March, 2005

Low regularity solutions to a gently stochastic nonlinear wave equation in nonequilibrium statistical mechanics

L Rey-Bellet, *University of Massachusetts - Amherst*
LE Thomas

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER Stochastic Processes and their Applications 115 (2005) 1041–1059
stochastic processes and their applications
www.elsevier.com/locate/spa

Low regularity solutions to a gently stochastic nonlinear wave equation in nonequilibrium statistical mechanics

Luc Rey-Bellet[a,*], Lawrence E. Thomas[b]

[a]*Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA*
[b]*Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA*

Received 26 August 2004; accepted 9 February 2005
Available online 10 March 2005

Abstract

We consider a system of stochastic partial differential equations modeling heat conduction in a non-linear medium. We show global existence of solutions for the system in Sobolev spaces of low regularity, including spaces with norm beneath the energy norm. For the special case of thermal equilibrium, we also show the existence of an invariant measure (Gibbs state).

Keywords: Stochastic nonlinear wave equation; Gibbs measures; Nonequilibrium statistical mechanics; Hamiltonian PDE's; Low regularity solutions; Heat conduction

1. Introduction

In this article we consider the following system of partial differential

$$\begin{aligned}
&\partial_t \phi(x,t) = \pi(x,t),\\
&\partial_t \pi(x,t) = (\partial_x^2 - 1)\phi(x,t) - \mu\phi^3(x,t) - r(t)\alpha(x),\\
&\mathrm{d}r(t) = -(r(t) - \langle \alpha, \pi(t)\rangle)\,\mathrm{d}t + \sqrt{2T}\,\mathrm{d}\omega(t).
\end{aligned} \tag{1}$$

*Corresponding author. Tel.: +4135456020; fax: +4135451801.
E-mail addresses: lr7q@math.umass.edu (L. Rey-Bellet), let@virginia.edu (L.E. Thomas).

0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.02.003

In Eqs. (1) (ϕ, π) is a pair of scalar fields satisfying periodic boundary conditions with $x \in [0, 2\pi]$. The vector-valued functions $\alpha = (\alpha_1, \dots, \alpha_K)$ has each component $\alpha_i(x)$ in the Sobolev space H^γ for some $\gamma > 0$. The vector $r(t) = (r_1(t), \dots, r_K(t))$ takes value in $\mathbf{R}^K$. Here $r(t)\alpha(x) = \sum_{i=1}^K r_i(t)\alpha_i(x)$ and $\langle \alpha, \pi(t) \rangle$ is the vector with values in $\mathbf{R}^K$ and with components $\langle \alpha_i, \pi(t) \rangle$ where $\langle \cdot, \cdot \rangle$ is the $L^2([0, 2\pi])$ inner product. Finally $\omega(t) = (\omega_1(t), \dots, \omega_K(t))$ is a standard K-dimensional Brownian motion, and $\sqrt{2T}\,\mathrm{d}\omega$ has components $\sqrt{2T_i}\,\mathrm{d}\omega_i$ and T_i is interpreted as a temperature. The parameter μ is a coupling constant; we will be primarily interested in the cases $\mu = 0$ (linear Klein–Gordon equation) and $\mu > 0$ (non-linear defocusing linear wave equation).

The system of equations (1) arises from a model for heat conduction in a nonlinear medium. It can be derived from first principles from a Hamiltonian system which consists of K linear wave equations in $\mathbf{R}$ coupled to a nonlinear wave equation in $[0, 2\pi]$. The total Hamiltonian is given by

$$
\begin{aligned}
H = {} & \sum_{j=1}^K \int_{\mathbf{R}} \frac{1}{2}(|\partial_x u_j(x)|^2 + |v_j(x)|^2)\,\mathrm{d}x \\
& + \int_{[0,2\pi]} \frac{1}{2}(|\partial_x \phi(x)|^2 + |\phi(x)|^2 + |\pi(x)|^2) + \frac{\mu}{4}\,|\phi(x)|^4\,\mathrm{d}x \\
& + \sum_{j=1}^K \left(\int_{\mathbf{R}} \partial_x u_j(x) \rho_j(x)\,\mathrm{d}x \right) \left(\int_{[0,2\pi]} \partial_x \phi(x) \alpha_j(x)\,\mathrm{d}x \right),
\end{aligned}
\tag{2}
$$

with the ρ_j's and the α_j's fixed coupling functions. One assumes further that the initial conditions of the (u_j, v_j), $j = 1, \dots, K$ ("the reservoirs") are distributed according to Gibbs measures at temperatures T_j. These measures are (formally) expressed as

$$
Z^{-1} \exp\left(-\frac{1}{2T_i} \int_{\mathbf{R}} (|\partial_x u_j(x)|^2 + |v_j(x)|^2)\,\mathrm{d}x \right) \prod_{x \in \mathbf{R}} \mathrm{d}u_j(x)\,\mathrm{d}v_j(x), \tag{3}
$$

and they are simply the product of a Wiener measure (for the position fields u_j) with a white noise measure (for the momenta fields v_j).

We refer to [11] or [20,18] for details on the derivation of equations (1) from the Hamiltonian system (2) with initial conditions (3), at least in the case where the nonlinear wave equation is replaced by a chain of nonlinear oscillators (formally a discrete wave equation). In that case one obtains a set of stochastic ordinary differential equations. The derivation is essentially the same as for the model considered here and will not be repeated. We simply remark that the derivation of Markovian equations is possible due to a particular choice of the ρ_j's.

In a series of papers [11,12,19–21,8,10,18] about the chain of nonlinear oscillators, the existence, uniqueness, and strong ergodic properties of invariant measures have been established. Moreover, a number of properties of these invariant measures have been elucidated, such as existence of heat flow, positivity of entropy production, and symmetry properties of entropy production fluctuations. These invariant measures represent stationary states which generalize Gibbs distributions to non-equilibrium situations where there is heat flow. Ultimately our goal is to establish similar properties for the systems of equations (1). But we study here the more immediate problems of existence of global solutions—a prerequisite for studying the existence of stationary states—and existence and invariance of an equilibrium (Gibbs) measure.

In the case of equilibrium, that is, when all temperatures are equal, $T_j = T$ for all $j = 1, \dots, K$, we will prove below that there is an invariant state given formally by the (non-Gaussian) Gibbs measure

$$\begin{aligned} \mathrm{d}\nu = Z^{-1} \exp\left(-\frac{1}{2T} \int_{[0,2\pi]} (|\partial_x \phi(x)|^2 + |\phi(x)|^2 + \frac{\mu}{2}\, |\phi(x)|^4 + |\pi(x)|^2)\, \mathrm{d}x \right) \\ \times \exp\left(-\frac{1}{2T}\, r^2 \right) \mathrm{d}r \prod_{x \in [0,2\pi]} \mathrm{d}\phi(x)\, \mathrm{d}\pi(x). \end{aligned} \tag{4}$$

To make sense of this measure, one considers first the Gaussian measure ν^0 for the case $\mu = 0$. Its support is contained in $H^s \times H^{s-1} \times \mathbf{R}^K$ for any $s < \frac{1}{2}$ and, with probability 1, ϕ is also a continuous function. Hence we can think of the measure ν as the measure which is absolutely continuous with respect to ν^0 with a Radon–Nikodym derivative proportional to $\exp(-\mu \int |\phi(x)|^4\, \mathrm{d}x/4T)$. We *expect*, but have by no means proved, that the invariant measure for different temperatures, if one exists, has similar support properties. But with this intuition, it is appropriate to seek solutions of (1) in spaces of rough data $H^s \times H^{s-1} \times \mathbf{R}^K$ with $s < \frac{1}{2}$. Indeed we show the global existence of strong solutions, for $1/3 \leqslant s < 1$ (see Corollary 3.4 and the remark following it). We believe that these spaces, with at least $1/3 \leqslant s < 1/2$, are natural to the invariant measure problem.

Clearly, in these spaces no energy conservation (or bounds on the energy growth/dissipation) is available. In recent years, however, Bourgain [2], Keel and Tao [13] and many others have developed techniques to show global existence for wave equations and other Hamiltonian PDE's in Sobolev spaces below the energy norm. A review of recent results with an extensive bibliography can be found in [6]. Here, we use and extend these methods to establish global existence of solutions for wave equations coupled to heat reservoirs, i.e., with noise and dissipation.

In the last section, we show that solutions to an ultra-violet cut-off version of our system of equations, Eq. (1), converge as the cut-off is removed. This result is then applied to show that the equilibrium Gibbs state ν described above, Eq. (4), is indeed an invariant measure in the case of equilibrium. Note that Gibbs measures for nonlinear wave equations (and nonlinear Schrödinger equations) have been constructed and studied by several authors, (Lebowitz, Rose and Speer [15], Zhidkov [22], McKean and Vaninsky [17], Bourgain [1,3], Brydges and Slade [5]) but for isolated systems only, i.e., without dissipation or noise. Note that in these works Gibbs measures for any temperature are invariant while in our case the temperature is selected by the coupling to the reservoir. Our work is also related in spirit to various recent works on the ergodic properties of randomly forced dissipative equations, see e.g. [4,7,9,14,16] and others. The main and very important differences are that our equation is Hamiltonian rather than parabolic so that there is no intrinsic smoothing in the equations, and that the dissipation is very weak.

