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Students' Mental Acts of Anticipating in Solving Problems involving Algebraic Inequalities and Equations

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STUDENTS' MENTAL ACTS OF ANTICIPATING IN SOLVING PROBLEMS INVOLVING ALGEBRAIC INEQUALITIES AND EQUATIONS

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The dissertation of Kien Hwa Lim is approved, and it is acceptable in quality and form for publication on microfilm:

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May the successful completion of my doctorate

encourage others to have faith in themselves and succeed in theirs.

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ABSTRACT OF THE DISSERTATION

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Anticipating is the mental act of conceiving a certain expectation without performing a sequence of detailed operations to arrive at the expectation. This dissertation seeks to characterize students' problem-solving in terms of two types of anticipating acts: (a) *foreseeing* an action, which refers to the act of conceiving an expectation that leads to an action, prior to performing the operations associated with the action, and (b) *predicting* a result, which refers to the act of conceiving an expectation for the result of an event without actually performing the operations associated with the event. Harel's (in press) triad of determinants—mental act, ways of understanding, and ways of thinking—is used to analyze students' acts of foreseeing and predicting.

This research has three objectives: (a) to categorize students' ways of thinking associated with foreseeing and predicting, (b) to identify the relationships between these

ways of thinking and students' ways of understanding inequalities/equations, and (c) to explore the potential for advancing students' ways of thinking associated with foreseeing/predicting. To accomplish these goals, fourteen 11th graders enrolled in various mathematics courses were interviewed. Four of them participated in one-on-one teaching interventions. Non-directive tasks were used to elicit students' anticipatory behaviors.

In this study, five ways of thinking associated with foreseeing were identified: impulsive anticipation, tenacious anticipation, explorative anticipation, analytic anticipation, and interiorized anticipation. Three ways of thinking associated with predicting were identified: association-based prediction, comparison-based prediction, and coordination-based prediction. In addition, five ways of understanding inequalities/equations (I/E) were identified: I/E-as-a-signal-for-procedure, I/E-as-a-static-comparison, I/E-as-a-proposition, I/E-as-a-constraint, and I/E-as-a-comparison-of-functions. Students' ways of thinking associated with foreseeing/predicting were found to be related to the quality of their solutions as well as the sophistication of their ways of understanding inequalities/equations.

One learner's improvement was summarized in terms of the change in the *sub-context* (Cobb, 1985) in which she operated, from manipulating symbols in the pre-interview to reasoning with symbols in the post-interview. Her operating in the sub-context of working with numbers helped her to achieve this transition. This finding underscores the importance of using numbers as a platform for algebra students to explore algebraic expressions and symbolic structures.

CHAPTER 1: INTRODUCTION

This chapter has five sections. The first section presents some students' problem-solving behaviors in a classroom setting to illustrate the diversity in the ways students anticipate as they solve problems in algebra. The second section highlights why students' act of anticipating deserves explicit attention within the field of mathematics education. The objectives of this research and the reasons for choosing algebraic inequalities and equations as a context for this investigation are discussed in the third and fourth sections. Finally, an organization of the dissertation is outlined in the fifth section.

1.1 Anticipation in Problem-solving Situations

In a pilot study, a 90-minute classroom interaction was conducted with a group of nine high-school calculus students. The students were told that the purpose of the session was to understand the way they think as they solve problems in algebra. They worked on four problems (see Figure 1.1) and were asked to write down their thoughts as they solved each problem. Initially, these students worked on each problem individually. This was followed by a class discussion in which the approaches they used to solve the problem were polled. Occasionally, the students were asked to share their solution and/or their struggles with the class.

P1. Find all values of q that make $|q - 2| < |q - 8|$ true.

P2. x and y can be any positive even numbers less than 25.
Is $x + 2y + 3y + 4x > 225$ always true, sometimes true, or never true?
If it is sometimes true, find the values of x and y that make it true.

P3. Are there any odd integers for n that satisfy $(n - 2)(n - 8) + 10 < 0$?
If yes, find all of them. If no, provide a convincing argument.

P4. Is $\frac{3n}{n - 2} < 1$ always true, sometimes true, or never true? If it is always true or never true, prove it. If it is sometimes true, find exactly when it is true.

Figure 1.1: Four tasks used in a pilot study

Item P2 and Item P3 are discussed below. For Item P2, two students listed all positive even numbers less than 25 before working the problem. One of them, Cindy, guessed that it was never true after trying three cases: (2, 4), (16, 24) and (22, 22). The other student, Cora, found that (2, 4) and (22, 24) did not make the equation true, but (24, 22) did; she wrote "*x* has to be greater than *y*" for the inequality to be true. Five students (Alex, Andy, Brad, Sera, and Sue) started with the largest possible numbers and worked downwards, for example, (24, 24), (22, 24), (24, 22) and (22, 22). They most likely predicted, prior to plugging in (24, 24), that the inequality could not be true for small values such as (2, 4). This act of *predicting* is what Cindy and Cora did not appear to do. Finally, the remaining two students, Adam and Sam, simplified the inequality to $5x + 5y > 225$ and $x + y > 45$ respectively, prior to substituting (24, 24). These two students possibly foresaw that a simplified version would ease their computations.

The problem-solving behaviors of Cindy and Cora were qualitatively different from the other seven students. Cindy and Cora seemed rather methodological in their approach. They first listed all possible values for x and y, selected and tested three pairs of values, observed a pattern, and formed a conclusion based on their generalization. One may ask why they did not check the largest possible values for x and y, (24, 24). Is it because they did not conceive inequality as a magnitude comparison between its two sides? Or is it because they anticipated that a pattern would arise from the results of checking a few cases? Or is it because they had a procedure to perform? This episode raises the question: how ubiquitous are students' tendencies to start doing and stop "thinking" once they have a procedure?

In the case of Item P3, one student, Alex, substituted n = 1, 5, 7, 9, 11 into the inequality. He found that the inequality was false for each substitution. He stated that he was "99.9% confident" that "there are no odd integers that will satisfy this." He then substantiated his claim by writing "I didn't try any [numbers] higher than 11 because the answer will just keep getting bigger and make the answer more untrue" and "any negative number when multiplied [with another negative number] will give you a positive number & when [you] add [that] by 10 it will make the answer larger." Alex was systematic in his exploration and plugged in numbers to help him see the structure of the inequality; after this, he predicted "never true" without having to plug in other numbers.

Another student, Andy, was goal-oriented in his choice of integers for substitution, aiming to make one factor positive and one factor negative. He also predicted that negative integers did not work because both the factors would be negative.

Both Alex and Andy were attending to the meaning and the quantitative value of the symbols.

In contrast, Cora simplified the inequality to $(n - 2)(n - 8) < -10$, from which she obtained $n - 2 = -10$ and $n - 8 = -10$, and then $n = -8$ and $n = 2$. She then re-simplifed the inequality to $n(n - 10) < -26$, from which she obtained $n = -26$ and $n = -16$. She rejected these four values because they were not odd integers. In this problem, Cora manipulated the symbols without attending to the quantitative comparison between the two sides of the inequality. Her reasoning with symbols without attending to the referents[1] of those symbols is what Harel (1998, in press b) calls *non-referential symbolic reasoning*. It is worth investigating whether there is a relationship between a student's tendency not to predict and her or his tendency to engage in non-referential symbolic reasoning.

The students in the pilot study exhibited different problem-solving behaviors: anticipating an empirical approach of plugging in numbers, predicting prior to performing any actions, anticipating the ease of reasoning with a simpler expression, exploring the inequality by plugging in numbers and reasoning with its structure, anticipating the plugging in of certain numbers in a goal-oriented manner, and anticipating different ways of manipulating symbols. Some of these approaches have desirable characteristics, such as being goal-oriented, while others have less desirable characteristics such as performing operations without attending to meaning.

In this research, I aim to categorize students' problem-solving behaviors in terms of the characteristics of their anticipation. The term "anticipation" used in this research is

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[1] In this case the referents are numbers whereas in contextualized problems the referents are usually quantities.

in accordance with Piaget's notion: "anticipation is nothing other than a transfer or application of the scheme … to a new situation before it actually happens" (1967/1971, p. 195). A scheme involves three components: the perceived situation, the activity, and the expected result (von Glasersfeld, 1995). The expected result component of a scheme is what allows a person to anticipate. During problem-solving, a few schemes may be enacted and coordinated before one arrives at an action. The verb "anticipate" refers to the mental act of conceiving a certain expectation[2] without performing a sequence of detailed operations to arrive at the expectation. If the expectation concerns an action to be performed, then I use the term *foresee*. If the expectation concerns a result or outcome, then I use the term *predict*. In other words, one foresees an action but predicts a result.

1.2 Why Does Anticipation Deserve Attention in Mathematics Education?

This section presents arguments for why anticipation deserves attention in the field of mathematics education. Three interrelated aspects of the importance of studying anticipation are discussed: (a) reasoning and sense-making in mathematics, (b) problem solving in mathematics, and (c) learning of mathematics.

Anticipation and Sense-making

The phenomenon of applying a newly learned procedure or algorithm to solve a problem occurs very often in mathematics classrooms. Many students apply procedures they have been taught without having to make sense of what they are doing (see Brown

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[2] Conceiving an expectation usually entails transforming *images* which are generally more compact and versatile than the *operations* they signify. The distinction between images and operations is discussed in Chapter 2.

et al. 1988). Teachers frequently witness inappropriate use of procedures, what Fischbein and Barash (1993) call improper application of algorithmic models. For example, consider the following problem that was posed by Cramer, Post and Currier (1993) to students in a mathematics methods class: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* 32 out of 33 college students in a mathematics methods class solved this problem by setting up a proportion, $\frac{9}{3} = \frac{x}{15}$, and obtained $x = 45$. These students had applied the proportion algorithm without making "sense", from an observer's perspective, of the problem situation. Because they possess a tool, the proportion algorithm, they have little need to make sense of the situation. Many students have already developed the habit of spontaneously applying algorithms and formulas to solve mathematics problems. Will helping students anticipate lessen their tendency to apply algorithms automatically? Are students more likely to make sense of the problem situation if they predict a solution and then check their prediction?

According to Sowder (1992), estimation and mental computation are curricular activities that aid students' development of *number sense*. Likewise, in algebra, certain activities could promote students' development of *symbol sense* (Fey, 1990; Arcavi, 1994) and *structure sense* (Hoch & Dreyfus, 2004). Just as number sense is an intuitive feel for numbers in arithmetic, symbol sense is an intuitive feel for structure in algebra. According to Arcavi (1994), symbol sense encompasses many features. It includes having an intuition for when to use and when to abandon an algebraic approach, reading symbolic expressions for reasonableness, being cautious of symbolic illusions (e.g., perceiving ($f(x))^2$ as always greater than | $f(x)$|), and choosing appropriate symbols and representations (e.g., using $2n - 1$ to represent odd numbers). The goals proposed by Fey (1990) for students' development of symbol sense include (a) the ability to scan an algebraic expression to predict the corresponding numeric or graphic pattern; (b) the ability to inspect algebraic operations and predict the form of the result; and (c) the ability to select the most appropriate form for a particular task, for example factored form for finding roots of a polynomial and standard form for differentiating or integrating a polynomial. These descriptions portray symbol sense as an ability related to having foresights or making predictions.

