\lim_{T \rightarrow +\infty} (f(T) - g(T)) = 0$. Recall that for $T > 0$, $\mathcal{P}(G, T)$ is the collection of piecewisely smooth paths defined be (1). Note in the case $l_{ij} = \infty$ for some i, j , any path in $\mathcal{P}(G, T)$ will have no edge of e_{ij} , or the number of paths containing e_{ij} always be zero.

2.1. Case 1

Firstly let us consider a special length matrix L with $l_{ij} = l_i$ for all $i, j = 1, \dots, p$. The case $l_{ij} = l_j$ for all i, j can be treated by revising the direction. We need to compute $N(L, T) = \#\mathcal{P}(G, T)$. For $i = 1, \dots, p$, let $n_i(\gamma)$ to be the number of times the path $\gamma \in \mathcal{P}(G, T)$ passing through the vertex v_i . Then define

$$\Gamma^1(T) = \{(n_i)_{i=1}^p : n_i \geq 0 \text{ and } \sum_{i=1}^p l_i n_i \in [T, T + l^*)\}. \quad (1)$$

For each choice $(n_i)_{i=1}^p \in \Gamma^1(T)$ there would be $\frac{(\sum_{i=1}^p n_i)!}{\prod_{i=1}^p n_i!}$ kinds of different patterns of paths which pass each vertex i for exactly n_i times. For T large

enough we have $N(L, T) \leq \sum_{(n_i)_{i=1}^p \in \Gamma^1(T)} N((n_i)_{i=1}^p) \leq N(L, l^*) \cdot N(L, T)$. Note that $\#\Gamma^1(T)$ has polynomial growth as $T \rightarrow \infty$. Then

$$\begin{aligned} \frac{1}{T} \log N(L, T) &\sim \frac{1}{T} \log \sum_{(n_i) \in \Gamma^1(T)} \frac{(\sum_{i=1}^p n_i)!}{\prod_{i=1}^p n_i!} \\ &\sim \max_{(n_i) \in \Gamma^1(T)} \left\{ \frac{1}{T} \log \frac{(\sum_{i=1}^p n_i)^{\sum_{i=1}^p n_i}}{\prod_{i=1}^p n_i^{n_i}} \right\} \\ &= \max_{(n_i) \in \Gamma^1(T)} \left\{ \frac{\sum_{i=1}^p n_i}{T} \log \frac{\sum_{i=1}^p n_i}{T} - \sum_{i=1}^p \frac{n_i}{T} \log \frac{n_i}{T} \right\}. \end{aligned}$$

Then we get the limit $\lambda(G) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log N(L, T)$ exists and equals to, via putting $x_i = n_i/T$ for each $i \in \{1, \dots, p\}$,

$$\lambda(G) = \max \left\{ \left(\sum_{i=1}^p x_i \right) \log \left(\sum_{i=1}^p x_i \right) - \sum_{i=1}^p x_i \log x_i : x_i \geq 0 \text{ and } \sum_{i=1}^p l_i x_i = 1 \right\}.$$

Solving this conditional maximal problem we easily get the exponential growth rate $\lambda(G)$ to be λ where λ is the unique positive solution of $\sum_{i=1}^p e^{-l_i \lambda} = 1$.

2.2. Case 2

Secondly let us consider a 2×2 length matrix L . Given $L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$. For a directed graph G with two vertices the collection of primitive cycles contains

- (1) two *1-cycles*, denoting as b_i consisting exactly one edge e_{ii} for $i = 1, 2$,
- (2) a *2-cycle* b_{12} , consisting of exactly one edge e_{12} and one edge e_{21} .