Our methods do not apply to the focusing case (i.e., for $\mu<0$) as we use repeatedly to prove global existence that the energy and the $H^1 \times L^2$ norm for the Klein–Gordon equation are equivalent. Also we have chosen periodic boundary conditions for mathematical convenience, although other boundary conditions, e.g., Dirichlet boundary conditions can be treated along the same line. The global existence of the flow can be proved also on the real line, but our analysis of the invariant measure is restricted to finite domains.

1.1. Notation

It is convenient to write our system as Bourgain does [2]. Set

$$u = \phi + \frac{i}{B}\pi, \tag{5}$$

where B is the operator defined $B = \sqrt{-\partial_x^2 + 1}$. Note that $\phi = \Re u$ and $\frac{1}{B}\pi = \Im u$ are respectively the real and imaginary parts of u. Thus our differential equations can be written,

$$\begin{aligned} i\partial_t u &= Bu + \frac{1}{B}(\mu\phi^3 + r\alpha), \\ \mathrm{d}r(t) &= -(r(t) - \langle \alpha, \pi(t)\rangle)\,\mathrm{d}t + \sqrt{2T}\,\mathrm{d}\omega(t). \end{aligned} \tag{6}$$

Let

$$\mathbf{u}(\omega, t) = (u, r) = \left(\phi + \frac{i}{B}\pi, r\right), \tag{7}$$

and let $\mathbf{u}_o(\omega, t) = (u_o, r_o)(\omega, t)$ be the corresponding solution to the differential equations but with the non-linearity turned off, $\mu = 0$.

For a vector quantity $\mathbf{u} = (u, r)$, we introduce the norms

$$\|\mathbf{u}\|_{H^s} = (r^2 + \|u\|_{H^s}^2)^{1/2}. \tag{8}$$

where H^s is the Sobolev space with norm $\|f\|_{H^s}^2 = \sum_k (1 + k^2)^s |\hat{f}(k)|^2$. The energy of a vector $\mathbf{u}$ is defined by

$$\mathscr{E}(\mathbf{u}) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}r^2 + \frac{\mu}{4}\int (\Re u)^4\,\mathrm{d}x. \tag{9}$$

2. Estimates for the linear wave equation

In this section we collect basic estimates for the linear system, $\mu = 0$. These estimates actually establish global existence for this system.

The first step is to consider the linear deterministic (dissipative) system obtained from (6) by omitting both the nonlinear term and the noise,

$$\begin{aligned} \frac{\mathrm{d}u_o}{\mathrm{d}t} &= -iBu_o - i\frac{1}{B}\alpha r_o, \\ \frac{\mathrm{d}r_o}{\mathrm{d}t} &= \langle B\alpha, \Im u_o\rangle - r_o, \end{aligned} \tag{10}$$

where $\Im u_o$ is the imaginary part of u_o. Set $L_o = B_o + P$ with

$$B_o = \begin{pmatrix} -iB & 0 \\ 0 & -1 \end{pmatrix},$$
$$P = \begin{pmatrix} 0 & -i\,\dfrac{1}{B}\,\alpha \\ \langle B\alpha| \Im & 0 \end{pmatrix}. \tag{11}$$

Here, the upper right matrix element of P acts as $-i\frac{1}{B}\alpha r = -i\frac{1}{B}\sum_i \alpha_i r_i$, and the lower left matrix element of P, $\langle B\alpha|\Im$, maps u to the vector in $\mathbf{R}^K$ with components $\langle B\alpha_i, \Im u\rangle$. Symbolically, the solution of this system Eqs. (10) is given by $\mathrm{e}^{tL_o}\mathbf{u}_o(0)$ with $L_o = B_o + P$. The system should be regarded as linear in a function space of complex functions over the reals (so that $\Im$ is linear).

Lemma 2.1. *Assume that $\alpha \in H^\gamma$ for some $\gamma > 0$ and $0 \leqslant s < 1$. For λ_0 sufficiently large depending on the α's only, $(B_o + \lambda_0)(L_o + \lambda_0)^{s-1}(B_o + \lambda_0)^{-s}$ acting in $L^2 \oplus \mathbf{R}^K$ is defined as a bounded invertible operator.*

Proof. It is no restriction to assume that $\gamma < 1$. We have the following operator estimates (the operators acting in $L^2 \oplus \mathbf{R}^K$):

$$\left\| P\frac{1}{B_o + \lambda_0 + \lambda} \right\| \leqslant \frac{c(\lambda_0)}{(1+\lambda)^\gamma},$$
$$\left\| P\frac{1}{(B_o + \lambda_0 + \lambda)(B_o + \lambda_0)^s} \right\| \leqslant \frac{c(\lambda_0)}{(1+\lambda)^{\gamma'}}, \tag{12}$$

with $\gamma' = \min\{\gamma + s, 1\}$ and with $c(\lambda_0) \to 0$ for $\lambda_0 \to \infty$. The first estimate in (12) is obtained by considering the off diagonal terms separately; the upper right term is estimated by

$$\left\| \frac{1}{B}\alpha \right\| \left\| \frac{1}{-1 + \lambda_0 + \lambda}\, r \right\| \leqslant c\,\frac{|r|}{-1 + \lambda_0 + \lambda} \leqslant \frac{c(\lambda_0)}{(1+\lambda)^\gamma}\,|r|, \tag{13}$$

and the norm of the lower left term is estimated by

$$\left\| \frac{B^{1-\gamma}}{(-iB + \lambda_0 + \lambda)^{1-\gamma}}\, B^\gamma \alpha \right\| \left\| \frac{1}{(-iB + \lambda_0 + \lambda)^\gamma}\, u \right\| \leqslant \frac{c(\lambda_0)}{(1+\lambda)^\gamma}\, \|u\|. \tag{14}$$

The second estimate of (12) is obtained similarly. By expanding the resolvent for L_o in a geometric series, convergent for $c(\lambda_0) < 1$, one finds from these estimates that

$$(B_o+\lambda_0)(L_o+\lambda_0+\lambda)^{-1}(B_o+\lambda_0)^{-s} = (B_o+\lambda_0)^{1-s}(B_o+\lambda_0+\lambda)^{-1}+\mathcal{O}\left(\frac{c(\lambda_0)}{(1+\lambda)^{\gamma'}}\right), \tag{15}$$

by splitting off the first term in the series and estimating the remainder. Using

$$(L_o+\lambda_0)^{s-1}=c_s\int_0^\infty \frac{\mathrm{d}\lambda}{(L_o+\lambda_0+\lambda)\lambda^{1-s}}, \tag{16}$$

with c_s a suitable normalizing constant, and integrating the previous equation, we obtain

$$(B_o+\lambda_0)(L_o+\lambda_0)^{s-1}(B_o+\lambda_0)^{-s}=1+\mathcal{O}(c(\lambda_0)), \tag{17}$$

which clearly is bounded. By choosing λ_0 large so that $c(\lambda_0)$ is sufficiently small, we see that $(B_o+\lambda_0)(L_o+\lambda_0)^{s-1}(B_o+\lambda_0)^{-s}$ is invertible. □

Lemma 2.2. *Assume* $\alpha\in H^\gamma$, *with* $\gamma>0$, $0<s\leqslant 1$. *There is a constant* c_3 *depending only on* s *and the* α*'s, such that*

$$\|\mathrm{e}^{tL_o}\mathbf{u}(0)\|_{H^s}\leqslant c_3\|\mathbf{u}(0)\|_{H^s} \tag{18}$$

for all time t.

Proof. We have that

$$\mathscr{E}_o(\mathbf{u})\equiv\tfrac{1}{2}(\|u\|_{H^1}^2+r^2) \tag{19}$$

is a (degenerate) Liapunov function for the linear system Eq. (10), since $\mathrm{d}\mathscr{E}_o(\mathbf{u})/\mathrm{d}t=-r_o^2(t)\leqslant 0$. The lemma follows if we can show that for a suitably large constant λ_0, $\mathscr{E}_o((L_o+\lambda_0)^{s-1}\mathbf{u})$ is equivalent to $\|(B_o+\lambda_0)^s\mathbf{u}\|_2^2$, which is in turn equivalent to $\|\mathbf{u}\|_{H^s}^2$. This is certainly the case if $s=1$. For $s<1$, this amounts to showing that $(B_o+\lambda_0)(L_o+\lambda_0)^{s-1}(B_o+\lambda_0)^{-s}$ is a bounded invertible operator, which is the content of the previous lemma. □

We now provide an estimate for the linear stochastic evolution $\mathbf{u}_o$ solving Eq. (6), with the nonlinearity turned off, $\mu=0$.