Number sense and symbol sense can be viewed as resources that allow students to predict prior to performing standard algorithms. Students with number sense can use their knowledge of number properties to estimate a computation or check the reasonableness of their answers (Sowder, 1992). For example, when asked to mentally estimate 922 × 0.34, a student with number sense recognizes 0.34 as approximately 1/3 and chooses to round 922 to 930, a third of which is 310. A student who is weak in number sense may have to mentally perform the paper-and-pencil algorithm by rounding 922 to 900 and 0.34 to 0.3, multiply 900 by 3 to get 2700, and then move one decimal to get 270. In algebra, students with symbol sense can mentally carry out transformations of images based on their knowledge of the algebraic structure, while students who lack symbol sense will have to perform the operations in an algorithm. For example, when asked to find a number that satisfies the inequality $6x + 10 > x + 90$, one student may compare the slopes of the functions and predict that a large number, such as $x = 1000$, will work; another student may perform a standard equation-solving procedure to obtain

$x > 16$ and give $x = 17$ as the answer. Just as students are more likely to develop number sense when they engage in estimation activities, I conjecture that students are more likely to develop symbol sense when they engage in tasks that require them to predict and then check their prediction (e.g., problems P2, P3, and P4 in Figure 1.1).

A desirable problem-solving behavior is one in which the student makes sense of the problem, makes a prediction, and then confirms her or his prediction. Consider the following problem: *Solve* $\frac{2x+3}{4x+6} = 2$ for x. Arcavi (1994) reported that a student noticed that the numerator is half the denominator and said “OK, so this problem has no solution, but what if I ‘solve’ it anyway?” (p. 27). When the student obtained $x = -1\frac{1}{2}$, he was puzzled because he expected no solution. He resolved the conflict when he tried substituting $x = -1\frac{1}{2}$ into $\frac{2x+3}{4x+6} = 2$ and realized it was not permissible because the denominator is zero; so $x = -1\frac{1}{2}$ was not a solution. The disposition of *first predicting and then performing to confirm* as exhibited by this student is a desirable goal for instruction.

Anticipation and Mathematical Problem Solving

According to Halmos (1980), the heart of mathematics is solving problems: “what mathematics *really* consists of is problems and solutions” (p. 519). Many mathematics educators (e.g., Brousseau, 1997; Harel, 2001) believe that the development of mathematical knowledge occurs through solving mathematical problems for which a procedure is not initially known. From a Piagetian perspective, learning involves cycles of experiencing disequilibrium, resolving cognitive conflicts, and re-establishing a new equilibrium (Piaget, 1975/1985). Solving problems is probably the best way to experience such a process, especially for learning mathematics. Thompson (1985) outlines five guiding principles for developing a mathematics curriculum, the first principle being that the curriculum must be problem-based. According to Schoenfeld (1994), in a problem-based curriculum, "problems are the major vehicles for introducing important issues and their solutions are the major carriers of curricular weight" (p. 67). If problem solving is the way to develop knowledge, then mathematics educators should gain an understanding of how students think as they solve problems in mathematics.

Schoenfeld (1985) identifies four categories of cognition that provide a framework for analyzing problem-solving behaviors: (a) knowledge base, which includes both mathematical content and access to the content; (b) use of heuristics, which are rule-of-thumb strategies employed in solving a problem; (c) control and metacognition, which includes planning and managing resources, selecting goals and sub-goals, monitoring and assessing progress, and revising and abandoning plans; and (d) belief systems about mathematics and doing mathematics. These categories allow mathematics educators to contrast behaviors of good problem solvers with those of poor problem solvers.

However, to account for the development of the above categories, a finer grain of analysis is needed to analyze students' mental acts[3] as they solve a problem. According to Harel (in press c), "mental acts are basic elements of human cognition. To describe, analyze, and communicate about humans' intellectual activities, one must attend to their

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[3] The term "mental act" and its associated terms, ways of understanding and ways of thinking, which are discussed in Chapter 2, are as defined in Harel (in press a).

mental acts." This current research is an attempt to study the ways students solve problems by focusing on their mental acts of foreseeing and predicting.

Anticipation and Mathematical Learning

Up to this point, I have discussed the importance of studying anticipations because of its relation with sense making and problem solving. Now I discuss the role of anticipation in mathematical learning.

Simon, Tzur, Heinz and Kinzel (2004) articulate a mechanism for conceptual learning that explains how more sophisticated mathematical conceptions are developed from less sophisticated ones. The learning mechanism involves (a) setting a goal; (b) enacting and re-enacting (mental or physical) activities, which are coordinated by anticipating and retroacting[4], to attain one's goal; (c) attending to the corresponding effects; (d) differentiating effects that advance toward one's goal from those that do not; (e) reflecting on one's mental records of the experience with a particular focus on the relationship between activities and their effects; (f) identifying regularities among those activity-effect relationships; and (g) eventually abstracting a new activity-effect relationship. Hence, Simon et al. conceive "mathematics learning as a reflection on activity-effect relationships" (p. 325). Two types of reflections are distinguished: (a) reflection on the results of an activity in relation to one's goal, and (b) reflection on the regularities in the activity-effect relationship which results in the formation of a new conception. The authors view a conception as "the ability to anticipate the effect of one's

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[4] Retroaction refers to the process of revising one's earlier actions in the light of new information (Inhelder & Piaget, 1964/1969). The verb *retroact* refers to going back to refine an earlier action based on one's anticipation of a new outcome for the to-be-renewed action.

activity, without mentally or physically running that activity" (p. 319). This definition is in keeping with Piaget's notion of scheme, which is anticipatory in nature.

Tzur (2003) identifies two stages in the abstraction of a new conception: the *participatory stage* and the *anticipatory stage*. At the participatory stage, the ability to anticipate the effects of an activity is confined to the context of the activity through which the activity-effect relationship was developed. As such, the conception is provisional and cueing may be needed in a context that differs from the context from which the conception was abstracted. At the anticipatory stage, the ability to anticipate extends beyond the original context. This means that the conception can be used autonomously in new situations. According to Tzur and Simon (2003), the distinction between these two stages can be used to account for the "next day phenomenon," in which the conception that was abstracted from an activity on one day is forgotten, in the absence of the activity, on the next day. However, the conception may be evoked if students are reminded of the previous day's activity. These students are considered to be at the participatory stage of development of the new conception. It is only at the anticipatory stage that students have established the connection between the activity-effect relationship and the situations where the new conception might apply. Tzur and Simon suggest that "the failure of educational interventions to promote structured (object) levels of concepts could be explained by the lack of explicit attention to fostering … the anticipatory stage." They acknowledge that the transition from participatory stage to anticipatory stage has yet to be explicated. Tzur, via personal communication, added that anticipation plays an essential role in the transition.

Prediction can be used as a pedagogical means to aid students' learning. For example, the *predict-observe-explain* instructional approach (White & Gunstone, 1992) used in physics education requires students to predict prior to observing a demonstration. In biology education, Lavoie (1999) found that the addition of *prediction-discussion phase* to a three-phase learning cycle—exploration, term introduction, and concept application—could improve students' process skills, logical-thinking skills, science concepts, and scientific attitudes. In mathematics education, Fischbein and Grossman (1997) comment that having students predict may improve their understanding of the underlying principle in a solution.

> Encouraging the learner to guess intuitively, one creates a challenging situation. Another way of achieving this is through facing the student with a conflict between a personal guess and a mathematically accepted solution. Such a conflict may stimulate the interest of the learner and may help him or her to overcome his or her intuitive obstacles. Moreover this understanding may contribute to the understanding of the mechanisms that shape the answer. (p. 43)

Getting students to predict results prior to performing calculations may also help them to notice certain relationships, generalize from specific cases, and expand the assimilatory range of a particular conception. For example, having students predict prior to computing whether the result of multiplying 9.29 by 7/6, or by 0.64, is greater or less than 9.29 tends to draw their attention to the effect of the multiplier. Upon further reflection, students may even advance their understanding of multiplication, moving from viewing multiplication as an algorithm-to-follow and/or multiplication as repeated-addition to viewing multiplication as enlargement/amplification. In algebra, a student who incorrectly predicts that there is no value of x that will make $3(2x - 9) = 6(2x - 9)$ true because 6 is always bigger than 3 is more likely to appreciate the notion of critical value, than another student who applies the standard equation-solving procedure and obtain the correct answer of $x = 4.5$.

The mental act of anticipating appears to be an important construct in many aspects of mathematics education. As discussed above, anticipation plays an important role in sense-making, problem solving, concept development, and mathematical learning. However, there is virtually no research in mathematics education that focuses explicitly on students' mental act of anticipating.

1.3 Research Objectives

This research focuses explicitly on students' mental act of anticipating (foreseeing or predicting) in the context of solving problems in algebra. One objective of this research is to identify and characterize students' foresights and predictions as they solve problems in the domain of algebraic inequalities and equations. This research explores the feasibility of using characteristics of students' anticipations as a means to communicate the quality of students' problem solving.

The second objective is to study the relation between students' anticipations (foresights or predictions) and their interpretations of inequalities and equations. Such a relation is expected to exist because how students interpret a problem situation should affect what and how they anticipate. For example, Cora's interpreting the inequality $(n - 2)(n - 8) < -10$ without attending to the quantitative comparison between the two sides of the inequality most likely influenced her anticipation of manipulating symbols.

The third objective is to explore the plausibility of improving the way students anticipate (foresee or predict) via a short-term one-on-one teaching intervention. The ways in which students anticipate may be resistant to change. Information on what factors can contribute to certain change, and why other change is difficult, will be valuable to mathematics educators who wish to improve the ways students work on problems.

Corresponding to these objectives are three research questions. Because the questions include technical terms that are an integral part of the theoretical framework, I will present them in Chapter 2 after I explicate the theoretical framework for studying students' mental act of anticipating.

1.4 Why Study Anticipation in the Domain of Algebraic Inequalities and Equations?

Algebraic inequalities and equations[5] were chosen for three main reasons: (a) they are foundational concepts in secondary mathematics curriculum, (b) they are not well understood among algebra students, and (c) they are rich and suitable contexts for studying students' mental act of anticipating.

The Prevalence of Inequalities and Equations in Mathematics

The notion of equivalence is one of the most important foundational concepts in mathematics. It is essential for understanding equality, congruence, and isomorphism. Equations and inequalities are a means for expressing the quantitative relationship between two or more mathematical objects. Inequalities and equations can be understood in many ways. An equation can be conceived of as (a) a constraint to determine the value of an unknown; (b) a function relating an output variable to its input variable(s), for

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[5] The term *inequalities* is placed in front of the term *equations* because students are more likely to attend to the variable attribute of a letter in an inequality as compared to an equation because the solution to a single-variable inequality is a range of unspecified numerical values, whereas the solution to a single-variable equation is usually a specific number. This means that inequalities tend to promote a variable conception of a letter, whereas equations tend to promote an unknown conception of a letter.

example, $z = x^2 + y^2$; (c) an expression to denote a family of functions/graphs, for example, $y = x^2 + 2x + c$; (d) a formula to model a physical phenomenon, for examples, $A = pr^2$ and $v_f = v_i + at$; (e) an identity to denote a mathematical property, for example, $a(b + c) = ab + ac$; and (f) a proposition-valued function, for example, $n^2 + 5 = (n + 5)^2$ is true only for n = -2. Fundamentally, an equation/inequality is a quantitative comparison between its two sides. However, this conception is not deeply understood by students. For example, some students have to re-solve an equation to determine if a solution is correct; they often do not know that an incorrect solution will yield different values for the two sides of the equation (Greeno, 1982, cited in Kieran, 1989).