Every path $\gamma \in \mathcal{P}(G, T)$ can be divided into some combination of these bricks and at most one extra e_{12} or e_{21} . For example the path along 122111221112 has: four bricks of b_1 ; two bricks of b_2 ; two bricks of b_{12} and one extra edge e_{12} . Let $\mathcal{J} = \{1, 2, 12\}$. Given a path γ of length T , we use n_* to denote the number of times that the primitive cycle b_* appears in γ for each $* \in \mathcal{J}$. Then

$$\begin{aligned} \Gamma^2(T) &= \{(n_*)_{* \in \mathcal{J}} : n_* \geq 0 \text{ for each } * \in \mathcal{J} \\ &\quad \text{and } l_{11}n_1 + l_{22}n_2 + (l_{12} + l_{21})n_{12} \in (T - l^*, T]\}. \end{aligned} \quad (2)$$

Let $N(n_1, n_2, n_{12})$ be the number of patterns of paths consisting of n_* copies of primitive cycle b_* . Then for T large enough we have

$N(L, T - l^*) \leq \sum_{(n_*) \in \Gamma^2(T)} N(n_1, n_2, n_{12}) \leq N(L, T)$. Now it is easy to see $N(n_1, n_2, n_{12}) = C_{n_1+n_{12}}^{n_1} \cdot C_{n_2+n_{12}}^{n_2}$, $\#\Gamma^2(T)$ has polynomial growth as $T \rightarrow \infty$ and

$$\begin{aligned} \frac{1}{T} \log N(L, T) &\sim \frac{1}{T} \log \sum_{(n_*) \in \Gamma^2(T)} N(n_1, n_2, n_{12}) \\ &= \frac{1}{T} \log \sum_{(n_*) \in \Gamma^2(T)} C_{n_1+n_{12}}^{n_1} \cdot C_{n_2+n_{12}}^{n_2} \\ &\sim \max_{(n_*) \in \Gamma^2(T)} \left\{ \frac{1}{T} \log \frac{(n_1 + n_{12})^{n_1+n_{12}}}{n_1^{n_1} n_{12}^{n_{12}}} + \frac{1}{T} \log \frac{(n_2 + n_{12})^{n_2+n_{12}}}{n_2^{n_2} n_{12}^{n_{12}}} \right\} \\ &= \max_{(n_*) \in \Gamma^2(T)} \left\{ \frac{n_1 + n_{12}}{T} \log \frac{n_1 + n_{12}}{T} - \frac{n_1}{T} \log \frac{n_1}{T} - \frac{n_{12}}{T} \log \frac{n_{12}}{T} \right. \\ &\quad \left. + \frac{n_2 + n_{12}}{T} \log \frac{n_2 + n_{12}}{T} - \frac{n_2}{T} \log \frac{n_2}{T} - \frac{n_{12}}{T} \log \frac{n_{12}}{T} \right\}. \end{aligned}$$

Let $x_* = n_*/T$ for each $*$ $\in \mathcal{J}$. Similarly to Case 1 we have the limit $\lambda(L) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log N(L, T)$ exists and satisfies

$$\begin{aligned} \lambda(L) &= \max \{ (x_1 + x_{12}) \log(x_1 + x_{12}) - x_1 \log x_1 - x_{12} \log x_{12} \\ &\quad + (x_2 + x_{12}) \log(x_2 + x_{12}) - x_2 \log x_2 - x_{12} \log x_{12} : \\ &\quad x_* \geq 0 \text{ for each } * \in \mathcal{J} \text{ and } l_{11}x_1 + (l_{12} + l_{21})x_{12} + l_{22}x_2 = 1 \}. \end{aligned}$$

Solving this conditional maximal problem we easily get the exponential growth rate $\lambda(G)$ to be λ where λ is the maxiaml positive solution of $(1 - e^{-l_{11}\lambda})(1 - e^{-l_{22}\lambda}) = e^{-l_{12}\lambda - l_{21}\lambda}$.

Remark 2.1. Above relation can be written as

$$\det \begin{pmatrix} 1 - e^{-l_{11}\lambda} & e^{-l_{12}\lambda} \\ e^{-l_{21}\lambda} & 1 - e^{-l_{22}\lambda} \end{pmatrix} = 0.$$

We will see that λ is the unique real number such that the principal eigenvalue of the matirx $e^{-L\lambda}$ is 1.

2.3. Case 3

Now we consider $p = 3$. Given $L = \begin{pmatrix} l_{11} & l_{12} & l_{12} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$. We are to compute

$N(L, T)$. For the directed graph G with three vertices, the collection of primitive cycles contains

- (1) three 1-cycles denoting as b_i consisting exact one edge e_{ii} for $i = 1, 2, 3$,

- (2) three 2-cycles b_{ij} consisting of exactly two edges $e_{ij} + e_{ji}$ for $ij = 12, 13, 23$,
- (3) two 3-cycles b_{123} and b_{132} consisting of $e_{12} + e_{23} + e_{31}$ and $e_{13} + e_{32} + e_{21}$.