Lemma 2.3. *Assume that* $\gamma>0$, $0<s\leqslant 1$ *and set* $\|\mathbf{u}_o(0)\|_{H^s}=\beta$. *There exist constants* c *and* C, *such that for* $\lambda\geqslant c\beta$,

$$P\left\{\sup_{t'<t}\|\mathbf{u}_o(t)\|_{H^s}\geqslant\lambda\right\}\leqslant C\exp\left(-\frac{(\lambda-c\beta)^2}{c^2t(1+t)^2}\right). \tag{20}$$

Remark. The estimate is certainly not optimal. It does not account for the rapid dissipation of energy for small k modes of $\mathbf{u}_o$. The lemma provides a *global bound* on the linear evolution, showing that it does not blow up, almost surely.

Proof. The Duhamel formula for $\mathbf{u}_o(t)$ is

$$\mathbf{u}_o(t) = \int_0^t \mathrm{e}^{(t-t')L_o}\mathbf{v}_o \,\mathrm{d}\omega(t') + \mathrm{e}^{tL_o}\mathbf{u}_o(0)$$
$$= \mathbf{v}_o\omega(t) + \int_0^t \mathrm{e}^{(t-t')L_o} L_o\mathbf{v}_o\omega(t')\,\mathrm{d}t' + \mathrm{e}^{tL_o}\mathbf{u}_o(0), \tag{21}$$

the latter line obtained by integration by parts, with L_o defined as in the deterministic case, Eq. (11), and

$$\mathbf{v}_o = \begin{pmatrix} 0 \\ \sqrt{2T} \end{pmatrix}. \tag{22}$$

By Lemma 2.2, there is a constant c_3 such that $\|\mathrm{e}^{tL_o}\mathbf{u}_o(0)\|_{H^s} \leqslant c_3\beta$, and $\|\mathrm{e}^{(t-t')L_o}\mathbf{v}_o\|_{H^s} \leqslant c_3\|\mathbf{v}_o\|_{H^s}$ (which is finite) so that from the integral equation Eq. (21) above, we obtain the estimate

$$\|\mathbf{u}_o(t)\|_{H^s} \leqslant \|\mathbf{v}_o\|_{H^s}|\omega(t)| + c_3 t\|L_o\mathbf{v}_o\|_{H^s} \sup_{t' \leqslant t} |\omega(t')| + c_3\|\mathbf{u}_o(0)\|_{H^s}. \tag{23}$$

Thus we can write for a suitable constant c that

$$\|\mathbf{u}_o(t)\|_{H^s} \leqslant c(1+t) \sup_{t' \leqslant t} |\omega(t')| + c\beta, \tag{24}$$

with β the H^s norm of the initial data $\mathbf{u}_o(0)$.

Now if at some time t', with $t' \leqslant t$, we have that $\|\mathbf{u}_o(t')\|_{H^s} > \lambda$, then evidently $\frac{\lambda - c\beta}{c(1+t)} \leqslant \sup_{t' \leqslant t} |\omega(t')|$, and so, for $\lambda > c\beta$,

$$P\left\{\sup_{t' \leqslant t} \|\mathbf{u}_o(t')\|_{H^s} > \lambda\right\} \leqslant P\left\{\sup_{t' \leqslant t} |\omega(t')| > \frac{\lambda - c\beta}{c(1+t)}\right\}$$
$$\leqslant 2P\left\{|\omega(t)| > \frac{\lambda - c\beta}{c(1+t)}\right\}$$
$$\leqslant C\exp\left(-\frac{(\lambda - c\beta)^2}{c^2 t(1+t)^2}\right), \tag{25}$$

by the reflection principle for Brownian motion, for yet another suitable constant C depending on the dimension of r. This concludes the proof of the lemma. $\square$

For later use, we also note here some simple Sobolev inequalities, all in one-dimension only. Here and in the sequel $\|\cdot\|_p$ denotes the L^p norm.

Lemma 2.4. *For $s > (1/2 - 1/p)$ and $p \geqslant 2$ there is a constant $c = c(s,p)$ such that*

$$\|\phi\|_p \leqslant c\|\phi\|_{H^s}. \tag{26}$$

Also, for $0 \leqslant \theta \leqslant (1-1/p)$, *there is a constant* c *such that*

$$\|\phi\|_\infty \leqslant c\|\phi\|^{\theta}_{2(p-1)}\|\phi\|^{1-\theta}_{H^1}. \tag{27}$$

Finally, for $s > 1/6$ *and* $s' \leqslant \min(0, 3s-1)$, *or* $s' = 0$ *and* $s > 1/3$, *there is a constant* $c = c(s, s')$ *such that for* $\phi_1, \phi_2, \phi_3 \in H^s$,

$$\|\phi_1\phi_2\phi_3\|_{H^{s'}} \leqslant c\|\phi_1\|_{H^s}\|\phi_2\|_{H^s}\|\phi_3\|_{H^s}. \tag{28}$$

Remark. The first inequality of the lemma actually holds with $s = 1/3$ and $p = 6$, as can be proved using the Hardy–Littlewood–Sobolev inequality. For convenience we will use this inequality as well, although it is not essential for our purposes. But as a consequence of this remark, the last inequality (28) holds for $s' = 0$, $s = 1/3$.

Proof. The first inequality of the lemma is proved by estimating

$$\begin{aligned}\|\hat{\phi}\|^{p'}_{\ell^{p'}} &= \sum_n \frac{1}{(1+n^2)^{sp'/2}}(1+n^2)^{sp'/2}|\hat{\phi}|^{p'}(n) \\ &\leqslant \|(1+n^2)^{-sp'/2}\|_{\ell^r}\|\phi\|^{p'}_{H^s},\end{aligned} \tag{29}$$

with p' conjugate to p and $r = 2/(2-p')$. The right side of this inequality is bounded provided that $sp'r > 1$, i.e., $s > \frac{1}{2} - \frac{1}{p}$. One then applies Hausdorff–Young to obtain the first assertion of the lemma.

The second inequality of the lemma is shown by first noting that

$$\phi(x) = \frac{1}{2\pi}\sum_n \frac{e^{inx}}{(1+n^2)^{1/2}}(1+n^2)^{1/2}\hat{\phi}(n), \tag{30}$$

which by the Schwarz inequality gives the special case ($\theta = 0$)

$$\|\phi\|_\infty \leqslant c\|\phi\|_{H^1}. \tag{31}$$

Also, we have that

$$|\phi|^p(x) \leqslant p\int_y^x |\phi|^{p-1}|\phi'(t)|\,dt + |\phi|^p(y). \tag{32}$$

Estimating the integral by $\|\phi\|^{p-1}_{2(p-1)}\|\phi\|_{H^1}$ and then integrating this inequality (32) with respect to y over $[0, 2\pi]$, we get

$$2\pi|\phi|^p(x) \leqslant 2\pi p\|\phi\|^{p-1}_{2(p-1)}\|\phi\|_{H^1} + \|\phi\|^{p-1}_{2(p-1)}\|\phi\|_2, \tag{33}$$

so that

$$\|\phi\|_\infty \leqslant c\|\phi\|^{1-1/p}_{2(p-1)}\|\phi\|^{1/p}_{H^1}. \tag{34}$$

The second inequality of the lemma is then obtained by interpolation between inequalities (31,34).