Students' Difficulties with Inequalities and Equations

In a traditional algebra curriculum, a substantial amount of time is dedicated to learning and practicing the techniques for solving equations (linear equations, systems of linear equations, quadratic equations, and equations involving absolute values, exponentials, and logarithm functions). These standard equation-solving procedures have probably removed students' need to attend to the quantitative comparison between the two sides of the equation and the preservation of solution set of the equation. As such, many algebra students perceive an equation as a "do something signal" (Behr et al., 1976, cited in Kieran, 1992); a conception for which they have procedures but not necessarily meanings. "Most of the time algebraic formulae are for some pupils not more than mere strings of symbols to which certain well-defined procedures are routinely applied" (Sfard & Linchevski, 1994a, p. 223). Such interpretations promote algorithm-oriented behaviors. To promote sense-making and deductive reasoning, mathematics teachers should use tasks that require students to compare quantities and cannot be done by mindlessly manipulating symbols. The following are four examples of such tasks:

T1. Given that $a = b + 2$ is always true, which is larger, a or b? (Falkner, Levi, & Carpenter, 1999)

T2. Which is larger, $2n$ or $n + 2$? Explain. (Küchemann, 1981)

T3. Consider $(x + 1)(2k - 5) = 3(2k - 5)$. Is there a value for k that makes this equation true for all values of x?

T4. Consider these inequalities: $4x + 11 > x + 50$ and $4x + 22 > x + 50$. Can you find a value for x that will make one of them true and the other false?

Figure 1.2: Tasks that can promote comparison of quantities

In a traditional curriculum, inequalities are taught as an extension to equations in the sense that the procedures for solving equations are applicable to solving inequalities, with a few exceptions such as flip the sign when multiplying/dividing by a negative number, and consider intervals when solving inequalities involving polynomials. Some students are even taught to treat inequalities as equations. Under such instruction, students do not need to grapple with the meaning of a solution set. A challenge for educators is to get students to experience the need for determining the solution set of an inequality. Task T4 has the potential of helping students appreciate the need for solving inequalities because it is easier to determine a value for x by comparing solution sets, $x > 13$ versus $x > 9\frac{1}{3}$, than by comparing the original inequalities, $4x + 11 > x + 50$ and $4x + 22 > x + 50$.

Inequalities/Equations as an Effective Context for Studying Students' Mental Act of Anticipating

The use of unconventional problems involving inequalities and equations allows mathematics educators to gain information about a variety of problem-solving behaviors: (a) whether students predict prior to performing a procedure; (b) whether they explore via trial-and-error substitution or try different ways to manipulate symbols; (c) whether they reason deductively, inductively, or non-referential-symbolically; (d) whether they consider the structure of the equation/inequality (e.g., noticing that both sides of the equation in T3 are multiples of $2k - 5$); and (e) whether their symbol manipulation is goal-oriented or algorithm-oriented. In addition, educators can observe what students do in unfamiliar situations. Do they analyze the problem situation? Do they make predictions? Do they resort to familiar procedures, such as trial-and-error substitution? Do they try to apply and/or adapt their equation-solving algorithms when solving inequalities? The context of inequalities and equations allows me to investigate these types of questions.

Most of the tasks used in this study (see Appendix C and Appendix D) encourage students to use trial-and-error substitution when they are in doubt. According to Kieran (1988), students who have used the trial-and-error substitution method possess a more developed notion of equality or balance between the two sides of an equation than those who have not. The trial-and-error substitution method requires students to select appropriate values for substitution. The selection process may involve prediction. It also may involve retroactive anticipation in the sense that the outcome of one substitution affects the choice of what to substitute next. Hence, the domain of algebraic inequalities and equations is suitable for initial investigation of students' mental act of anticipating.

1.5 Organization of the Dissertation

In this chapter, I have demonstrated that students' problem-solving behaviors can be described in terms of the characteristics of their anticipation. I have also discussed the objectives of this study, the first of which is to identify and characterize students' anticipations. The theoretical framework within which this research is conducted and the research questions are discussed in Chapter 2. The research design of a two-part study that is used to answer those research questions is discussed in Chapter 3. The results of the study in relation to the three research questions are reported in Chapters 4, 5 and 6. The final chapter offers a summary of the major results, a discussion of instructional implications, and suggested avenues for future research.

CHAPTER 2: THEORETICAL FRAMEWORK

This chapter is organized into five sections. In the first section, the theoretical construct of anticipation is discussed from a Piagetian perspective. In the second and third sections, a theoretical framework for analyzing students' mental act of anticipating and the pedagogical principles guiding the teaching intervention are elaborated. The fourth section presents the research questions that guide this study. Issues related to the learning and teaching of algebra is discussed in the fifth section.

2.1 Anticipation: A Piagetian Perspective

In this section, several aspects of anticipation in the context of mathematics are discussed. Piaget's notion of anticipation is introduced by contrasting intellectual adaptation and physiological adaptations. Riegler's notion of anticipation is presented to highlight the reasons for adopting Piagetian's notion for this research. The relation between anticipation and knowing is discussed next. Following that, von Glasersfeld's (1998) elaboration of Piaget's notion of anticipation and Cobb's (1985) identification of three hierarchical levels of anticipation are discussed.

Piaget's Notion of Anticipation

For Piaget (1936/1952), "intelligence is a particular instance of biological adaptation" (p. 3-4). Piaget (1967/1971) drew parallels between intellectual adaptation and organic (i.e., physiological) adaptation; an essential characteristic of both is that they strive towards equilibrium. Organic adaptation refers to the readjustment of the organic or sensorimotor structures in response to pressures from the changing environment for survival. Intellectual adaptation refers to the reorganization of the conceptual structures to eliminate cognitive conflicts.

Piaget (1967/1971) highlighted two essential differences between the intellectual and organic adaptations: (a) intellectual structures are more conserving, that is, intellectual accommodation is somewhat permanent; and (b) "the second striking characteristic of intellectual accommodation is its capacity for anticipation" (p. 184); this means that our capacity for anticipation is much more pronounced in the realm of thoughts than in the domain of reflexes and sensorimotor actions. For Piaget, intelligence is not limited to observation of the immediate present and re-presentation of the past, it also applies to foresight into the future. This foresight or prediction is possible because of our ability to assimilate situations into our existing scheme(s); "anticipation is nothing other than a transfer or application of the scheme … to a new situation before it actually happens" (p. 195). A scheme, as outlined by von Glasersfeld (1995), involves three components: *the perceived situation*, *the activity*, and *the expected result*. The expected result component provides the anticipatory feature of a scheme. This component constitutes the fundamental difference between a Piagetian scheme and a condition-action pair in information processing or a stimulus-response association in behaviorism.

Piaget's Notion of Anticipation versus Riegler's Notion of Anticipation

Riegler (2001) offers an alternative conception for anticipation. He argues that anticipations are the result of internal canalizations, which are involuntary responses that bypass consciousness. His notion of canalizations was inspired by Waddington's (1957) epigenetic landscape whereby the epigenetic system―system in which embryonic development is canalized toward certain attractors by control genes which encode programs that interpret structural genes—can be visualized as a ball rolling down through a ramifying system of valleys. According to Riegler, "our thinking is canalized (or fixed) with respect to the way we have learned to deal with things … we implicitly anticipate that similar issues have similar causes, and thus similar solutions" (p. 535).

Riegler (2001) assumes that "the mind is populated by schemata which consist of merely two parts, a set of conditions and a sequence of actions" (p. 539). This assumption is consistent with the information processing perspective whereby an expert's knowledge is viewed as being conditionalized (Simon, 1980). For examples, a medical symptom is paired with possible courses of treatment, a chess configuration is linked to possible effective moves, and a symbolic form in algebraic expressions is linked to certain heuristics. With a repertoire of *chucks*, which are perceptual configurations that are familiar and recognizable, experts are can notice meaningful patterns and key features in a domain-specific situation more readily than novices (NRC, 2000).

I find Riegler's view of anticipation to be inappropriate for this research. His view implies that helping students improve their problem-solving ability would be limited to helping them expand their repertoire of knowledge. Piaget's notion of anticipation, on the other hand, involves expectation on the part of the problem solver. From this perspective, problem solvers are viewed as active players and problem solving is viewed as more exploratory than procedural, and occasionally serendipitous instead of always deterministic.

Anticipation as a Function of Knowing

"Anticipation … derives from a capacity for inference based on information previously acquired" (Piaget, 1967/1971, p. 185). This means that without information extracted from past experience, anticipation will not be possible. According to Piaget, there are two functions of knowing: a *conservation-of-information function* and an *anticipation function*. The conservation-of-information function is associated with memory, an instrument of which is a scheme. The anticipation function, on the other hand, deals with the application of a scheme to a new situation in which the individual conceives certain expectations prior to the unfolding of the events. The anticipation function will be discussed first, followed by the conservation-of-information function.

The anticipation function is found at every level of cognitive mechanisms (Piaget, 1967/1971). In the domain of sensorimotor actions, the conditioned reflex is anticipatory; for example, an infant anticipates to be fed upon being embraced in her or his mother's arm, or an adult anticipates the presence of another person when her or his shoulder is tapped. In the domain of perception, perceptual illusions presuppose anticipation. For example, a magician's floating object illusion presupposes the anticipation of the object falling when released in mid-air. Without an ability to anticipate what should be perceived, a person with mental deficiencies or a very young child will not find magic acts fascinating. In the domain of scientific thought, scientific investigation is essentially anticipatory. For example, anticipation is needed to organize an experiment so as to produce certain results that can strengthen or falsify the hypotheses related to the law under investigation. In the domain of mathematical thinking, formulation of conjectures presupposes anticipation. In the realm of imagination, anticipation allows us to engage in thought experiments that have yet to be, or can never actually be, realized. For example, Zeno's paradox of a person traversing half the remaining distance to a wall in every minute presents an interesting phenomenon that eventually gives rise to two notions of infinity: potential infinity and actual infinity. With respect to planning, anticipation is a fundamental component in *mental simulation*—the process of envisioning possibilities in the future and developing plans to bring about those possibilities (Taylor et al., 1998). The function of anticipation seems to extend into almost every aspect of cognition.