Every path of length T can be divided into some combination of these bricks and at most two extra edges e_{ij} . For example the path along 12233111323313213112323 has: three bricks of b_1 ; one brick of b_2 ; two bricks of b_3 ; two bricks of b_{13} ; two bricks of b_{23} ; one brick of b_{123} ; one brick of b_{132} ; two extra edges e_{12} and e_{23} . Let $\mathcal{I} = \{1, 2, 3, 12, 13, 23, 123, 132\}$. Given a path γ of length T , we use n_* to denote the number of times that the primitive cycle b_* appears in γ for each $* \in \mathcal{I}$. Then

$$\begin{aligned} \Gamma^3(T) = \{ & (n_*)_{* \in \mathcal{I}} : n_* \geq 0, \sum_{i=1}^3 l_{ii} n_i + \sum_{ij \in \{12, 13, 23\}} (l_{ij} + l_{ji}) n_{ij} \\ & + \sum_{ijk \in \{123, 133\}} (l_{ij} + l_{jk} + l_{ki}) n_{ijk} \in (T - l^*, T] \}. \end{aligned} \quad (3)$$

Let $N((n_*)_{* \in \mathcal{I}})$ be the number of patterns of paths consisting of n_* copies of primitive cycle b_* for each $* \in \mathcal{I}$. Clearly each path $\gamma \in \mathcal{P}(G, T)$ has a unique type $((n_*)) \in \Gamma^3(T)$. We have $N(L, T - l^*) \leq \sum_{(n_*) \in \Gamma^3(T)} N((n_*)_{* \in \mathcal{I}}) \leq N(L, T)$. With (n_*) bricks we can build a path by firstly arranging the 3-cycles, secondly adding 2-cycles, and finally adding the rest 1-cycles. Using combinatorial method it is easy to get

$$\begin{aligned} N((n_*)_{* \in \mathcal{I}}) = & C_{n_{123}+n_{132}}^{n_{123}} C_{n_{123}+n_{132}+n_{12}}^{n_{12}} C_{n_{123}+n_{132}+n_{12}+n_{13}}^{n_{13}} C_{n_{123}+n_{132}+n_{12}+n_{13}+n_{23}}^{n_{23}} \\ & C_{n_{123}+n_{132}+n_{12}+n_{13}+n_{21}}^{n_{21}} C_{n_{123}+n_{132}+n_{12}+n_{23}+n_{21}}^{n_{23}} C_{n_{123}+n_{132}+n_{13}+n_{23}+n_{31}}^{n_{31}} \end{aligned}$$

and similarly we get, via $x_* = n_*/T$ for each $* \in \mathcal{I}$ as in Case 1 and 2,

$$\begin{aligned} \lambda(G) &= \lim_{T \rightarrow +\infty} \frac{1}{T} \log N(L, T) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \sum_{(n_*) \in \Gamma^3(T)} N((n_*)_{* \in \mathcal{I}}) \\ &= \max\{ (x_{123} + x_{132} + x_{12} + x_{13} + x_{23}) \log(x_{123} + x_{132} + x_{12} + x_{13} + x_{23}) \\ &\quad - x_{123} \log x_{123} - x_{132} \log x_{132} - x_{12} \log x_{12} - x_{13} \log x_{13} - x_{23} \log x_{23} \\ &\quad + (x_{123} + x_{132} + x_{12} + x_{13} + x_{21}) \log(x_{123} + x_{132} + x_{12} + x_{13} + x_{21}) \\ &\quad - (x_{123} + x_{132} + x_{12} + x_{13}) \log(x_{123} + x_{132} + x_{12} + x_{13}) - x_{21} \log x_{21} \\ &\quad + (x_{123} + x_{132} + x_{12} + x_{23} + x_{21}) \log(x_{123} + x_{132} + x_{12} + x_{23} + x_{21}) \end{aligned}$$