To prove the last inequality of the lemma (28), we suppose each of the ϕ_i's is in H^s with $s > 1/6$. Pick p' with $s > \frac{1}{p'} - \frac{1}{2}$ and, for later purposes, $\frac{6}{5} < p' \leqslant \frac{3}{2}$. By inequality (29) above, each $\hat{\phi}_i$ is in $\ell^{p'}$, and the double convolution $\hat{\phi}_1 * \hat{\phi}_2 * \hat{\phi}_3$ is in ℓ^r for

$\frac{1}{r} = \frac{3}{p'} - 2$ by Young's inequality. Note that $2 < r \leqslant \infty$. It is then easy to check that $(1+n^2)^{s'/2}\hat{\phi}_1 * \hat{\phi}_2 * \hat{\phi}_3$ is in ℓ^2 provided $s'r' < -1$ where $\frac{1}{r'} + \frac{1}{r} = \frac{1}{2}$. This is so if $s' < 0$ (r' is positive) and $s' < -\frac{1}{r'} = \frac{1}{r} - \frac{1}{2} = \frac{3}{p'} - \frac{5}{2} < 3(s+\frac{1}{2}) - \frac{5}{2} = 3s - 1$. The special case with $s' = 0$, $s > 1/3$ is an immediate consequence of the first inequality (26). □

3. Estimates for the non-linear equations

3.1. Local existence

The Duhamel integral representation of the system equations for $\mathbf{u}$, Eq. (6), is

$$\mathbf{u}(t) = \int_0^t \mathrm{e}^{(t-t')L_o} \begin{pmatrix} \frac{\mu}{B}\,\phi^3(t')\,\mathrm{d}t' \\ \sqrt{2T}\,\mathrm{d}\omega(t') \end{pmatrix} + \mathrm{e}^{tL_o}\mathbf{u}(0). \tag{35}$$

Fix s with $\frac{1}{6} < s < 1$, and for $R > 1$ let $\mathscr{D}_R(\beta, t)$ be the set of functions defined

$$\mathscr{D}_R(\beta,t) \equiv \left\{ \mathbf{u}(\cdot) \in \mathscr{C}([0,t], H^s) \mid \|\mathbf{u}(0)\|_{H^s} \leqslant \beta \text{ and } \sup_{t' \leqslant t} \|\mathbf{u}(t')\|_{H^s} \leqslant R\beta \right\}, \tag{36}$$

and let $\mathscr{F}_R(\beta, t)$ be the (probabilistic) event that the Duhamel integral equation Eq. (35) has a unique *strong* solution in $\mathscr{D}_R(\beta, t)$. We have the following local existence result.

Proposition 3.1. *Assume $\frac{1}{6} < s < 1$. There exist constants c_1, c_2, c_3 and C such that if $\mathbf{u}(0)$ satisfies $\|\mathbf{u}(0)\|_{H^s} \leqslant \beta$, $R > 3c_3$ and $t \leqslant c_1/(R^2\beta^2)$, then*

$$P\{\mathscr{F}_R(\beta,t)\} \geqslant 1 - C\exp\left(-\frac{c_2 R^2 \beta^2}{t(1+t)^2}\right). \tag{37}$$

Clearly, the sets $\mathscr{F}_R(\beta, t)$ are nested, $\mathscr{F}_R(\beta, t_2) \subset \mathscr{F}_R(\beta, t_1)$ if $t_1 \leqslant t_2$. The event $\mathscr{F}_R(\beta) \equiv \bigcup_n \mathscr{F}_R(\beta, t/n)$ is the event that $\mathbf{u}(\cdot)$ exists for some positive time, and, in this time, has H^s norm no bigger than $R\beta$. An immediate corollary of the above proposition is that $\mathscr{F}_R(\beta)$ occurs with probability one.

Corollary 3.2. *For $s > \frac{1}{6}$, local existence of the solution $\mathbf{u}(\cdot)$ in H^s holds almost surely,*

$$P\{\mathscr{F}_R(\beta)\} = 1. \tag{38}$$

Proof of Proposition 3.1. We have

$$\begin{aligned} \left\| \int_0^t \mathrm{e}^{(t-t')L_o} \begin{pmatrix} \frac{\mu}{B}\,\phi^3(t')\,\mathrm{d}t' \\ 0 \end{pmatrix} \right\|_{H^s} &\leqslant c_3 t\mu \sup_{t' \leqslant t} \|\phi^3(t')\|_{H^{s-1}} \\ &\leqslant c\mu t \sup_{t' \leqslant t} \|\mathbf{u}(t')\|_{H^s}^3 \leqslant c\mu t (R\beta)^3 < \frac{1}{3} R\beta, \end{aligned} \tag{39}$$

for a suitable constant c. Here, we have used Lemma 2.2 and the Sobolev inequality (28) of Lemma 2.4 to estimate $\|\phi^3\|_{H^{s-1}}$, assuming that $s > 1/6$ and using

$s-1<\min(0,3s-1)$. Also, we have chosen $t<(3c\mu R^2\beta^2)^{-1}$. Furthermore

$$\left\| \int_0^t \mathrm{e}^{(t-t')L_o} \begin{pmatrix} 0 \\ \sqrt{2T}\,\mathrm{d}\omega(t') \end{pmatrix} \right\|_{H^s} < \frac{1}{3} R\beta, \tag{40}$$

using that the left side inequality (40) is bounded by $c(1+t)\sup_{t'\leqslant t}|\omega(t')|$ for a suitable constant c (cf. the proof of Lemma 2.3); this condition holds with probability exceeding

$$P\left\{ \sup_{t'\leqslant t} |\omega(t')| \leqslant \frac{R\beta}{3c(1+t)} \right\} \geqslant 1 - C\exp\left(-\frac{c_2 R^2\beta^2}{t(1+t)^2} \right), \tag{41}$$

for suitable constants $c_2,\ C$. Finally, since $R>3c_3$ we have $\|\mathrm{e}^{tL_o}\mathbf{u}(0)\|_{H^s} \leqslant c_3\beta \leqslant \frac{1}{3}R\beta$. Together with inequalities (39,40) this implies that the right side of the Duhamel integral equation Eq. (35) is a map of $\mathscr{D}_R(\beta,t)$ into itself.

It remains to check that the right side of the Duhamel equation Eq. (35) is contractive for small t. But clearly for two functions $\mathbf{u}_1,\ \mathbf{u}_2 \in \mathscr{D}_R(\beta,t)$, with real field parts ϕ_1 and ϕ_2 respectively, and for $t_1 \leqslant t$,

$$\begin{aligned} &\left\| \int_0^{t_1} \mathrm{e}^{(t_1-t')L_o} \begin{pmatrix} \frac{\mu}{B}(\phi_1^3 - \phi_2^3)(t') \\ 0 \end{pmatrix} \mathrm{d}t' \right\|_{H^s} \\ &\quad \leqslant c_3\mu t \sup_{t'\leqslant t} \|(\phi_1^2 + \phi_1\phi_2 + \phi_2^2)(\phi_1 - \phi_2)(t')\|_{H^{s-1}} \\ &\quad \leqslant 3c\mu t (R\beta)^2 \sup_{t'\leqslant t} \|(\mathbf{u}_1 - \mathbf{u}_2)(t')\|_{H^s}, \end{aligned} \tag{42}$$

by inequality (28) of Lemma 2.4, for a suitable constant c. Thus the Duhamel integral is a contraction for $t<1/(3c\mu R^2\beta^2)$.

In summary, if $t<c_1/\mu R^2\beta^2)$, for a suitably small constant c_1, and if the stochastic integral estimate inequality (40) holds, the right side of the Duhamel expression maps $\mathscr{D}_R(\beta,t)$ into itself and it is a contraction, so that by the contraction mapping theorem, Eq. (6) has a unique strong solution in $\mathscr{D}_R(\beta,t)$. Inequality (40) holds with probability at least that given in inequality (41). □

3.2. Global existence

Finally, in this section we provide a *global estimate* for the non-linear stochastic evolution $\mathbf{u}(t)$. Following Bourgain's methods for the non-linear wave equation [2], we set

$$\tilde{\mathbf{u}}_N(t) = (\tilde{u}_N(t), \tilde{r}(t)) = ((u(t) - P_{>N}u_o(t)), (r - r_o)(t)), \tag{43}$$

with the positive integer N to be chosen later, $P_{>N}$ projection onto the Fourier modes $\{k : |k|>N\}$. Here, $\mathbf{u}(t)$ and $\mathbf{u}_o(t)$ (the linear evolution) are assumed to begin with the same initial data, $\mathbf{u}(0) = \mathbf{u}_o(0)$, and are driven by the same stochastic driving terms, so that they are *not* independent: they are *coupled*. The quantity $I_N(t)$ is defined as it would be in the pure deterministic case,

$$I_N(t) = \mathscr{E}(\tilde{\mathbf{u}}_N(t)), \tag{44}$$

with $\mathscr{E}$ the energy defined by Eq. (9). Set

$$\theta_* \equiv \min\{\tfrac{1}{3}(4s-1), \tfrac{1}{3}(1-s), \gamma\}. \tag{45}$$

We will assume below that $\alpha \in H^\gamma$ with $\gamma > 0$, and that $1/3 \leqslant s < 1$.

Our main result is the following.