As for the conservation-of-information function, Piaget (1967/1971) identified two processes: (a) recognition—perception of an object or re-presentation of the object upon perceiving its index, and (b) evocation—re-construction of a prior experience in the absence of the object. "Evocation is something of a much higher order than recognition and presupposes a symbolic function (mental images or language) as well as the processes of inference and logical organization necessary for the mental reconstruction of the past" (p. 187). In terms of images, which are conceived by Piaget and Inhelder (1966/1971) as interiorized imitation of actions rather than as stored perception, recognition of an object in its absence can be viewed as the enactment of *reproductive images*. Evocation, on the other hand, can be viewed as the construction of *anticipatory images*. A reproductive image corresponds to the re-presentation of an object or an event that is already known. In contrast, an anticipatory image corresponds to the imagination of an event that results in a combination that has not been previously perceived. An example is visualizing, for the first time, the process of obtaining a rhombus-hole in the center of a paper by folding the paper twice to reduce it to a quarter of its original size, cutting a little triangle off its prime corner, and unfolding it back.

In discussing how anticipation affects children's action, Piaget and Inhelder (1966/1971) distinguish between *executional anticipation* and *evocational anticipation*. In executional anticipation, one foresees the gesture required to reproduce an event that is already known or is currently perceived: for example, a child dropping a rubber ball to see the bouncing effect again. In evocational anticipation, one foresees an event that is not already known; for example, a child anticipates a bouncing-on-the-water effect when he or she drops his or her rubber ball into the swimming pool for the first time. The distinction between executional anticipation and evocational anticipation may be adapted to differentiate between two types of anticipatory behaviors in mathematics. One type is the anticipation of using a known algorithm to solve a problem. For example, one anticipates the use of quadratic formula upon seeing $3x^2 - |x| + 2 = 0$. The second type is the anticipation of exploring an idea to solve a problem. For example, one explores $3x^2 - |x| + 2 = 0$ by comparing the terms and concludes accordingly: $3x^2$ dominates $|x|$ for $|x| > 1$ and 2 dominates $|x|$ for $|x| < 2$, so $3x^2 - |x| + 2$ is always positive. This distinction is particularly important in mathematics education because the former type promotes memorization while the latter promotes sense-making.

Solving problems in mathematics presupposes activation of schemes, or *anticipatory schemas*, which allow us to anticipate the consequence of an action prior to performing it. Bergson and Selz (cited in Piaget & Inhelder, 1948/1956) defined an anticipatory schema as one that provides an individual with an answer prior to "filling in" the details in the actual process of arriving at the answer. Piaget and Inhelder explained that an anticipatory schema is a 'grouping' of operations where the operations can be arranged in direct or reverse order. The ability to mentally arrange and rearrange operations in a different order is what makes a schema anticipatory. The authors described the genesis of anticipatory schemata as follows: the repetition of past successes in motor activity allows a child to anticipate her or his goal and form a schema[6]; the continuation of motor adaptation extends into imitation, interiorization of which constitutes images; the evocation of images provides the imaginal dimension in anticipation and reconstruction processes; these processes allow actions to be coordinated and become reversible and thereby constitute operations; the arrangement of these reversible operations constitutes an anticipatory schema.

Von Glasersfeld's Three Types of Anticipation

Von Glasersfeld (1998) elaborates on Piaget's notion of anticipation by identifying three general types of anticipation: (a) implicit expectations that are present in our actions, e.g., the preparation and control of our movements when we grope in the dark; (b) explicit expectation of an outcome based on certain cause-effect relationships (e.g., predicting that it will soon rain upon noticing that the sky is being covered by dark clouds); and (c) anticipation of a desired event and the means for attaining it (e.g., a child's foresight of the means to get his parent to give in, say by throwing a temper tantrum in public). In my attempt to apply von Glasersfeld's categories to problem-solving in mathematics, I encountered three aspects of anticipation: the regulatory aspect, the predictive aspect, and the volitive aspect (i.e., goal-related aspect).

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[6] Piaget (1970) differentiated schema and scheme based on a figurative-operative distinction: "the term scheme (plural: schemes) is used to refer to operational activities, whereas schema (plural: schemata) refers to the figurative aspects of thought—attempts to represent reality without attempting to transform it (imagery, perception, and memory)" (p. 705). "A schema is a simplified image (for example, the map of a town), whereas a scheme represents what can be repeated and generalized in an action (e.g., the scheme is what is common in the actions of 'pushing' an object with a stick or any other instrument)." (p. 719). A schema is the result of schematizing an image, a process in which some characteristics of the image are retained, some are distorted, while others are discarded (Piaget, 1970).

Regulatory aspect of anticipation

Piaget (1947/1950) conceived intelligence as a form of equilibrium, towards which all the cognitive structures that arise out of sensorimotor actions, imitations, and perceptions tend. According to Piaget (1975/1985), the process of intellectual development can be explicated in terms of the equilibration of cognitive structures.

Anticipation and retroaction are necessary for equilibration to occur. Inhelder and Piaget (1964/1969) found, in their studies on children's development of classification, that "the development from graphic structures[7] to operational structures depends on a complex interplay of retroactive and anticipatory activities" (p. 232).

> Hindsight, or 'retro-action', is the process whereby a subject is led to revise his earlier actions in the light of those that have followed: he goes back on his moves, or corrects his mistakes. Foresight or 'anticipation' is the process of internally *carrying out* [italics added] actions which will not be actually *performed* [italics added] until a later stage, and thereby modifying the action that is in fact carried out in the present. (*ibid*, p. xix, translators' notes)

Initially, retroaction and anticipation arise as a result of growing coordination between successive actions in the course of a child's ongoing exploration. At this stage, the child's foresight is local. The interplay between anticipation and retroaction provides the regulation that sooner or later results in a state of equilibrium. With this equilibrium, the child begins to foresee more globally. Eventually, the child can anticipate the several phases necessary for the complete classification of a set of objects. In summary, the continuous development of reversible operations (e.g., classification, cross-classification,

[7] Graphic structures refer to the figural aspects of thought (i.e., imagery). Piaget and Inhelder (1966/1971) maintained that graphic structures are essentially irreversible. Inhelder and Piaget (1964/1969) asserted that operational structures which are reversible do not derive directly from imagery, instead they involve reflective abstraction.

seriation) based on elementary actions (e.g., putting things into piles, separating piles into lots, making alignments) necessarily involves regulatory processes of retroaction and anticipation.

In this research, I found that anticipations which are regulatory in nature occur at a rudimentary level that can rarely be inferred from students' actions and statements. Nevertheless, an awareness of this aspect of anticipation allows us to appreciate the complexity of students' reasoning. The idiosyncrasy we perceive in students' reasoning is an indication of our lack of understanding of the regulatory processes that occur in their minds, to which we have no access.

Predictive aspect of anticipation

From a Piagetian perspective, the act of predicting may be conceptualized as transforming anticipatory images. Anticipatory images and operations are coupled in that anticipatory images correspond to operations and facilitate the functioning of operations (Piaget, 1970; Piaget & Inhelder, 1966/1971). However, anticipatory images and operations differ in that "the operations carry out the transformations; the image represents them" (Piaget & Inhelder, 1966/1971, p. 228). Anticipatory images, being figurative,[8] are subordinate to operations; their role is to imitate rather than to construct. As "figural signifiers" (p. 383), anticipatory images can undergo transformations that demand much less cognitive load as compared to the execution of the operations, or the *signifieds*. Because they contain fewer details, images are generally more compact and

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[8] As opposed to operative which concerns an attempt to transform *reality*, figurative concerns an attempt to represent reality as it appears without transforming it (Piaget, 1970).

versatile than the operations they signify. The distinction between anticipatory images and operations is helpful for contrasting predicting and performing.

Piaget and Inhelder (1948/1956) have identified, in a study of students' conception of projective space, three distinct types of images: (a) a static image of an object without consideration for possible transformations even when movements are perceived; (b) an image that expresses a phase of an action performed on the object, but the image is unable to keep pace with the action because the image constitutes an imitation of the action itself; and (c) a dynamic image that is capable of anticipating the results of yet-to-be-performed actions because it depicts the coordination of, rather than an imitation of, the actions.

Thompson (1996) connects three levels of coordination of actions to the above three types of images: (a) coordination is absent in the first type of images, (b) actions are not well-coordinated in the second type, and (c) actions are well-coordinated in the third type. Thompson (1994a; 1994b) outlines three stages of development of images of rate: (a) image of change in some quantity, say displacement of position or rise in water level; (b) loosely coordinated image of two quantities, say distance traveled and time taken; and (c) a dynamic image of covariation of two quantities whose measures are in constant ratio.

The following example illustrates how these three levels of coordination can be useful in characterizing students' predictions. Consider the following problem: Given that $2/a = b/2$, which is larger: a or b? A student at the first level may conceive the image of $2/2 = 2/2$ and conclude that $a = b = 2$, or perceive proper fractions and conclude (association-based) that $a > 2$ and $b = 1$ without attending to the equality between the two sides. A student at the second level may plug in numbers, say 1 for *a* and 4 for *b*, that make $2/a = b/2$ true, and infer inductively based on numbers that work. A student at the third level will be able to coordinate the changes in values for *a* and *b* by reasoning (coordination-based), say with $2/2 = 2/2$ as the starting point, that as *a* increases *b* decreases, and vice versa.

Volitive aspect of anticipation

According to von Glasersfeld (1998), will is involved in the third type of anticipation—foresight of a desired event or goal and the means for attaining it. The notion of goal appears in Piaget's (1936/1952) discussion of infants' coordination between means and ends, a stage in the child's development of sensorimotor intelligence. According to Piaget, this stage marks the emergence of intelligence, whose two characterizing elements are now present: the differentiation of goals and the coordination of schemes to attain a goal.

In general, our cognitive actions and operations are goal-driven or volitive, although the goal may be implicit. At the most basic level, the goal is to seek cognitive equilibrium. At a higher level of consciousness, the goal could be a solution to a problem, an explanation for a phenomenon, or a good grade in an examination. To attain one's goal, one has to take physical and/or mental actions. As a volitive process, anticipation allows one to disregard unrelated actions, focus on viable ones, and choose certain actions to attain one's goal.

In problem solving, foresight of action is related to strategy selection, planning, and control. Most literature on planning and control involves metacognition (see Schoenfeld, 1987, 1992; Rickey & Stacy, 2000). The literature that incorporates anticipation into planning and control tends to be in artificial intelligence (AI). The underlying strategy in AI models of cognition is mean-ends analysis (Newell & Simon, 1972). The means-ends analysis refers to the process in which one determines the differences between the problem states and the goal state, and looks for ways to eliminate those differences. It allows us to attain a goal rapidly, with fewer irrelevant moves and minimal excursions into dead ends (Sweller, 1989). According to Sweller, when schemas are available for solving the problem, one uses a working-forward strategy; otherwise one uses a working-backward strategy (i.e, means-ends analysis).

In the domain of algebra, the standard procedures for solving linear equations of the form $ax + b = cx + d$ are essentially based on the application of means-ends analysis (Kieran, 1989). According to Simon (1980), one identifies the desired form of an algebraic expression, detects a difference between the current expression and the desired form, determines an appropriate algebraic transformation, and examines if the transformed expression has the desired form. Anticipation can provide a form towards which transformations are directed (Boero, 2001; Steiner, 1994).