$$\begin{aligned}
& - (x_{123} + x_{132} + x_{12} + x_{23}) \log(x_{123} + x_{132} + x_{12} + x_{23}) - x_2 \log x_2 \\
& + (x_{123} + x_{132} + x_{13} + x_{23} + x_3) \log(x_{123} + x_{132} + x_{13} + x_{23} + x_3) \\
& - (x_{123} + x_{132} + x_{13} + x_{23}) \log(x_{123} + x_{132} + x_{13} + x_{23}) - x_3 \log x_3 : \\
& x_* \geq 0 \text{ and } \sum_{i=1}^3 l_{ii} x_i + \sum_{ij \in \{12, 13, 23\}} (l_{ij} + l_{ji}) x_{ij} \\
& + \sum_{ijk \in \{123, 132\}} (l_{ij} + l_{jk} + l_{ki}) x_{ijk} = 1 \}.
\end{aligned}$$

Solving this conditional maximal problem we easily get the exponential growth rate $\lambda(G)$ to be λ where λ is the maximal positive solution of

$$\begin{aligned}
& (1 - e^{-l_{11}\lambda})(1 - e^{-l_{22}\lambda})(1 - e^{-l_{33}\lambda}) - e^{-l_{12}\lambda - l_{23}\lambda - l_{31}\lambda} - e^{-l_{13}\lambda - l_{32}\lambda - l_{21}\lambda} \\
& \quad \quad \quad (4) \\
& = (1 - e^{-l_{33}\lambda})e^{-l_{12}\lambda - l_{21}\lambda} + (1 - e^{-l_{22}\lambda})e^{-l_{13}\lambda - l_{31}\lambda} + (1 - e^{-l_{11}\lambda})e^{-l_{23}\lambda - l_{32}\lambda}.
\end{aligned}$$

Remark 2.2. Similarly as in Remark 2.1 we observe that (4) can be written as $\det(I - e^{-L\lambda}) = 0$. We will see that λ is the unique real number such that the principal eigenvalue of the matrix $e^{-L\lambda}$ is 1.

Above two remarks lead us to conjecture the relation of the exponential growth rate of matrix L of length information and the principal eigenvalue of the matrix $e^{-L\lambda}$, as we will see in next section (See Theorem 3.5).

Remark 2.3. Generally for a directed graph G with p vertices, we can find $C_n^i \cdot (i-1)!$ kinds of i -cycles for $i = 1, \dots, p$. So for a $p \times p$ matrix of length information L with $p \geq 4$, we can compute $\lambda(L) = \lambda(G)$ similarly by analyzing the primitive cycles. Clearly things along this line would be some more complex.

3. Exponential Growth Rate, Topological Entropy and Principal Eigenvalue

In this section we show the exponential growth rate of the associated directed graph G with length information L equals to the topological entropy of special suspension flow associated to G (or L), and equals to the unique real number for which the principal eigenvalue of the matrix $e^{-L\lambda}$ is 1. The later two notations have been treated by various authors.

Firstly we recall the construction of suspension flow associated to a continuous ceiling function $c : X \rightarrow (0, \infty)$ over base system (X, T) (see⁴). Consider the quotient space $\tilde{X} = \{(x, t) \in X \times \mathbb{R} : 0 \leq t \leq c(x)\} / \sim$, where

$\sim$ is the equivalence relation $(x, c(t)) \sim (Tx, 0)$. The suspension with c is the semiflow $\tilde{T}^t : \tilde{X} \rightarrow \tilde{X}$, given by $\tilde{T}^t(x, s) = (T^n x, s')$ where n and s' satisfy

$$\left\{ \begin{array}{l} \text{if } t + s < c(x), \text{ then } n = 0, s' = t + s; \\ \text{if } t + s \geq c(x), \text{ then } n \geq 1, 0 \leq s' < c(T^n x) \\ \text{and } \sum_{i=0}^{n-1} c(T^i x) + s' = t + s. \end{array} \right.$$