Proposition 3.3. *Let* $\mathbf{u}(0) = \mathbf{u}_o(0)$, *with* $\beta = \|\mathbf{u}(0)\|_{H^s} = \|\mathbf{u}_o(0)\|_{H^s}$. *Fix* $R > 1$, $\theta > 0$ *and* $\delta > 0$ *so that* $\theta + \delta < \theta_*$. *There exist constants* c, C *and an* $N_o = N_o(R, \beta)$ *and* $\tau = \tau(R, \beta)$, *such that if* $N \geqslant N_o$ *and* $t \leqslant \tau N^\delta$, *then*

$$P\left\{\sup_{t' \leqslant t} I_N(t') > R\beta^2 N^{2(1-s)}\right\} \leqslant C \exp\left(-\frac{cN^{2\theta}}{t(1+t)^2}\right). \tag{46}$$

This proposition and Lemma 2.3 give us a global bound:

Corollary 3.4. *Let* β, $\theta < \theta_*$, $R \geqslant 2$ *be fixed, as in the above Proposition. There exist constants*, c, C *and* $N_1 = N_1(\beta, R, t)$, *such that for any time* t, *and* $N \geqslant N_1$,

$$P\left\{\sup_{t' \leqslant t} \|\mathbf{u}(t')\|_{H^s} > R\beta N^{1-s}\right\} \leqslant C \exp\left(-\frac{cN^{2\theta}}{t(1+t)^2}\right). \tag{47}$$

Proof of Proposition 3.3. The stochastic differential of $I_N(t)$ is given by

$$\begin{aligned} \mathrm{d}I_N(t) = (&\Im\langle B\tilde{u}_N, \mu(\Re u)^3 - \mu(\Re\tilde{u}_N)^3 + r_o P_{\leqslant N}\alpha\rangle - \tilde{r}^2 \\ &+ \tilde{r}\Im\langle Bu_o, P_{\leqslant N}\alpha\rangle)\,\mathrm{d}t. \end{aligned} \tag{48}$$

In particular, there are no $\mathrm{d}\omega$ or $\mathrm{d}\omega^2 = \mathrm{d}t$ terms, hence the differential is the same as if we were just considering a wave equation with dissipation. We proceed to estimate the terms on the right side.

We have that

$$\begin{aligned} \|(\Re u)^3(t) - (\Re\tilde{u}_N)^3(t)\|_2 &\leqslant c''(\|P_{>N}u_o(t)\|_2 \|\tilde{u}_N\|_\infty^2 + \|P_{>N}u_o(t)\|_6^3) \\ &\leqslant c'(\|P_{>N}u_o(t)\|_2 \|\tilde{u}\|_4^{4/3} \|\tilde{u}\|_{H^1}^{2/3} + \|P_{>N}u_o(t)\|_6^3) \\ &\leqslant c(N^{-s}\|u_o(t)\|_{H^s} I_N(t)^{2/3} + \|u_o(t)\|_{H^s}^3), \end{aligned} \tag{49}$$

for suitable constants c'', c', c. Here the first line is obtained by factoring the difference of cubes and then using $u = \tilde{u}_N + P_{>N}u_o$; the remaining two lines are obtained by the Sobolev inequalities, first inequality (27), then inequality (26) of Lemma 2.4. The other factors in Eq. (48) are readily estimated, and we get

$$\begin{aligned} \mathrm{d}I_N(t) \leqslant c(&N^{-s}\|u_o(t)\|_{H^s} I_N(t)^{7/6} + (|r_o(t)| + N^{1-\gamma-s}\|u_o(t)\|_{H^s} \\ &+ \|u_o(t)\|_{H^s}^3) I_N(t)^{1/2})\,\mathrm{d}t. \end{aligned} \tag{50}$$

Now assuming that $I_N(t) \leqslant R\beta^2 N^{2(1-s)}$ and $\|\mathbf{u}_o(t)\|_{H^s} \leqslant N^\theta$, with $0 < \theta < \theta_* \equiv \min\{\frac{1}{3}(4s-1), \frac{1}{3}(1-s), \gamma\}$, we obtain

$$\begin{aligned} \mathrm{d}I_N(t) &\leqslant N^{2(1-s)}(N^\theta \mathcal{O}(N^{\frac{1}{3}(1-4s)}) + N^{3\theta}\mathcal{O}(N^{-(1-s)}) + N^\theta \mathcal{O}(N^{-\gamma}))\,\mathrm{d}t \\ &\leqslant N^{2(1-s)}\mathcal{O}(N^{\theta-\theta_*})\,\mathrm{d}t. \end{aligned} \tag{51}$$

It follows that $I_N(t) < R\beta^2 N^{2(1-s)}$ for $t \leqslant T$ with $T = \mathcal{O}(N^\delta)$, $\delta < \theta_* - \theta$, provided that in this time interval, $\|\mathbf{u}_o(t)\|_{H^s} < N^\theta$. Said more precisely, given β, R, there exist an $N_o(\beta, R)$ and a $\tau(\delta, \theta, \beta, R)$ such that for $N \geqslant N_o$ $I_N(t)$ remains less than $R\beta^2 N^{2(1-s)}$ for a time t, $0 \leqslant t \leqslant \tau N^\delta$, *provided* that $\|\mathbf{u}_o(t)\|_{H^s}$ remains less than N^θ in this same time interval.

Thus we have that

$$\begin{aligned} &P\left\{\sup_{t' \leqslant t} I_N(t') > R\beta^2 N^{2(1-s)}\right\} \\ &\quad = P\left\{\sup_{t' \leqslant t} I_N(t') > R\beta^2 N^{2(1-s)} \text{ and } \sup_{t' \leqslant t} \|\mathbf{u}_o(t')\|_{H^s} \geqslant N^\theta\right\} \\ &\quad \leqslant P\left\{\sup_{t' \leqslant t} \|\mathbf{u}_o(t')\|_{H^s} \geqslant N^\theta\right\} \\ &\quad \leqslant C \exp\left(-\frac{(N^\theta - c\beta)^2}{c^2 t(1+t)^2}\right) \leqslant C_1 \exp\left(-\frac{c_1 N^{2\theta}}{t(1+t)^2}\right), \end{aligned} \tag{52}$$

for $t \leqslant \tau N^\delta$, by inequality (20) of Lemma 2.3 and appropriate new constants C_1 and c_1. After renaming of constants and taking N_o still larger so that $N_o^\theta > 2c\beta$, the proof of the proposition is complete. $\square$

Proof of Corollary 3.4. We have that

$$\begin{aligned} P\left\{\sup_{t' \leqslant t} \|\mathbf{u}(t')\|_{H^s} > R\beta N^{1-s}\right\} &\leqslant P\left\{\sup_{t' \leqslant t} \|\tilde{\mathbf{u}}_N(t')\|_{H^1} > \frac{1}{2} R\beta N^{1-s}\right\} \\ &\quad + P\left\{\sup_{t' \leqslant t} \|\mathbf{u}_o(t')\|_{H^s} > \frac{1}{2} R\beta N^{1-s}\right\}. \end{aligned} \tag{53}$$

Now the first probability on the right side is bounded by

$$P\left\{\sup_{t' \leqslant t} I_N(t') > \frac{1}{2} R^2\beta^2 N^{2(1-s)}\right\} \leqslant C \exp\left(-\frac{cN^{2\theta}}{t(1+t)^2}\right), \tag{54}$$

for $N > N_o$ by Proposition 3.3. The second probability on the right side of inequality (53) is bounded by the estimate given in Lemma 2.3, with $\lambda = \frac{1}{2} R\beta N^{1-s} \geqslant \mathcal{O}(N^\theta)$. Thus this probability is negligible compared to the first term on the right side of inequality (53). Enlarging C completes the proof of the corollary. $\square$

Remark. The same ideas, in particular estimating $I_N(t)$ for any N, can be used to prove global existence almost surely in the energy norm $s=1$, but we do not write out precise statements here.

4. Large k cut-off systems and an equilibrium invariant measure

4.1. Convergence of finite dimensional cut-off systems

We consider a cut-off version of our system (1) where we retain Fourier modes $\{k\}$ with $|k| \leqslant M$, M a positive integer. Let $\mathbf{u}_M(t) = (u_M, r_M)(t) = (\phi_M + \frac{1}{B}\pi_M, r_M)(t)$ denote a solution to the finite dimensional system

$$
\begin{aligned}
&i\partial_t u_M(t) = Bu_M(t) + \tfrac{1}{B} P_{\leqslant M}(\mu\phi_M^3(t) + r_M(t)\alpha),\\
&\mathrm{d}r_M(t) = -(r_M(t) - \langle P_{\leqslant M}\alpha_i, \pi_M(t)\rangle)\,\mathrm{d}t + \sqrt{2T}\,\mathrm{d}\omega(t),
\end{aligned} \tag{55}
$$

for initial data $u_M(0) \in P_{\leqslant M}L^2$. The solution $\mathbf{u}_M(t)$ remains in $P_{\leqslant M}L^2 \oplus \mathbf{R}^K$ and is clearly in H^s (for any s), since all Fourier coefficients $\hat{u}_{M,k} = 0$ for $|k| > M$. We remark that under the same assumptions on the coupling functions α, the conclusions of the previous section, Propositions 3.1, 3.3 and Corollary 3.4, hold for solutions $\mathbf{u}_M$ *uniformly* in M with respective initial data $\mathbf{u}_M(0) = P_{\leqslant M}\mathbf{u}(0)$ for an initial $\mathbf{u}(0) \in H^s$, $s > 1/3$. In particular the arguments used there are equally valid for the cut-off systems.