> In order to direct the transformation in an efficient way, the subject needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This 'anticipation' allows planning and continuous feed-back. (p. 99)

Consider solving $(x – 2)^2 = (x – 2)(x – 5)$ as an example. A student who anticipates the usefulness of the quadratic formula will manipulate symbols to obtain the standard form $ax^2 + bx + c = 0$, say by expanding the factors, moving everything to one side, and simplifying. A student who anticipates the usefulness of factored form may notice the common factor $x – 2$ on both sides and manipulate symbols towards the factored form

$(x - r_1)(x - r_2) = 0$, say by moving everything to one side and factoring out the common factor $x - 2$. Steiner (1994) analyzed the cognitive requirements for factoring trinomials and found that it was important to foresee the transformation before initiating the factoring algorithm: "anticipations are, thus, the core processes in handling transformations" (p. 253). He claims that "good teaching helps [a] student to generate predictions, hypotheses, or anticipations" (p. 253) and to test them.

To recapitulate, I have identified three aspects of anticipation that correspond to von Glasersfeld three general types of anticipation. The first type of anticipation, which has a regulatory function, is extremely difficult to infer from students' actions and statements. The second type, which is predictive in nature, can be inferred from the students' stated predictions. The third type, which is volitive in nature, can be inferred from the actions students take to solve a problem. In this research, I only examine the latter two types: the act of predicting results and the act of foreseeing actions. I define them as follows:

- *Predicting (a result) is the mental act of conceiving an expectation for the result of an event without actually performing the operations associated with the event*.
- *Foreseeing (an action) is the mental act of conceiving an expectation that leads to the volition for an action, prior to performing the operations associated with the action*.

From a Piagetian perspective one's anticipation depends on the scheme(s) that one evokes. By definition, schemes are anticipatory in nature. So I use the phrase *anticipatory scheme* to refer to the scheme that governs one's act of anticipating and *predictive scheme* to refer to the scheme that governs one's act of predicting.

Cobb's Three Hierarchical Levels of Anticipation

Cobb (1985) suggests viewing children's mathematical problem-solving as an expression of anticipations. According to Schoenfeld's framework, four categories of cognition (content knowledge, heuristics, metacognition, and beliefs) influence problem-solving. Cobb identifies three hierarchical levels of anticipation, which seem to correspond to three of the four categories of knowledge, namely beliefs, heuristics, and content knowledge. At the global level, students' beliefs about mathematics influence their anticipation of a certain kind of activity or "practice". In classroom situations, such a practice is termed *sociomathematical norm* (Cobb & Yackel, 1996). An example of anticipating a norm is when a student thinks that he must write down every step of the solution even when a step is trivial. A contrasting example is when a student capitalizes on previous results and skips trivial steps. At the intermediate level, a child anticipates using a heuristic for solving a problem. "A heuristic can be viewed as a metacognitive prompt which delimits a subcontext within which the child anticipates she can elaborate and solve the problem" (Cobb, 1985, p. 124). For example, the anticipation of a guess-and-check strategy may result in a student operating in the sub-context of plugging in numbers. At the most specific level, children's expressed conceptual structures dictate their anticipations within the heuristically constrained sub-context.

According to Cobb (1985), higher-level anticipations constrain lower-level anticipations. Nevertheless, anticipation does not necessarily occur in a top-down fashion. For example, one must interpret the problem before applying a heuristic. This initial interpretation is then elaborated and refined within the resulting sub-context. Cobb contends that there is a dialectical relationship between the reorganization/refinement of a conceptual structure and the activity of expressing it. While conceptual structure delimits anticipations and thereby constrains problem-solving activities, it can be restructured as one reflects on the activity, and it can be abandoned when one switches to another sub-context.

Cifarelli (1989, 1998) builds on Cobb's work by focusing on the connection between anticipation and conceptual structures. In his study (1998) on students' construction of mental representations, he found a gradual buildup of conceptual structure as students progressed through solving a series of related word problems. He views these structures as "purposeful organizations of the solvers' prior solution activity" (p. 259) that can guide subsequent solution activity by enabling solvers to anticipate while interpreting a new situation. In the initial task, students had to perform the solution activity. As students progressed, three levels of solution activity were observed. These levels correspond to three levels of conceptual structure: (a) *recognition*, at which students could identify the similarity between the new task and the previous task; (b) *re-presentation*, at which students could mentally 'run through' prior activity and use it to anticipate potential difficulties; and (c) *structural abstraction*, at which students could mentally "run through" potential solution activity and draw inferences without performing the solution activity. According to this hierarchy, the level of one's conceptual structure dictates one's anticipation.

In this research, I focused on the relation between students' anticipation and students' interpretation of inequalities and equations. I also attended to the relation between students' anticipation and the sub-context in which they operate. However, I did not attend to the relation between their beliefs about mathematics and their anticipation.

2.2 Framework for Analyzing Students' Mental Act of Anticipating

As stated in Chapter 1, the goal of this research is to explore the feasibility and usefulness of focusing on students' anticipation as means to study the ways students solve problems. However, students' anticipations are mental processes which can only be inferred from their actions and statements. To study and characterize students' anticipation (foresight of action and prediction of result), I employ Harel's notions of *mental act*, *way of understanding*, and *way of thinking* in his *DNR framework*[9] (in press a, in press c, 2001).

Mental Acts

In this research, students' anticipations are conceived as *mental acts*, which constitute humans' reasoning: "mental acts are basic elements of human cognition. To describe, analyze, and communicate about humans' intellectual activities, one must attend to their mental acts" (in press c). He provides the following as examples of mental acts: interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem solving.

In this framework, the analysis of a particular mental act involves identifying a product of the act, which is termed a *way of understanding* associated with the act, and inferring a characteristic of the act, which is termed a *way of thinking* associated with the act.

> Mental acts can be studied by observing peoples' statements and actions. A person's statements and actions are products of her or his mental acts; they represent the person's ways of understanding associated with those mental acts. Repeated observations of one's

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[9] DNR is an acronym for three pedagogical principles, namely the *Duality Principle*, the *Necessity Principle*, and the *Repeated-reasoning Principle*.

> ways of understanding associated with a given mental act may reveal certain characteristics—persistent features—of the act. These characteristics are referred to as ways of thinking associated with that act. (Harel, in press c).

Ways of Understanding

Harel (in press c) defines a way of understanding as “a particular product of a mental act carried out by an individual.” Hence, a way of understanding must be associated with a mental act. For the mental act of interpreting, what the student actually interprets is a way of understanding. For example, one student may interpret the equation $3x + 7 = 14$ as a signal to isolate x, while another student may interpret it as a constraint on the value x can assume. These students display two ways of understanding associated with the act of interpreting the equation $3x + 7 = 14$: equation-as-a-signal-to-isolate-a-variable and equation-as-a-constraint.

Likewise, the proof a student produces for an assertion is a way of understanding associated with the mental act of proving, and the solution a student produces for a problem is a way of understanding associated with the mental act of problem-solving. Regarding the act of foreseeing an action, the action a student performs, or says he or she will perform, constitutes a way of understanding. Regarding the act of predicting, the prediction one makes is the way of understanding.

Ways of Thinking

Harel (in press c) defines a way of thinking as “a characteristic of a mental act. Such a characteristic is always inferred from observations of ways of understanding.” If the student interprets $3x + 7 = 14$ as a signal to isolate x without attending to the referent of x, then the student’s mental act can be characterized as devoid of quantitative referent, in contrast to a view in which x could represent a quantity such as weight of an marble or a numerical value that makes the equation true. Harel (in press c) calls the former behavior the *non-referential symbolic* way of thinking, which he defines as "the behavior of operating on symbols as if they possess a life of their own, not as representations of entities in a coherent reality"

Examples of ways of thinking associated with the mental act of proving are the *authoritative proof scheme*, in which one derives conviction mainly from the authority of the teacher or textbook; the *empirical proof scheme*, in which one derives conviction from empirical evidence or visual perceptions; and the *deductive proof scheme*, in which one derives conviction based on the application of rules of logic (Harel & Sowder, 1998). Ways of thinking associated with the mental act of problem solving are problem-solving approaches, examples of which include a backward-strategy approach involving means-ends analysis, a forward-strategy approach involving a straightforward application of a procedure, a look-for-keyword strategy, and a look-for-a-simpler-problem strategy.

This research aims to identify categories of ways of thinking associated with the mental act of foreseeing and ways of thinking associated with the mental act of predicting. For instance, consider the following response of an 11th grader, Nick, in my pilot study. Having ascertained that $x > 4$ made $3.14(6x - 24) > 2.4(6x - 24)$ true by finding the critical value to be 4, Nick found the critical value of $2(2x - 6) < 7(2x - 6)$ to be 3, and predicted that it would be true for $x < 3$: "my critical point here is 3, so in this case I need to be less than. So I put in $x < 3$." His prediction is characterized as association-based way of thinking because he presumably associated the "<" in the inequality with the "<" in his solution. His prediction was probably a consequence of his non-referential symbolic way of thinking in his act of interpreting the inequality. Once he attended to the quantitative comparison between the two sides of the inequality, he caught his mistake and inferred that $x > 3$: "oh crap … in here ($2x - 6$) I [should] put a positive number because I see a 7 is greater than a 2." His way of thinking associated with inferring is considered coordination-based because he coordinated $x > 3$ with the factor $2x - 6$ being positive, a condition that would make the inequality true. This example suggests that the association-based way of thinking and the non-referential symbolic way of thinking are related, as are the coordination-based way of thinking and the *referential symbolic* way of thinking.

Reasons for Using these Constructs

The three constructs—mental act, way of understanding, and way of thinking—form a triad, as depicted in Figure 2.1. The MA-WoU-WoT triad serves two purposes: one for research and one for teaching and learning. For research, the MA-WoU-WoT triad provides a researcher with a means to analyze, based on students' actions and statements, what a student *understands* of a particular thing or phenomenon (i.e., what the product of the student's mental act is), and subsequently infer the manner (i.e., character of the act) in which the student engages a particular mental act to arrive at that way of understanding. By focusing on one mental act at a time, though with the awareness that mental acts do not occur in isolation, a researcher can breakdown a student's thinking into manageable components. Having analyzed a few mental acts that are related to one another, the researcher can construct a more complete picture of the student's thinking. This divide-and-conquer strategy enables researchers to manage the complexity in students' thinking or problem solving. One advantage of this approach is that the analysis is extremely fine-grained. The accompanying drawback is that this mode of analysis is time-consuming.

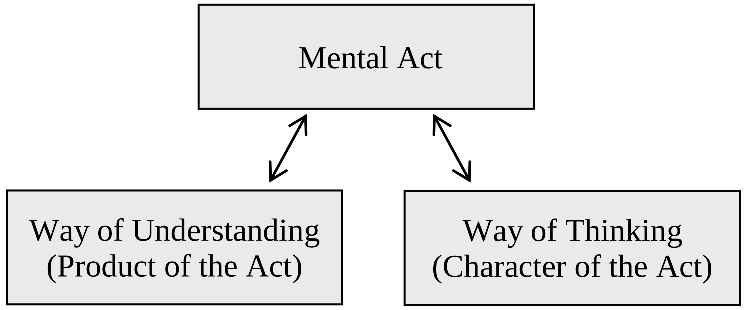

Figure 2.1: Harel's MA-WoU-WoT Triad

The term "way of understanding" is neutral in the sense that it merely a product of a mental act and does not suggest what a student understands or does not understand. Nevertheless, a student's way of understanding may be desirable or undesirable in relation to those that have been accepted by the mathematical community at large.