Let $\Sigma_p = \{1, \dots, p\}^{\mathbb{Z}_+}$ and σ be the shift over Σ_p . Now given a matrix L of length information, we associate a ceiling $c_L : \Sigma_L \rightarrow (0, \infty), x \mapsto l_{x_0 x_1}$ over base system (Σ_L, σ) where $\Sigma_L = \{x \in \Sigma_p : l_{x_n, x_{n+1}} < \infty \text{ for all } n \geq 0\}$ and σ is the shift restricted on Σ_L . There is a suspension semiflow $\tilde{\sigma}^t : \tilde{\Sigma}_L \rightarrow \tilde{\Sigma}_L$. Note that $\tilde{\Sigma}_L$ is a compact metrizable space (see³ and¹). Recall the metric on Σ_L as $d(x, y) = \sup\{2^{-n} : n \geq 0 \text{ and } x_n \neq y_n\}$. The metric on $\tilde{\Sigma}_L$ satisfies

$$\min\{d(x, y), d(\sigma x, \sigma y)\} \leq \rho((x, s), (y, s')) \leq d(x, y) + |s - s'|.$$

We will make the following reduction. If there is a vertex v_i at where $l_{ij} = \infty$ for all $j = 1, \dots, p$, then any path through v_i will stop passing new vertex and make no contribution to $\lambda(G)$. So we will eliminate all such vertices for the original graph. Similarly we can eliminate the vertex v_j if $l_{ij} = \infty$ for all $i = 1, \dots, p$. Up to finite reduction steps we can assume:

Each vertex has both finite length edges come to and leave that vertex.

Then every path of finite length represents a nonempty open set of $\tilde{\Sigma}_L$.

Notation 3.1. Consider $T > 0$ large and a point $(x, 0) \in \tilde{\Sigma}_L$. We will assign it a path $\gamma_x \in \mathcal{P}(G, T)$ by following steps. Define $t_0 = 0$ and $t_k = \sum_{j=0}^{k-1} l_{x_j, x_{j+1}}$ for $k \geq 1$. There is a unique $n \geq 1$ such that $t_n < T \leq t_{n+1}$. Then we define a piecewisely smooth path $\gamma_x \in \mathcal{P}(G, T)$ as

$\gamma_x|_{[t_j, t_{j+1}]}$ is exactly the edge $e_{x_j, x_{j+1}}$ for each $j = 0, \dots, n-1$ and

$\gamma_x|_{[t_n, T]}$ totally lies in the edge $e_{x_n, x_{n+1}}$.

Conversely for each $\gamma \in \mathcal{P}(G, T)$ we can extend γ arbitrarily (by reduction assumption), certainly avoiding the edges e_{ij} with $l_{ij} = \infty$, to generate a path $\tilde{\gamma} : [0, \infty) \rightarrow G$ with $\tilde{\gamma}|_{[0, T]} = \gamma$. Then set $x_j = \gamma(t_j)$ for $0 = t_0 < \dots < t_j < \dots$, and designs a point $x^\gamma \in \Sigma_L$ with $(x^\gamma)_n = x_n$ for every $n \geq 0$. Although the choice of x^γ is not unique, the map $\gamma \mapsto x^\gamma$ is injective from $\mathcal{P}(G, T)$ to Σ_L .

Proposition 3.2. *Let L be a matrix of length information, G a directed graph associated to L and $(\tilde{\Sigma}_L, \sigma_L^t)$ the semiflow associated to L . Then we have $\lambda(G) = h_{top}(\tilde{\sigma}) \triangleq h_{top}(\tilde{\sigma}^1)$, i.e., the exponential growth rate on paths in G exists and coincides with the topological entropy of $\tilde{\sigma}^t$.*

Proof. Let $r(\epsilon, T) = \max\{\#E : E \subset \tilde{\Sigma}_L \text{ and } E \text{ is } (\epsilon, T)\text{-seperated}\}$. The topological entropy of $\tilde{\sigma}$ is defined as (see⁸)

$$h_{top}(\tilde{\sigma}) = \lim_{\epsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \frac{1}{T} \log r(\epsilon, T) = \lim_{\epsilon \rightarrow 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log r(\epsilon, T).$$