Fix s and let $\mathscr{D}_R(\beta, t)$ be the set of continuous functions defined in Eq. (36), in particular functions $\mathbf{u}$ bounded in the H^s-norm by $R\beta$ with $\|\mathbf{u}(0)\|_{H^s} \leqslant \beta$, and let $\mathscr{G}_R(\beta, t)$ be the probabilistic event defined

$$
\mathscr{G}_R(\beta, t) \equiv \{\mathbf{u}(\cdot),\ \mathbf{u}_M(\cdot) \in \mathscr{D}_R(\beta, t) \text{ for each } M\}, \tag{56}
$$

with $\mathbf{u}(\cdot)$ the solution to Eq. (1) and $\mathbf{u}_M(\cdot)$ the solution to Eq. (55).

Proposition 4.1. *Fix $s > 1/3$, a time $t > 0$, and $s_o > s$. Then $\{\mathbf{u}_M(\cdot)\}$ converges strongly to $\mathbf{u}(\cdot)$ in H^s uniformly on $\mathscr{G}_R(\beta, t) \cap \{\mathbf{u} \mid \|\mathbf{u}(0)\|_{H^{s_o}} \leqslant \beta\}$.*

Proof. For notational convenience we will replace the t of the proposition statement by t_1, and work on the time interval $t \in [0, t_1]$. We will assume that in this time interval we have the a priori bounds $\|\mathbf{u}(t)\|_{H^s} \leqslant R\beta$ and $\|\mathbf{u}_M(t)\|_{H^s} \leqslant R\beta$, for all M and t. Given these bounds, we will actually show the stronger result that $P_{\leqslant M}\mathbf{u} - \mathbf{u}_M \to 0$ strongly in H^1 and $P_{>M}\mathbf{u} \to 0$ strongly in H^s, uniformly in t and the initial data with $\|\mathbf{u}(0)\|_{H^{s_o}} \leqslant \beta$.

The quantities $\mathbf{u}$ and $\mathbf{u}_M$ satisfy the respective Duhamel relations

$$
\mathbf{u}(t) = \int_0^t \mathrm{e}^{(t-t')B_o} \begin{pmatrix} \dfrac{-i}{B}(\mu\phi^3 + r\alpha)\,\mathrm{d}t' \\ \langle \alpha, \pi \rangle\,\mathrm{d}t' + \sqrt{2T}\,\mathrm{d}\omega(t') \end{pmatrix} + \mathrm{e}^{tB_o}\mathbf{u}(0), \tag{57}
$$

and, for $\mathbf{u}_M$ with initial data $P_{\leqslant M}\mathbf{u}(0)$,

$$\mathbf{u}_M(t) = \int_0^t \mathrm{e}^{(t-t')B_o} \begin{pmatrix} \frac{-i}{B} P_{\leqslant M}(\mu\phi_M^3 + r_M\alpha)\,\mathrm{d}t' \\ \langle \alpha, \pi_M \rangle\,\mathrm{d}t' + \sqrt{2T}\,\mathrm{d}\omega(t') \end{pmatrix} + \mathrm{e}^{tB_o} P_{\leqslant M}\mathbf{u}(0). \tag{58}$$

We proceed to estimate $(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t)$ in the H^1-norm, for $0 \leqslant t \leqslant t_1$.

From the above integral formulae Eqs. (57,58) one sees that there will be three contributions to $(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t)$ coming from integrals involving the non-linearity $\phi^3 - \phi_M^3$, $(r - r_M)\alpha$, and $\langle \alpha, (\pi - \pi_M) \rangle$.

(i) $\phi^3 - \phi_M^3$-*term*. We bound the contribution to $(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t)$ by

$$\begin{aligned} &\left\| \int_0^t \mathrm{e}^{(t-t')B_o} \begin{pmatrix} \frac{-i\mu}{B} P_{\leqslant M}(\phi^3 - \phi_M^3)\,\mathrm{d}t' \\ 0 \end{pmatrix} \right\|_{H^1} \\ &\quad \leqslant c(R\beta)^2 \int_0^t \|P_{>M}\mathbf{u}(t')\|_{H^s}\,\mathrm{d}t' + c(R\beta)^2 \int_0^t \|(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t')\|_{H^1}\,\mathrm{d}t' \end{aligned} \tag{59}$$

for a suitable constant c by the Sobolev inequality (28) with $s' = 0$ and $s > 1/3$.

(ii) $(r - r_M)\alpha$-*term*. The corresponding contribution to $(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t)$ is bounded by

$$\begin{aligned} &\left\| \int_0^t \mathrm{e}^{(t-t')B_o} \begin{pmatrix} \frac{-i(r - r_M)}{B} P_{\leqslant M}\alpha\,\mathrm{d}t' \\ 0 \end{pmatrix} \right\|_{H^1} \\ &\quad \leqslant c\|\alpha\|_2 \int_0^t \|P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t')\|_{H^1}\,\mathrm{d}t'. \end{aligned} \tag{60}$$

(iii) $\langle \alpha, (\pi - \pi_M) \rangle$-*term*. Let $\tilde{\mathbf{u}}(t) = \mathbf{u}(t) - \mathrm{e}^{tL_o}\mathbf{u}(0)$ be the difference between $\mathbf{u}(t)$ and the solution of the linear equation with dissipation but without noise. We then bound the contribution to $(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t)$ from the integral of $\langle \alpha, (\pi - \pi_M) \rangle$ by

$$\begin{aligned} &\left\| \int_0^t \mathrm{e}^{(t-t')B_o} \begin{pmatrix} 0 \\ \langle \alpha, (\pi - \pi_M) \rangle\,\mathrm{d}t' \end{pmatrix} \right\|_{H^1} \\ &\quad \leqslant \|P_{>M}\alpha\|_2 \int_0^t \|P_{>M}\tilde{\mathbf{u}}(t')\|_{H^1}\,\mathrm{d}t' + \left| \int_0^t \mathrm{e}^{-(t-t')} \langle \alpha, BP_{>M}\mathrm{e}^{t'L_o}\mathbf{u}(0) \rangle\,\mathrm{d}t' \right| \\ &\qquad + \|\alpha\|_2 \int_0^t \|(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t')\|_{H^1}\,\mathrm{d}t'. \end{aligned} \tag{61}$$

To estimate the second of these integrals on the right side we use the identity

$$\int_0^t \mathrm{e}^{-(t-t')}\mathrm{e}^{t'L_o}\,\mathrm{d}t' = \frac{\mathrm{e}^{tL_o} - \mathrm{e}^{-t}}{L_o + \lambda} - (-\lambda + 1) \int_0^t \mathrm{e}^{-(t-t')}\mathrm{e}^{t'L_o}\,\mathrm{d}t', \tag{62}$$

and the bound

$$\left\| \frac{e^{t' L_o}}{L_0 + \lambda} \mathbf{u}(0) \right\|_{H^1} \leqslant c_3 \left\| \frac{1}{L_o + \lambda} \mathbf{u}(0) \right\|_{H^1} \leqslant c \|\mathbf{u}(0)\|_2, \tag{63}$$

using Lemma 2.2 (with $s = 1$), then Lemma 2.1 (with $s = 0$). Then the second integral in inequality (61) is bounded by $c\|P_{>M}\alpha\|_2 \|\mathbf{u}(0)\|_2 \leqslant c\|P_{>M}\alpha\|_2 \beta$.

Adding these contributions, inequalities (59,60,61), and using the last estimate, we arrive at the bound

$$\begin{aligned} \|(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t)\|_{H^1} \leqslant c(\alpha, R\beta) & \int_0^t \|(P_{\leqslant M}\mathbf{u} - \mathbf{u}_M)(t')\|_{H^1} \, dt' \\ & + c(R\beta)^2 \int_0^t \|P_{>M}\mathbf{u}(t')\|_{H^s} \, dt' \\ & + \|P_{>M}\alpha\|_2 \int_0^t \|P_{>M}\tilde{\mathbf{u}}(t')\|_{H^1} \, dt' \\ & + cR\beta\|P_{>M}\alpha\|_2, \end{aligned} \tag{64}$$

where $c(\alpha, R\beta)$ is linear in $\|\alpha\|_2$ and quadratic in $R\beta$.

The inhomogeneous terms on the right side of this inequality (64), i.e., the second, third, and fourth terms, each go to zero, $M \to \infty$ uniformly in t and the data.