For teaching and learning purposes, students' ways of understanding and ways of thinking can be used by mathematics teachers as descriptors of students' mathematical knowledge. Helping students' to advance their mathematical knowledge can be conceived as helping students to progress from less desirable ways of understanding and ways of thinking to more desirable ones. Harel (in press c) proposes that teachers "must attempt to identify students' current ways of understanding and ways of thinking, regardless of their quality, and help students gradually refine and modify them toward those that have been institutionalized—those the mathematics community at large accepts as correct and useful."

This research aims to identify students' ways of thinking associated with foreseeing and predicting as well as their ways of understanding associated with interpreting algebraic inequalities and equations. When these ways of thinking and ways of understanding are made explicit, teachers can be more sensitive to their students' ways of thinking and ways of understanding. They can then design instructional tasks to help their students overcome erroneous ways of understanding or deficient ways of thinking and progress toward desirable ways of understanding and ways of thinking.

2.3 DNR-Based Instruction for the Teaching Intervention

Harel's *DNR-based instruction in mathematics* (in press a, 2001) is a theoretical perspective that provides pedagogical principles for helping students advance their ways of understanding and ways of thinking.

> DNR-based instruction in mathematics stipulates the conditions for achieving critical goals such as provoking students' intellectual need to learn mathematics, helping them acquire mathematical ways of understanding and ways of thinking, and assuring that they internalize and retain the mathematics they learn. (Harel, in press a)

The DNR framework stipulates three foundational principles: the *Duality Principle*, the *Necessity Principle*, and the *Repeated-reasoning Principle*, hence the acronym DNR.

The Duality Principle. Harel (in press a) asserts that "students develop ways of thinking only through the construction of ways of understanding, and the ways of understanding they produce are determined by the ways of thinking they possess." This principle underscores the importance of incorporating complementary ways of understanding and ways of thinking into cognitive objectives for instruction, which should help students reason independently, be in control, and "create" mathematics. However, many mathematics teachers focus on imparting ways of understanding such as definitions, rules, algorithms, solutions, theorems, and proofs, without attempting to help students develop desirable ways of thinking. Some teachers may teach certain ways of thinking (e.g., drawing-a-diagram, checking-your-answer, looking-for-pattern) directly to students, which is generally ineffective.

> We have observed that teachers often form, at least implicitly, cognitive objectives in terms of ways of thinking, but their efforts to teach ways of thinking is often counterproductive because these efforts do not build on ways of understanding. Conversely, teachers often focus on ways of understanding but overlook the goal of helping students abstract effective ways of thinking from these ways of understanding. (Harel and Sowder, 2005, p. 3)

Implementing the Duality Principle involves (a) attending to students existing ways of understanding and ways of thinking; (b) identifying appropriate cognitive objectives, appropriate in the sense that they are aligned with students' current ways of understanding and ways of thinking, and that they preserve the mathematical integrity of the content; and (c) designing activities, with an understanding of the interplay among various ways of understanding and ways of thinking, to meet those objectives.

The Necessity Principle. The Necessity Principle stipulates that "for students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need, not social or economic need" (Harel, in press c, p. 501). Intellectual need refers to students' intrinsic desire to resolve a cognitive conflict, which may arise from the incompatibility between their existing knowledge and the problem situation, or from their inability to solve the problem with their existing knowledge. The resolution of the conflict can potentially lead students to advance their ways of understanding and ways of thinking. Hence, ways of understanding and ways of thinking should emerge out of a need to solve a problem rather than being taught directly to students.

> The implementation of the necessity principle involves (a) recognizing what constitutes an intellectual need for a particular population of students relative to the concept to be learned; (b) developing a system of problem situations that correspond to their intellectual need, and from whose solutions the concept can be elicited; and (c) creating an instructional environment in which the student can elicit the concept through engagement with the system. (Harel, 2001, p. 208).

The *Repeated-reasoning Principle*. Repeated reasoning is not drill and practice of routine problems, but rather requires providing opportunities for students to repeat the reasoning that is useful for solving a set of seemingly different problems. The Repeated-reasoning Principle asserts that "students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking" (Harel, in press c, p. 209). Internalizing a piece of knowledge means being able to apply it autonomously and spontaneously in a variety of situations.

The implementation of the Repeated-reasoning Principle involves (a) sequencing problems that can potentially lead students to abstract a certain way of understanding (e.g., completing-the-square algorithm) or a way of thinking (e.g., changing-the-form-without-changing-the-value-to-attain-a-certain-form); (b) incorporating problems that allow students to apply and/or adapt their newly learned ways of understanding and ways of thinking, and to realize the affordances and limitations of those ways of understanding and ways of thinking; and (c) assigning adequate amounts of homework that require students to repeat their reasoning and thus retain those ways of understanding and ways of thinking.

2.4 Research Questions

Recall from Chapter 1 that the objectives of this research are (a) to identify and characterize students' anticipations (foresights and predictions) as they solve problems in the domain of algebraic inequalities and equations, (b) to study the relation between students' anticipations and their interpretations of inequalities and equations, and (c) to explore the plausibility of improving the way students anticipate via a short-term one-on-one teaching intervention. Corresponding to these objectives are the following three research questions.

1. What are students' ways of thinking associated with the mental acts of foreseeing and predicting when they solve problems involving algebraic inequalities/equations? Are these ways of thinking related to the quality of their solutions?
2. What relationships exist between students' ways of thinking associated with foreseeing/predicting and their ways of understanding inequalities and equations?
3. What is the potential for advancing students' ways of thinking through an instructional intervention informed by DNR-based instruction? What factors can contribute to students' improvement?

As mentioned earlier, Cobb (1985) contends that one's anticipations are embodied in one's expressed conceptual structures. An expressed conceptual structure can be viewed as a way of understanding associated with the mental act of interpreting. In the domain of algebraic inequalities and equations, the association between conceptual structures and anticipations suggests a relationship between students' ways of understanding inequalities/equations and their ways of thinking associated with the mental act of anticipating (foreseeing or predicting). This relationship is depicted by a dotted curve in Figure 1, which is a schematic representation of the framework for analyzing students' act of problem-solving in terms of mental acts of predicting, foreseeing, and interpreting.

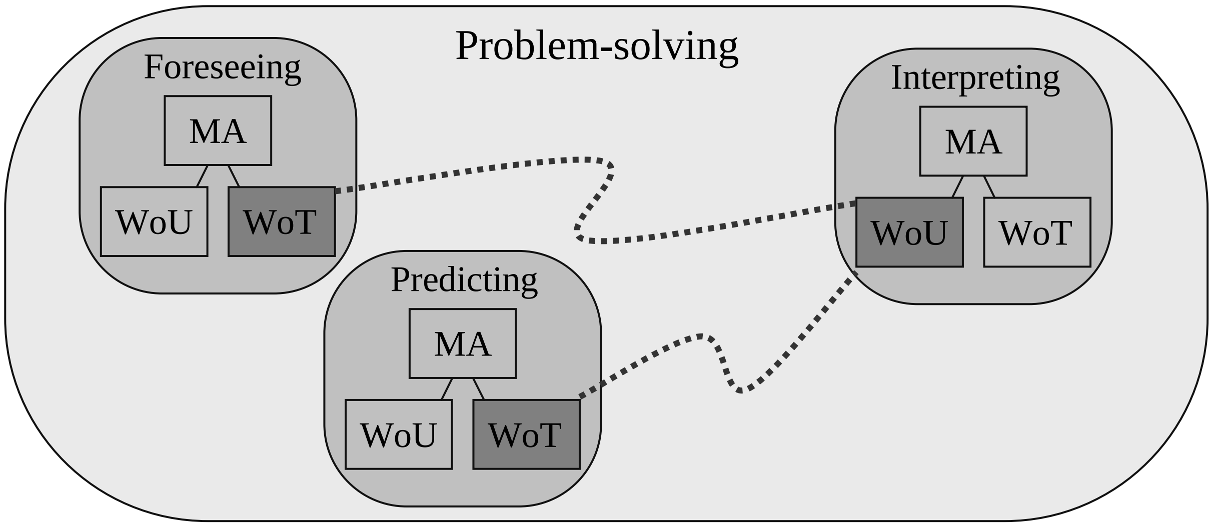

Figure 2.2: A schematic representation of the framework for this research

2.5 The Learning of Elementary Algebra

Since this research is conducted in the domain of algebraic inequalities and equations, it is appropriate to discuss issues related to the learning of algebra, focusing on equivalence and structure sense. The first sub-section presents various aspects of algebra. The second sub-section focuses on two approaches to the learning and teaching of algebra. The third sub-section discusses the notion of structure sense. The fourth sub-section discusses introduces the notion of process-object duality. The fifth sub-section presents some of the challenges algebra students face. The final sub-section highlights some desirable ways of understanding and ways of thinking for instruction regarding algebraic inequalities and equations.

Conceptualizations of Algebra

The power of algebra lies in its symbolism which took more than 1300 years to develop. By leaving out contextual details, algebra provides us a way to solve certain problems efficiently, especially those that involve functions. An essential characteristic of algebraic reasoning is the possession of control over the symbols (Balacheff, 2001); that is, one is able to manipulate the symbols in a goal-oriented and anticipatory manner, rather than in a directionless and haphazard manner. However, most students lack this control when they are solving problems in algebra (Sfard & Linchevski, 1994a; Vinner, 1997).

For students to have control over the symbols with which they work, the symbols must be meaningful to them. Symbols become meaningful only when they are associated with referents, which are usually numbers or quantities. Thus, a cognitive objective for instruction should be to help algebra students to develop a *referential symbolic* way of thinking. Related to this objective are attempts to introduce algebra as a means to generalize numerical and geometrical patterns (Kaput, 1999; Carpenter & Franke, 2001), or as a means to solve realistic problems that involve two related quantities in functional situations (Kieran et al., 1996; Chazan, 2000). The use of variables in each approach is rather different: variables as pattern-generalizers in the former and variables as quantity-referents in the latter. Usiskin (1988) presents four conceptions of algebra and highlights for each conception the way variables are used and the key instructions associated with the use of variables. These are summarized in the table below.