Let $X_0 = \{(x, 0) : x \in \Sigma_L\} \subset \tilde{\Sigma}_L$. Since $\tilde{\sigma}^t$ flows in unit speed in t -direction, it suffices to consider the (ϵ, T) -seperated subset of X_0 . Let $r(\epsilon, T, X_0) = \max\{\#E : E \subset X_0 \text{ is } (\epsilon, T)\text{-seperated}\}$. Then we have

$$h_{top}(\tilde{\sigma}) = \lim_{\epsilon \rightarrow 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log r(\epsilon, T, X_0) = \lim_{\epsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \frac{1}{T} \log r(\epsilon, T, X_0)$$

(1) Let $\epsilon > 0$ small, $T > 0$ large and E_T be a maxmial (ϵ, T) -seperated subset of X_0 with $\#E_T = r(\epsilon, T, X_0)$. Pick $N = \lfloor \frac{-1}{\log \epsilon} \rfloor + 1$. Then we have $d(x, y) \leq e^{-1-N} < \epsilon/2$ if $x_n = y_n$ for $n = 0, \dots, N$. Let $C = N \cdot l^*$. For each $(x, 0) \in E_T$ we consider a path $\gamma_x \in \mathcal{P}(G, T + C)$ assigned to x as in Notation 3.1 for every $(x, 0) \in E_T$.

Claim 1: The mapping $E_T \rightarrow \mathcal{P}(G, T + C), (x, 0) \mapsto \gamma_x$ is injective.

Justification. Let $(x, 0), (y, 0) \in E_T$ and $(x, 0) \neq (y, 0)$. If $\gamma_x = \gamma_y \in \mathcal{P}(G, T + C)$, then we have $x_k = y_k$ for all $k = 0, \dots, n$, where $\sum_{k=0}^{n-1} l_{x_k, x_{k+1}} < T + C \leq \sum_{k=0}^n l_{x_k, x_{k+1}}$.

Since $(x, 0) \neq (y, 0)$ and E_T is (ϵ, T) -seperated, $\rho(\tilde{\sigma}^t(x, 0), \tilde{\sigma}^t(y, 0)) > \epsilon$ for some $t \in [0, T)$. Let $q \geq 0$ such that $\tilde{\sigma}^t(x, 0) = (\sigma^q x, s)$ and $\tilde{\sigma}^t(y, 0) = (\sigma^q y, s)$ for some $0 \leq s < l_{x_q, x_{q+1}}$. Note that $n - N > q$ by our choice of C . So the first N coordinates of $\sigma^q y$ and $\sigma^q x$ coincide. Then

$$\epsilon < \rho(\tilde{\sigma}^t(x, 0), \tilde{\sigma}^t(y, 0)) = \rho((\sigma^q x, s), (\sigma^q y, s)) \leq d(\sigma^q x, \sigma^q y) < \epsilon/2,$$

which contradicts in itself. This finishes the proof of Claim 1.

By Claim 1 we have $\#\mathcal{P}(G, T + C) \geq \#E_T = r(\epsilon, T, X_0)$ for all $T > 0$ and hence

$$\lambda^-(G) \triangleq \liminf_{T \rightarrow \infty} \frac{1}{T} \log \#\mathcal{P}(G, T) \geq \liminf_{T \rightarrow \infty} \frac{1}{T} \log r(\epsilon, T, X_0).$$

Since $\epsilon > 0$ can be arbitrary small we have $\lambda^-(G) \geq h_{top}(\tilde{\sigma})$.

(2) Let $T > 0$ be large enough and make a choice of the point $x^\gamma \in \Sigma_L$ assigned to each path $\gamma \in \mathcal{P}(G, T)$ as in Notation 3.1. We have

Claim 2: The set $\{(x^\gamma, 0) : \gamma \in \mathcal{P}(G, T) \text{ and } \gamma(0) = v_j\}$ is $(1/2, T)$ -separated for each $j = 1, \dots, p$.