(i) *Second term of* (64): Consider the projection of the integral formula of Eq. (57) above onto $P_{>M}H^s$. To control the non-linear contribution to $\|P_{>M}\mathbf{u}(t')\|_{H^s}$, we use

$$\begin{aligned} \left\| \frac{1}{B} P_{>M}\phi^3 \right\|_{H^s} &= \|P_{>M}\phi^3\|_{H^{s-1}} \\ &\leqslant M^{s-s''} \|P_{>M}\phi^3\|_{H^{s''-1}} \\ &\leqslant M^{s-s''} \|\mathbf{u}\|_{H^s}^3 \leqslant M^{s-s''}(R\beta)^3, \end{aligned} \tag{65}$$

by the Sobolev inequality (28), with s'' chosen, $s < s'' < 1$. The α-contribution presents little difficulty and is $\mathcal{O}(M^{-\gamma} R\beta\|\alpha\|_{H^\gamma})$, while the inhomogeneous term is estimated $\|P_{>M} e^{tB_o}\mathbf{u}(0)\|_{H^s} \leqslant c_3 M^{s-s_o} \|P_{>M}\mathbf{u}(0)\|_{H^{s_o}} \leqslant c_3 M^{s-s_o}\beta$, by Lemma 2.2.

(ii) *Third term of* (64): We have that

$$\begin{aligned} \|P_{>M}\tilde{\mathbf{u}}(t)\|_{H^1} &\leqslant \int_0^t \left\| P_{>M}\left(\frac{\mu}{B}\phi(t')^3 + \frac{(r - r_o)(t')}{B}\alpha \right) \right\|_{H^1} dt' \\ &\leqslant \int_0^t \|P_{>M}(\mathbf{u}(t')^3)\|_2 \, dt' + (R + c_3)\beta t \|P_{>M}\alpha\|_2. \end{aligned} \tag{66}$$

Here, $P_{>M}(\mathbf{u}(t')^3) = P_{>M}((P_{\leqslant[M/3]}\mathbf{u}(t') + P_{>[M/3]}\mathbf{u}(t'))^3)$, where $[M/3]$ denotes the greatest integer $\leqslant M/3$. Expanding this out, one sees that terms containing a factor $P_{>[M/3]}\mathbf{u}(t')$ go to zero uniformly for $M \to \infty$ as in our analysis of the second term of (64), and the term $P_{>M}(P_{\leqslant[M/3]}\mathbf{u}(t')^3)$ is identically zero. The α-term is $\mathcal{O}(M^{-\gamma}R\beta\|\alpha\|_{H^\gamma})$. It follows that $\|P_{>M}\tilde{\mathbf{u}}(t)\|_{H^1}$ goes to 0 uniformly.

(iii) *Fourth term of* (64): This term, proportional to α, is $\mathcal{O}(M^{-\gamma}R\beta\|\alpha\|_{H^\gamma})$.

Thus inequality (64), with each of its inhomogeneous terms going to zero uniformly, $M\to\infty$, implies via Gronwall's inequality that $(P_{\leqslant M}\mathbf{u}-\mathbf{u}_M)(t)$ goes to zero in H^1 uniformly in t and the data. Since $P_{>M}\mathbf{u}(t)$ goes to zero uniformly in H^s as we have seen above, we have that $\mathbf{u}_M(t)$ converges to $\mathbf{u}(t)$ uniformly in H^s, *provided* that $\|\mathbf{u}(t)\|_{H^s}$ and $\|\mathbf{u}_M(t)\|_{H^s}$ stay less than $R\beta$ for $t\leqslant t_1$, i.e., are in $\mathcal{G}_R(\beta,t_1)$, and the data $\mathbf{u}(0)$ satisfies $\|\mathbf{u}(0)\|_{H^{s_0}}\leqslant\beta$. □

4.2. Equilibrium invariant measure

We proceed now to show the existence of an invariant measure for the complete system Eqs. (1), but in equilibrium where all temperatures are equal to a common T. Let v^0 be the Gaussian measure referred to in the introduction. For v_0, φ,π and r are independent, of mean zero, and with respective covariance $\frac{1}{B^2}$, 1, and 1. With respect to v^0, $\|u\|^2_{H^s}$ has finite expectation for $s<1/2$,

$$\int \|u\|^2_{H^s}\,\mathrm{d}v^0 = 2T\sum_k (1+k^2)^{s-1}<\infty, \tag{67}$$

hence $\|\phi\|_{H^s}$ is finite, v^0-a.s. By Sobolev inequality (26) of Lemma 2.4, $\|\phi\|_4\leqslant c\|\phi\|_{H^s}$ for $s>1/4$, so that as random variables $\|\phi\|_4$ and $\|P_{\leqslant M}\phi\|_4$ are also finite v^0- a.s. (Actually v_0 is supported on continuous functions, but we do not need this here.) Set

$$\mathrm{d}v_M(\pi,\phi) = Z_M^{-1}\exp\left(-\frac{\mu}{4T}\int\phi_M^4\,\mathrm{d}x\right)\mathrm{d}v^0(\pi,\phi),$$
$$\mathrm{d}v(\pi,\phi) = Z^{-1}\exp\left(-\frac{\mu}{4T}\int\phi^4\,\mathrm{d}x\right)\mathrm{d}v^0(\pi,\phi), \tag{68}$$

with appropriate normalizations

$$Z_M = \int\exp\left(-\frac{\mu}{4T}\int\phi_M^4\,\mathrm{d}x\right)\mathrm{d}v^0,\quad Z=\exp\left(-\frac{\mu}{4T}\int\phi^4\,\mathrm{d}x\right)\mathrm{d}v^0, \tag{69}$$

and $\phi_M = P_{\leqslant M}\phi$. Since $\|\phi\|_4$ and $\|P_{\leqslant M}\phi\|_4$ are finite a.s., the Radon–Nikodym factors $\exp(-\frac{\mu}{4T}\int\phi_M^4\,\mathrm{d}x)$ and $\exp(-\frac{\mu}{4T}\int\phi^4\,\mathrm{d}x)$ are bounded and positive a.s., and the normalizations Z and Z_M are positive. The measures v_M and v are absolutely continuous with respect to v^0.

The semigroup associated with the cut-off system Eq. (55) acts invariantly on functions $\{f(\mathbf{u})\}$ of the form $f(\mathbf{u}) = g(\hat{u}_{-M},\ldots,\hat{u}_M,r)$, with g integrable. The measure v_M is an invariant measure for this semigroup, as can be checked by computing the generator of the process and showing that its adjoint annihilates v_M; we leave this exercise to the reader.

We also have that $\lim_{M\to\infty} Z_M = Z$. This is the case by the bounded convergence theorem: Clearly the exponentials in (69) are bounded by one, and

$$|\|\phi\|_4 - \|\phi_M\|_4| \leqslant \|\phi - \phi_M\|_4 \leqslant c\|\phi - \phi_M\|_{H^s} \to 0, \quad M \to \infty \text{ a.s.} \tag{70}$$

since ϕ is in H^s a.s. Thus

$$\exp\left(-\frac{\mu}{4T}\int \phi_M^4\,\mathrm{d}x\right) \to \exp\left(-\frac{\mu}{4T}\int \phi^4\,\mathrm{d}x\right), \text{ a.s.} \tag{71}$$

Let f be a function in the norm closure $\bar{\mathscr{X}}$ of functions depending continuously on only a finite number of modes,

$$\mathscr{X} \equiv \bigcup_M \{f \mid f = g(\hat{u}_{-M}, \dots, \hat{u}_M, r),\ g \text{ bounded continuous}\}. \tag{72}$$

Then again we have by bounded convergence that

$$\int f \exp\left(-\frac{\mu}{4T}\int \phi_M^4\,\mathrm{d}x\right)\mathrm{d}v^0 \to \int f \exp\left(-\frac{\mu}{4T}\int \phi^4\,\mathrm{d}x\right)\mathrm{d}v^0, \tag{73}$$

and so

$$Z_M^{-1}\int f \exp\left(-\frac{\mu}{4T}\int \phi_M^4\,\mathrm{d}x\right)\mathrm{d}v^0 \to Z^{-1}\int f \exp\left(-\frac{\mu}{4T}\int \phi^4\,\mathrm{d}x\right)\mathrm{d}v^0. \tag{74}$$

Thus, v_M converges to v in a weak-$\star$ sense.

For later use, we also note a kind of tightness for the measures $\{v_M\}$; for $s<1/2$,

$$\begin{aligned}\int_{\{\|\mathbf{u}\|_{H^s}>\beta\}} \mathrm{d}v_M &\leqslant \frac{1}{Z_M}\int \frac{\|\mathbf{u}\|_{H^s}}{\beta}\exp\left(-\frac{\mu}{4T}\int \phi_M^4\,\mathrm{d}x\right)\mathrm{d}v^0 \\ &\leqslant \frac{1}{\beta Z_M}\left(\int \|\mathbf{u}\|_{H^s}^2\,\mathrm{d}v^0\right)^{1/2},\end{aligned} \tag{75}$$

which is arbitrarily small for β large, uniformly in M, by inequality (67).