Table 2.1: Usiskin's (1988) Four Conceptions of Algebra

Conception of Algebra	**Use of Variables**	**Key Instructions in the Use of Variable**
Algebra as the study of procedures for solving certain types of problems	Variables as unknowns and constants, e.g., $5x + 3 = 40$, $(x - 2)^2 = 25$	Simplify, solve
Algebra as generalized arithmetic	Variables as pattern generalizers, e.g., $-x \times y = -xy$, $n \times 1/n = 1$	Translate, generalize
Algebra as the study of relationships among quantities, including functions.	Variables as quantities whose value varies, e.g., $V = \pi r^2 h$. Variables as arguments and names for functions, e.g., $P(n) = 1000 \times 1.05^n$	Relate, formulate, graph
Algebra as the study of structures	Variable as symbols for manipulation, e.g., prove $\sum_{k=1}^{n} 2k - 1 = n^2$	Manipulate, prove

Algebra as the study of procedures for solving certain types of problems dominates the other conceptions in a traditional curriculum for algebra. School algebra has traditionally been taught as a set of procedures for solving word problems and rules for manipulating symbols. The emphasis is on syntactic and procedural aspects at the expense of semantic and structural aspects of algebra (Kieran, 1992). This tends to promote the belief that algebra is a subject devoid of meaning or is merely a collection of "meaningless" rules and procedures to follow. Such a belief is undesirable because it promotes a non-referential symbolic way of thinking.

To address this issue, reform efforts have advocated referential approaches. These approaches include: (a) generalization of numerical and geometrical patterns and generalization about fundamental properties of arithmetic (Kaput, 1999; Carpenter & Franke, 2001); (b) a functional and graphical approach with use of technology (Kieran et al., 1996; Yerushalmy & Schwartz, 1993); (c) modeling of physical and mathematical phenomena (e.g., Realistic Mathematics Education, see Freudenthal, 1991); and (d) use of concrete models like the balance (Filloy and Rojano, 1989; Linchevski & Herscovics, 1996; Vlassis, 2002), the geometrical model (Filloy and Rojano, 1989), and the arithmagon (Pirie & Martin, 1997).

Referential Approach and Structural Approach

On the one hand, algebra initially derives its meaning from the referent it represents. On the other hand, it derives its power by freeing itself from the referent. In Kieran et al.'s (1996) study on using a technology-supported, functional approach to introduce algebra, students were found to be reluctant to simplify expressions that were tied to a specific context because they wanted to preserve recognizable links between the symbols and the contextual situation. Resnick (1986) argues that the referential aspect of algebra is essential for beginning algebra students, but a divorce from referents is necessary for complete mastery of algebra.

Kirshner (2001) distinguishes between the structural approach and the referential approach: "The structural approach builds meaning internally from the connections generated within a syntactically constructed system. Referential approaches import meaning into the symbol system from external domains of reference" (p. 84). Balacheff (2001) introduces the term *symbolic arithmetic* to highlight that in referential approaches the control over the solving process is provided by the external domain.

> Algebra is not there, but instead we see the functioning of what I would call symbolic arithmetic which has its own rules and domain of validity. In some way I would say that symbolic arithmetic is to algebra what quantities are to numbers. Both may use the same representation system, and even common tools, but they are not

> submitted to the same control system (this phenomenon is well illustrated by studies such as that of the carpenter and the apprentice by Nunes, 1993). Without making this explicit, teaching practices tend to turn symbolic arithmetic into *de facto* content to be taught instead of algebra – a choice often made by ‘realistic mathematics educator’. (p. 255)

Both Balacheff (2001) and Kirshner (2001) doubt that an agenda of contextually rich applications could help students develop deductive rigor in their algebraic reasoning. According to Balacheff, a good command of algebra involves mastery over its representation systems, which include operators for manipulating symbols and meta-rules for monitoring, strategizing, and decision-making. This implies that the control one has should reside in the algebraic world of symbols and not in the concrete world of referents. Such control is necessary for doing algebraic proofs. Since symbolic arithmetic provides only pragmatic control, doing proofs in algebra, according to Balacheff, will be outside the realm of symbolic arithmetic.

The tension between referential approaches and structural approaches is evident in Carolyn Kieran’s presentation at the 10th International Congress on Mathematical Education (ICME) 2004 conference. She raises three open questions pertaining to research in the teaching and learning of algebra, two of which incorporate the tension: (a) to what extent do instructional approaches that provide students with pragmatic, referential control of solution processes prevent students from developing the theoretical control that is afforded by syntactic algebra? and (b) to what extent does integrating a functional approach and a non-functional equation-based approach exacerbate students’ difficulties in making sense of algebra? These questions reflect an ongoing challenge in integrating the referential aspect and the formal aspect of algebra.

Integrating the two aspects of algebra would involve mastering the dual role of mathematical symbols, namely as signifier and as signified. According to Resnick (1986), algebraic expressions derive their meanings from two sources: (a) the referents they represent, and (b) the formal system which consists of definitions, principles, and rules. “A meaningful expression is one that is legal within the formal system, and the application of the correct transformation rules insures that all expressions that are generated will be legal” (p. 190-191). Resnick hypothesizes that good mathematics learners take the time and effort to make sense of the formal rules and link them to their intuitive knowledge. She believes that understanding the nature of the links between formal representations and intuitive knowledge can expand our understanding of how cognitive development proceeds. Investigating students’ anticipation is probably a viable avenue to study the process in which students link the structural aspect of algebra to their arithmetic-based intuition. Inequalities and equations provide an excellent domain for such studies.

Symbol Sense and Structure Sense

Having a good command of algebra requires *symbol sense*. Symbol sense in algebra is analogous to number sense in arithmetic. According to Arcavi (1994), “having symbol sense should include the intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools" (1994, p. 26). A concept closely related to symbol sense is what Hoch and Dreyfus (2004) call structure sense.

> Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to: see an algebraic expression or sentence as an entity, recognize an algebraic

> expression or sentence as a previously met structure, divide an entity into sub-structures, recognize mutual connections between structures, recognize which manipulations it is possible to perform, and recognize which manipulations it is useful to perform. (p. 51)

In their conference paper presentation, Hoch and Dreyfus propose three hierarchical levels of abilities associated with structure sense: (a) ability to deal with a complex literal term as a single entity; (b) ability to recognize equivalent and identical structures; and (c) ability to choose appropriate manipulations to make best use of the structure. These abilities are considered desirable ways of thinking associated with the mental act of symbol-transforming.

Associated with the mental act of interpreting a function as a conceptual entity is the *capitalizing-on-structure* way of thinking. For example, a student who recognizes the structure of a quadratic equation may think of using the substitution $x = 2r + 1$ to solve $2(2r + 1)^2 + 3(2r + 1) – 9 = 0$. However, most students do not use structure sense to solve problems. In a study on the effects of parentheses on students' use of structure sense, Hoch and Dreyfus found that very few students used structure sense, and that the presence of parentheses did help some students to attend to structure. For example, none of the 31 eleventh-graders in the study solved $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$ using structure sense (i.e., by noticing that the terms on the left side cancel out) but five out of 29 eleventh-graders solved $\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n+3}\right)=\frac{1}{132}$ using structure sense.

The capitalizing-on-structure way of thinking hinges on the ability to recognize equivalent and identical structures, which in turn depends on having a profound way of understanding equivalence. For example, consider the following three equivalent equations: $2(x-3)^2-32=0$, $2(x+1)(x-7)=0$, and $2x^2-12x-14=0$. Some students view equivalence in terms of transformability (Sfard & Linchevski, 1994b); that is, two forms are equivalent if one form can be transformed into the other. More advanced students view equivalence in terms of having the same solution set. Using Frege's (1892, cited in Arzarello et al., 1993) notions of *sense* and *denotation*, we can say that the three equations provide three different senses, each highlighting a certain feature: vertex, x-intercepts, and y-intercept respectively. However, they all have the same denotation: the set {-1, 7}. According to Bazzini, Boero and Garuti (2001), algebraic transformation involves activation of senses: “doing algebra means interpreting expressions and relating them with senses, coherently with the given denotation” (p. 122). However, many students do not have the *sense* necessary for manipulating symbolic expressions. This is probably due to their lack of an *equivalent-equations-have-same-solution-set* way of understanding.

According to Hoch and Dreyfus (2004), structure sense requires the ability to recognize and capitalize on equivalent structures. Mastery over this ability presupposes versatility in interpreting something (e.g., a function) as a process and an object, and a profound understanding of the notion of equivalence.

Process-Object Duality

According to Sfard (1991), a mathematical conception has a dual nature. It can be conceived operationally as a process or structurally as an object. The processes of learning and problem-solving involve an intricate interplay between the two conceptions. “The structural approach invites contemplation; the operational approach invites action; the structural approach generates insight; the operational approach generates results”

(p. 28, Sfard paraphrasing Henrici, 1974). Sfard theorizes, based on Piaget's scheme theory and on historical analysis of mathematical concepts, that the operational conception is developed prior to the structural conception during the acquisition of a mathematical notion. For example, one has to first conceptualize *potential infinity*, which is conceived as a never ending process, prior to conceptualizing *actual infinity*, which is conceived metaphorically as a "result of a process without end" (Lakoff & Núñez , 2000, p. 158). A student is said to truly understand the meaning of infinity only when he or she has encapsulated (Dubinsky, 1991) or reified (Sfard, 1991) the on-going process (potential infinity) as an object (actual infinity, which is symbolized by ¥) on which he or she can operate. Sfard contends that even if a new concept is introduced to a student structurally, the student will have to interpret it operationally in order to make sense of it. One cognitive objective for instruction should be to help students develop an *acting it out* way of thinking; that is, acting it out operationally to make sense of something abstract.

Sfard (1991) proposes a theory that accounts for the development of a structural conception from its operational conception. It involves three phases: (a) interiorization, which is marked by the ability to carry out a process via mental operations without explicitly performing its operations; (b) condensation, which refers to reduction of a complicated sequence of operations into an entity of manageable units; this is marked by the ability to think of a "process as a whole, without feeling an urge to go into details" (p. 19); and (c) reification, which is said to have occurred when the entity can be thought about independently of its process; this is marked by "a sudden ability to see something familiar in a totally new light" (p. 19). The reification process is an instance of reflective abstraction, a process in which the entity "proceeds from the actions or operations of the knowing subject and transfers to a higher plane … it leads to differentiations that necessarily imply new, generalizing compositions at the higher level" (Piaget, 1977/2001, p. 29). This reification process helps explain the construction of new mathematical objects out of existing ones. The reified object, what Greeno (1983) calls a conceptual entity, can be taken as input for other processes. According to Harel and Kaput (1991), conceptual entities alleviate working-memory load, facilitate comprehension (e.g., seeing the relationship between $f(x)$ and $f(x-a)$), and assist focus of attention on appropriate structure in problem solving. In essence, conceptual entities enable us to reason with symbols. Hence, symbols (e.g., $f(x)$, T_n, $\frac{d^n}{dx^n}$) that are signifiers of conceptual entities facilitate algebraic reasoning (e.g., $f(x) + f(y) = f(x + y)$ indicates linearity). One cognitive objective for instruction should be to help students develop the *minimizing-cognitive-load* way of thinking. Related to this way of thinking are other ways of thinking such as the *treating-a-collection-of-objects-as-one-entity* technique, *leaving-out-the-details* strategy, and *looking-for-a-simpler-problem* approach.