Justification. Fix $j \in \{1, \dots, p\}$. Let $\gamma, \eta \in \mathcal{P}(G, T)$ be two different paths starting at the same vertex v_j . Let $x^\gamma = (x_n)_{n \geq 0}$, $y^\eta = (y_n)_{n \geq 0}$ denote the correspondin points in Σ_L . There exists a unique $q \geq 0$ such that $x_j = y_j$ for $j = 0, \dots, q$ and $x_{q+1} \neq y_{q+1}$. Let $t = \sum_{j=0}^{q-1} l_{x_j, x_{j+1}}$. Then $t \leq T$ and $\tilde{\sigma}^t(x^\gamma, 0) = (\sigma^q x, 0)$, $\tilde{\sigma}^t(y^\eta, 0) = (\sigma^q y, 0)$. So we have

$$\begin{aligned} \rho_T((x^\gamma, 0), (y^\eta, 0)) &\geq \rho(\tilde{\sigma}^t(x^\gamma, 0), \tilde{\sigma}^t(y^\eta, 0)) \\ &\geq \min\{d(\sigma^q x^\gamma, \sigma^q y^\eta), d(\sigma^{q+1} x^\gamma, \sigma^{q+1} y^\eta)\} = 1/2. \end{aligned}$$

Thus $\{(x^\gamma, 0) : \gamma \in \mathcal{P}(G, T) \text{ and } \gamma(0) = v_j\}$ is $(1/2, T)$ -separated. This finishes the proof of Claim 2.

By Claim 2 we have $\#\mathcal{P}(G, T) \leq p \cdot r(1/2, T, X_0)$ for all $T > 0$ and hence

$$\lambda^+(G) \triangleq \limsup_{T \rightarrow \infty} \frac{1}{T} \log \#\mathcal{P}(G, T) \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log r(1/2, T, X_0) \leq h_{top}(\tilde{\sigma}).$$

Combining (1) and (2) we have the limit $\lambda(G) = \lim_{T \rightarrow \infty} \frac{1}{T} \log \#\mathcal{P}(G, T)$ exists and equals to $h_{top}(\tilde{\sigma})$. This complete the proof of proposition. $\square$

Given a positive matrix A , we use $\tau(A)$ to denote the principal eigenvalue of A . Let $U = U(i, j)$, $1 \leq i, j \leq n$ be a positive matrix, then U defines a nearest pair potential. This is regarded as an energy of the interaction between i th and j th state in a Markov chain. For any invariant probability measure $\mu \in \mathcal{M}(\sigma)$, one can define, as in statistical mechanics, the so-called free energy. The free energy is roughly the total energy associated with the invariant measure minus the entropy of the invariant measure $e_U(\mu) - s(\mu)$. One consequence of the main results of Spitzer⁷ is that, for a finite positive potential $U(x) = U(i, j)$ over the shift dynamics $\sigma : \Sigma \rightarrow \Sigma$, we have the following inequality $e_U(\mu) - s(\mu) \geq -\log \tau(e^{-U})$. Moreover, there is a unique Markov measure attaining the minimum. We note that by standard abusing notation, we use the same U to indicate the matrix $U = (U(i, j))_{1 \leq i, j \leq p}$. Also we write $e^{-U} = (e^{-U(ij)})_{1 \leq i, j \leq p}$ as explained in the introduction.

To make this result suitable for our purpose, we need the notion of topological pressure. For a continuous function $\phi : X \rightarrow \mathbb{R}$ over an dynamical system (X, T) , the topological pressure $P(T, \phi)$ is defined by

$$P(T, \phi) = \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} e^{\phi_n(x)} : E \text{ is } (\delta, n)\text{-separated} \right\},$$

where

$$\phi_n(x) = \sum_{k=0}^{n-1} \phi(T^k x).$$

The variational principle implies that $P(T, \phi) = \sup_{\mu \in \mathcal{M}(T)} (h_\mu(f) + \int_X \phi d\mu)$ (see⁸). Then the conclusions of⁷ show that $P(\sigma, -U) = \log \tau(e^{-U})$ and there is a unique equilibrium state μ which is exactly the Parry measure associated to e^{-U} . We will show some of above results hold for our L with some $l_{ij} = \infty$.