Finally, we address the invariance of v. Define the semigroups

$$S^t f(\mathbf{u}) \equiv E_{\mathbf{u}}[f(\mathbf{u}(t))], \quad S_M^t f(\mathbf{u}) \equiv E_{\mathbf{u}}[f(\mathbf{u}_M(t))], \tag{76}$$

where $f \in \bar{\mathscr{X}}$. (We will assume here for definiteness that the Fourier modes $\hat{\mathbf{u}}_{M,k}(t)$ are simply constant in time for modes $|k|>M$.)

Proposition 4.2 (*Equilibrium case*). *The measure v is invariant with respect to the semigroup S^t in the sense that for $f \in \bar{\mathscr{X}}$,*

$$\int S^t f\,\mathrm{d}v = \int f\,\mathrm{d}v. \tag{77}$$

Proof. Choose s, $1/3<s<1/2$, and let $\mathscr{G}_R(\beta, t)$ be the event defined in Eq. (56) (using the H^s-norm). By Corollary 3.4, we have that for an initial $\mathbf{u}$ with $\|\mathbf{u}\|_{H^s}\leqslant\beta$,

$$P_{\mathbf{u}}\{\mathscr{G}_R^c(\beta, t)\} \leqslant C\exp\left(-\frac{cR^{2\theta/(1-s)}}{t(1+t)^2}\right). \tag{78}$$

We identify the R here with $2N^{1-s}$ in the corollary statement (the R of the corollary being chosen equal to 2) and appropriately redefine the constant c.

Now let s_o be chosen, with $s<s_o<1/2$ and let $f(\mathbf{u})$ be a bounded function continuous in the H^s-norm of $\mathbf{u}$. (Such functions are dense in $\bar{\mathscr{X}}$). For $\|\mathbf{u}\|_{H^{s_o}}\leqslant\beta$ and any $\varepsilon>0$,

$$\begin{aligned}
&|S^t f(\mathbf{u}) - S^t_M f(\mathbf{u})| \\
&\quad\leqslant |E_{\mathbf{u}}[\chi_{\mathscr{G}_R(\beta,t)}(f(\mathbf{u}(t)) - f(\mathbf{u}_M(t)))]| + 2\|f\|_\infty P_{\mathbf{u}}\{\mathscr{G}^c_R(\beta,t)\} \\
&\quad<\varepsilon,
\end{aligned}\tag{79}$$

for R and then M chosen sufficiently large, by the above probability estimate, and by the uniform convergence of $\mathbf{u}_M$ to $\mathbf{u}$ on $\mathscr{G}_R(\beta,t)\cap\{\mathbf{u}\,|\,\|\mathbf{u}(0)\|_{H^{s_o}}\leqslant\beta\}$, Proposition 4.1. Consequently, $S^t_M f(\mathbf{u})\to S^t f(\mathbf{u})$, for $M\to\infty$ uniformly in $\mathbf{u}$, $\|\mathbf{u}\|_{H^{s_o}}\leqslant\beta$.

Finally,

$$\begin{aligned}
&\left|\int S^t f\,\mathrm{d}\nu - \int S^t_M f\,\mathrm{d}\nu_M\right| \\
&\quad\leqslant\left|\int S^t f(\mathrm{d}\nu - \mathrm{d}\nu_M)\right| + \int \chi_{\{\|\mathbf{u}\|_{H^{s_o}}\leqslant\beta\}}|S^t f - S^t_M f|\,\mathrm{d}\nu_M \\
&\qquad + 2\|f\|_\infty \int_{\{\|\mathbf{u}\|_{H^{s_o}}>\beta\}} \mathrm{d}\nu_M.
\end{aligned}\tag{80}$$

The first term on the right side goes to zero by weak-$\star$ convergence of $\{\nu_M\}$ to ν, the last term can be made arbitrarily small for β suitably large by tightness inequality (75), and the middle term then goes to zero by uniform convergence of $S^t_M f$ for $\|\mathbf{u}\|_{H^{s_o}}\leqslant\beta$. Thus

$$\int S^t f\,\mathrm{d}\nu = \lim_{M\to\infty}\int S^t_M f\,\mathrm{d}\nu_M = \lim_{M\to\infty}\int f\,\mathrm{d}\nu_M = \int f\,\mathrm{d}\nu,\tag{81}$$

by the above inequality (80), by invariance of ν_M under $(S^t_M)^*$, and again by weak-$\star$ convergence of the $\{\nu_M\}$. This completes the proof of invariance of ν for functions depending continuously on $\mathbf{u}$ with respect to the H^s-norm and, by density, invariance for all $f\in\bar{\mathscr{X}}$. $\square$

Remark. We emphasize that the question of ergodicity for this equilibrium measure ν remains open, as does the existence of non-equilibrium invariant measures for differing temperatures.

Acknowledgements

L.E.T. wishes to thank the Department of Mathematics, U. Mass Amherst, for the hospitality extended to him. L.R-B. was supported in part by NSF Grant DMS-0306540, L.E.T. by NSF Grant DMS-0245511.

References

[1] J. Bourgain, Periodic nonlinear Schrödinger equation and invariant measures, Comm. Math. Phys. 166 (1994) 1–26.

[2] J. Bourgain, Global solutions of nonlinear Schrödinger equations, American Mathematical Society Colloquium Publications, vol. 46. American Mathematical Society, Providence, RI, 1999 (Chapter IV, Section 2).

[3] J. Bourgain, Invariant measure for the non-linear Schrödinger equation, in infinite volume, Commun. Math. Phys. 210 (2000) 605–620.

[4] J. Bricmont, A. Kupiainen, R. Lefevere, Exponential mixing of the 2D stochastic Navier–Stokes dynamics, Comm. Math. Phys. 230 (2002) 87–132.

[5] D.C. Brydges, G. Slade, Statistical mechanics of the 2-dimensional focusing non-linear Schrödinger equation, Commun. Math. Phys. 182 (1996) 485–504.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Local and global well-posedness for non-linear dispersive and wave equations, http://www.math.ucla.edu/%7Etao/Dispersive/

[7] W. E, J.C. Mattingly, Ya. Sinai, Gibbsian dynamics and ergodicity for the stochastically forced Navier–Stokes equation, Comm. Math. Phys. 224 (2001) 83–106.

[8] J.-P. Eckmann, M. Hairer, Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators, Commun. Math. Phys. 212 (2000) 105–164.

[9] J.-P. Eckmann, M. Hairer, Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise, Comm. Math. Phys. 219 (2001) 523–565.

[10] J.-P. Eckmann, M. Hairer, Spectral properties of hypoelliptic operators, Commun. Math. Phys. 235 (2003) 233–253.

[11] J.-P. Eckmann, C.-A. Pillet, L. Rey-Bellet, Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures, Commun. Math. Phys. 201 (1999) 657–697.

[12] J.-P. Eckmann, C.-A. Pillet, L. Rey-Bellet, Entropy production in non-linear, thermally driven Hamiltonian systems, J. Statist. Phys. 95 (1999) 305–331.

[13] M. Keel, T. Tao, Local and global well-posedness of wave maps in $\mathbf{R}^{1+1}$ for rough data, Int. Math. Res. Not. 21 (1998) 1117–1156.

[14] S. Kuksin, A. Shirikyan, Ergodicity for the randomly forced 2D Navier–Stokes equations, Math. Phys. Anal. Geom. 4 (2001) 147–195.

[15] J.L. Lebowitz, H. Rose, E. Speer, Statistical mechanics of the non-linear Schrödinger equation, J. Statist. Phys. 50 (1988) 657–687 See also, Statistical mechanics of the non-linear Schrödinger equation II, Mean field approximation, J. Statist. Phys. 54 (1989) 17–56.

[16] N. Masmoudi, L.-S. Young, Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDEs, Comm. Math. Phys. 227 (2002) 461–481.

[17] H.P. McKean, K.L. Vaninsky, Statistical mechanics of non-linear wave equations, in: Stochastic Analysis, Ithaca, NY, 1993, pp. 457–463 Proc. Symp. Pure Math 57, American Mathematical Society, Providence, RI, 1995.

[18] L. Rey-Bellet, Statistical Mechanics of anharmonic lattices, in: Advances in Differential Equations and Mathematical Physics, Contemporary Mathematics, vol. 327, AMS, Providence, RI, 2003.

[19] L. Rey-Bellet, L.E. Thomas, Asymptotic behavior of thermal non-equilibrium steady states for a driven chain of anharmonic oscillators, Commun. Math. Phys. 215 (2000) 1–24.

[20] L. Rey-Bellet, L.E. Thomas, Exponential convergence to non-equilibrium stationary states in classical statistical mechanics, Commun. Math. Phys. 225 (2002) 305–329.

[21] L. Rey-Bellet, L.E. Thomas, Fluctuations of the entropy production in anharmonic chains, Ann. H. Poinc. 3 (2002) 483–502.

[22] P.E. Zhidkov, An invariant measure for a nonlinear wave equation, Nonlinear Anal. 22 (1994) 319–325.