Lemma 3.3. *For a matrix U we consider the e^{-U} by setting $(e^{-U})_{ij} = 0$ if $U_{ij} = \infty$. Let $\tau(e^{-U})$ be the maximal norm of all eigenvalues of e^{-U} . Then $P(\sigma, -U|\Sigma_U) = \log \tau(e^{-U})$.*

Proof. The first part of the proof in⁷ shows that $P(\sigma, -U|\Sigma_U) \leq \log \tau(e^{-U})$. In the following we assume $\log \tau(e^{-U}) > -\infty$. Consider K large and $(U_K)_{ij} = \min\{K, U_{ij}\}$. In this case we can directly apply Spitzer's result to get

- (1) a finite positive matrix e^{-U_K} with principal eigenvalue $\tau(e^{-U_K})$,
- (2) a Markov measure ν_K on Σ_p with $h_{\nu_K}(\sigma) - \int_{\Sigma_p} U_K(x) d\nu_K(x) = \log \tau(e^{-U_K})$.

Clearly $\tau(e^{-U_K}) \rightarrow \tau$ for some nonnegative real number τ as $K \rightarrow \infty$. Since the eigenvalues depend continuously on the matrix, we have $\tau = \tau(e^{-U})$. Pick a sequence $K_n \rightarrow \infty$ such that $\nu_n = \nu_{K_n}$ converges weakly to a σ -invariant measure μ (not necessarily unique). Note that U_K is monotone with respect to K , we have for each $m \geq n > 1$

$$h_{\nu_m}(\sigma) - \int_{\Sigma_p} U_{K_n}(x) d\nu_m(x) \geq h_{\nu_m}(\sigma) - \int_{\Sigma_p} U_{K_m}(x) d\nu_m(x) = \log \tau(e^{-U_{K_m}}). \quad (1)$$

By the uppersemicontinuity of entropy, we let $m \rightarrow \infty$ to get $h_\mu(\sigma) - \int_{\Sigma_p} U_{K_n} d\mu \geq \log \tau(e^{-U})$ for each $n \geq 1$. So we have $h_\mu(\sigma) - \int_{\Sigma_p} U(x) d\mu \geq \log \tau(e^{-U}) > -\infty$. This also implies that μ is supported on Σ_U since μ is σ -invariant. So we have $P(\sigma, -U|\Sigma_U) \geq \log \tau(e^{-U})$. This finishes the proof of lemma. $\square$

Remark 3.4. In above proof we see for nonnegative matrix A there also exists an eigenvalue $\lambda(A)$ of maximal norm among all eigenvalues of A . By abusing notations we still call $\lambda(A)$ principal in this case.

Finally, we are ready to state and prove our main theorem.

Theorem 3.5. *Let L be a matrix of length information, G a directed graph associated to L . Then the exponential growth rate on paths in G equals to unique real number λ such that the principal eigenvalue of the matrix $e^{-\lambda L}$ is 1.*

Proof. Denote $\mathcal{M}(\sigma, \Sigma_L)$ the set of invariant probability measures on Σ_L and $\mathcal{M}(\{\sigma^t\}_{t \geq 0}, \tilde{\Sigma}_L)$ the set of flow-invariant probability measures on $\tilde{\Sigma}_L$. There exists a one-to-one correspondence $\mu \leftrightarrow \mu_L$ between $\mathcal{M}(\sigma, \Sigma_L) \leftrightarrow \mathcal{M}(\{\sigma^t\}_{t \geq 0}, \tilde{\Sigma}_L)$ (see^{1,2}). Then by the Abramov formula (see^{2,6}) we have $h_{\mu_L}(\tilde{\sigma}) = \frac{h_\mu(\sigma)}{\int c_L(x) d\mu}$ for every $\mu \in \mathcal{M}(\sigma, \Sigma_L)$. This is the same as to say $h = h_{top}(\tilde{\sigma})$ is exactly the unique solution of $P(\sigma, -hc_L|\Sigma_L) = 0$.

Now applying above lemma to $U = Lh$ we have $\tau(h) = 1$ where $\tau(h) = \tau(e^{-Lh})$ is the maximum norm of eigenvalues of e^{-Lh} . Since $\tau(x)$ is strictly decreasing with respect to $x \in \mathbb{R}$, we have $x = h$ is the unique real number such that $\tau(h) = 1$. Return to our matrix L . By Proposition 3.2 we know $\lambda(G) = h_{top}(\tilde{\sigma}) = h$. This finishes the proof of theorem. $\square$

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