In reality, many students fail to make the connection between the structural conception and the operational conception. For example, they see an equation as an object to be transformed into "$x =$ ______". The only source of meaning is the rules for solving the equation (Sfard & Linchevski, 1994b). They operate on the symbols as if the symbols "possess a life of their own" (Harel & Sowder, 1998, p. 250) without associating to them any quantitative reference. Instantiations of the non-referential symbolic way of thinking are commonly observed in students' errors such as $\frac{3a+5b}{2a+5c} = \frac{1}{2} \Rightarrow \frac{3+b}{2+c} = \frac{1}{2}$

and $(x - 6)(x - 9) < 0 \Rightarrow x < 6$ or $x < 9$ (Matz, 1980). These ways of understanding (e.g., treating inequalities as equations) are what Sfard and Linchevski's (1994b) call *pseudostructural conceptions*.

> *Pseudostructural* conceptions are the conceptions which develop when the student, unable to think in the terms of abstract objects, uses symbols as things in themselves and, as a result, remains unaware of the relations between the secondary and primary processes.[10] In the case of equations (or inequalities), … the primary processes are the arithmetic operations encoded in the formulae, the secondary processes are those which one must perform on equations in order to solve them, and the abstract objects behind the symbols are the truth-sets. (p. 279).

According to Sfard and Linchevski (1994b), students with pseudostructural conceptions exhibit behaviors such as: (a) making judgments based on the form of expressions because the meanings of symbols are the symbols themselves, (b) performing operations arbitrarily without justifications, (c) not making connections among different representations such as graphs, equations, and schematic diagrams, (d) relying on superficial similarities, and (e) confusing one mathematical entity with another. These can be viewed as undesirable ways of thinking associated with the mental act of interpreting or manipulating symbols. In summary, students' lack of understanding of an algebraic entity is probably the result of a disconnection between the structural conception and the operational conception of the entity. This disconnection contributes to students' difficulties in algebra.

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[10] Primary processes refer to those from which the reified object originated, while secondary processes refer to those which operate on the reified object.

Students' Challenges in Algebra

In this sub-section, I discuss students' challenges (a) in accepting lack of closure, (b) with object-process duality, (c) in interpreting literal symbols, (d) in working with unknowns, (e) in connecting arithmetic and algebra, (f) in interpreting the equal sign, (g) with the notion of equivalence, (i) in solving inequalities, and (j) in being goal-oriented.

Many studies have documented the difficulties students face in the transition from arithmetic to algebra. One explanation for the difficulties is the shift in focus. The focus in arithmetic is on obtaining numerical answers, while the immediate focus in algebra is on formulating and manipulating algebraic expressions without necessarily obtaining a numerical result (Booth, 1988).

Students' difficulties in accepting lack of closure. Accepting a lack of closure (Collis, 1974, cited in Kieran, 1992) is necessary in algebra. Accepting-lack-of-closure is a way of thinking that allows one to work with expressions without reducing them to a resultant number. Many students have not developed this way of thinking because they are not able to treat expressions as conceptual entities. Herscovics and Chalouh (1984) observed students' difficulties in accepting $8 \times a$ as the result for a quantity such as the area of a particular rectangle. Booth (1984) observed students' tendency to have single-term answers; for example, writing $7ab$ for $2a + 5b$, and introducing a new variable z to denote the answer $x + y$.

Students' difficulties in the dual interpretation of algebraic expressions. Students have difficulties in simultaneously interpreting an algebraic expression such as $x + 3$ as both a process, add 3 to n, and a product, a number that is 3 more than n (Booth, 1984). Overcoming this cognitive obstacle, which is commonly known as the *name-process* *dilemma* (Davis, 1975) or the *process-product dilemma* (Sfard & Linchevski, 1994a), requires students to reify the process into a conceptual entity.

Students' interpretation of literal symbols. While literal symbols represent values in algebra, they tend to represent objects in arithmetic, such as *m* for meters and *a* for apples (Booth, 1988). The phenomenon of literally translating word-for-word from a problem statement to symbols is not uncommon among algebra students. For example, "there are six times as many students as professors" is interpreted by a student as "6 times S is equal to professors" (Clement, 1982, p. 19). The student is said to be interpreting the letters as objects rather than as unknowns. Küchemann (1981) found that the majority of the 13-, 14- and 15-year-old students he studied either treated letters as concrete objects or ignored the letters. Only a very small percentage of them interpreted letters as a generalized number or as variables. For this item: Is $L + M + N = L + P + N$ true: always, sometimes (when), or never? Küchemann found that 51% of the 14-year-old students answered "never" and only 25% answered "sometimes" (true when $M = P$). Most students probably thought that different letters stands for different numbers. Such students are said to be unaware of the *unspecified* characteristic of variables (Fujii, 2003). Fujii also found that some students hold another view: "the same letter does not necessarily stand for the same number" (p. 52). He found that students accepted both 2, 5, 5 and 4, 4, 4 as the answer for $x + x + x = 12$. Such students are aware of the *definite* characteristic of variables. Lacking the way of understanding that variables are both *definite* and *unspecified*, students may find algebra to be inconsistent.

Students' difficulties with unknowns/variables. Beginning algebra students generally encounter difficulties while working with unknowns or variables. For example, when a 7th grader was asked to find the result of multiplying $\frac{6}{3x+1}$ by *x*, he responded, "How can we multiply by *x* when we don't know what *x* is?" (Davis, 1975, p. 8). Herscovics and Linchevski (1994) coined the term *cognitive gap* to refer to students' inability to operate on unknowns. In their study on the strategies used by students who have not been taught equation solving, only two out of 22 seventh-graders used the "grouping unknown" strategy to solve equations that have double occurrence of the unknown (e.g., $17n - 13n = 32$ and $4n + 9 = 7n$). Most of them used trial-and-error substitution. Overcoming this cognitive gap would require a student to be able to conceive an expression such as $7n$ as both an object and a process.

Arithmetic-algebra disassociation. According to Lee and Wheeler (1989), many of the difficulties students face when learning and doing algebra arise from the disassociation between the world of algebra and the world of arithmetic. "Students behaved as though algebra were a closed system untroubled by arithmetic" (p. 46). Lee and Wheeler found that some students were willing to accept algebraic solutions that differed from arithmetic solutions and felt no need to resolve the discrepancies. In their study, students were asked to determine if an algebraic statement, such as $\frac{2x+1}{2x+1+7} = \frac{1}{8}$, $\frac{1}{6n} - \frac{1}{3n} = \frac{1}{3n}$, and $(a^2 + b^2)^3 = a^6 + b^6$ is definitely true, possibly true, or never true. Only 10 out of 268 students made attempts to check with numbers. Only one student used a counter-example to prove that $(a^2 + b^2)^3 = a^6 + b^6$ is false. None of the students seemed to have established a substitution reflex to check their work. These behaviors are considered manifestations of the non-referential symbolic way of thinking.

Students' way of understanding the equal sign. Behr, Erwanger, and Nichols (1975) and Falkner, Levi, and Carpenter (1999) documented that elementary graders view the equal sign as a signal to perform the computation that precedes it and write the result after it. In Falkner et al.'s study, all 145 sixth-graders thought that either 12 or 17 should go into the box in $8 + 4 = \square + 5$. Knuth et al. (2006) found that relatively few middle school students, when asked to give meaning for the equal sign in the statement "$3 + 4 = 7$", provided a relational definition (e.g. "same value") as compared to operational view (e.g. "add the numbers"). Students who held a relational interpretation of the equal sign, as compared to those who held an operational interpretation, were found to perform better in solving linear equations like $4m + 10 = 70$ (Knuth et al., 2006) and recognizing equivalence in a pair of equivalent equations: $2 \times \square + 15 = 31$ and $2 \times \square + 15 - 9 = 31 - 9$ (Knuth et al., 2005). Hence, teachers should help students advance from an operational interpretation of the equal sign to a relational interpretation.

Falkner et al. recommend instructional tasks (e.g., $4898 + 3 = 4897 + \square$; given that $a = b + 2$ is true, which is larger, a or b?) that involve comparison rather than computation as a means to encourage students to conceive equality as a relationship. Lack of understanding equality as a relationship, according to Kieran (1981), is one of the obstacles students encounter in their transition from arithmetic to algebra. Conceiving an equation as a relation is reified from the "procedural" experience of comparing both sides of an equation such as using trial-and-error substitution. Kieran (1988) finds that pre-algebra students who have used the trial-and-error substitution method possess a more developed notion of the balance between the two sides of an equation than those who prefer the *undoing method* of working backwards with inverse operations (e.g., solving

$4n + 17 = 65$ by finding $4n = 65 – 17 = 48$ and then $n = 48/4 = 12$). Failing to conceive an equation as a relation, high school and college students may continue to interpret the equal sign as a signal to do something: for example, to solve for a variable or to find a derivative (Kieran, 1981).

Students' difficulties with the notion of equivalence. Equivalence is a foundational concept that underlies all transformations of algebraic expressions. However, this concept is inherently difficult for students. In a study on students' conservation of equation, Wagner (1981) found that some 12-17 year-old students perceived that a change in a letter within an equation results in a completely new equation. For example, they did not see that the value of W that makes $7 \times W + 22 = 109$ true is the same as the value of N that makes $7 \times N + 22 = 109$ true. Research has shown that many students do not relate the validity of a transformation on an equation to the preservation of the solution set of the equation. As such, they do not know that only the correct solution will yield equivalent values for the two sides in any equation of the equation-solving chain (Greeno, 1982, cited in Kieran, 1989). Steinberg, Sleeman and Ktorza (1990) found that many students were unable to associate equivalence of equations with valid transformations. For example, the students needed to compute the solutions to both equations, $x + 2 = 5$ and $x + 2 – 2 = 5 – 2$, even though they had generated the second equation in the process of solving the first.

On the other hand, many students confound equivalence with transformability (Sfard & Linchevski, 1994b). In their study involving 280 students (14-17 year-old students in Israel), Sfard and Linchevski found that only 17% conceived $4x – 11 = 2x – 7$ as being equivalent to $(x – 2)^2 = 0$, and only 12% conceived $4x^2 > 9$ as being nonequivalent to $2x > 3$. For these students, two equations are equivalent if one can be "transformed" into the other. Hence, one of the cognitive objectives for instruction should be to help students, by capitalizing on their referential symbolic reasoning, to develop the way of understanding that the referent of an equation is its solution set.

Students' treating inequalities as equations. Garuti, Bazzini and Boero (2001) point out that "the traditional teaching of inequalities … reduces the difficulties inherent in the variable concept and the complexity of the solution process by treating inequalities as a special case of equations" (p. 10). Because of this, many students do not understand the conceptual difference between inequalities and equations. Vaiyavutjamai and Clements (2006) found that 44% of 9th graders in Thailand gave a single number as the answer when asked to solve $3 - 4x \leq 6x - 7$. Many students tended to believe that a number that appeared on one side of the final line (the 1 in $1 < x$) was the solution to the inequality. Students also exhibit a variety of procedural errors in solving inequalities. The prevalent source of difficulty comes from the inappropriate analogies between equations and inequalities (Tsamir & Almog, 2001; Tsamir et al., 1998). For example, students may make incorrect deductions such as $x^2 < 16$ implies $x < \pm 4$ and $\frac{2x - 2}{x + 1} < 1$ implies $2x - 2 < x + 1$. The range of inequalities students can solve depends on the specific procedures that have been taught to them. For example, most students in Italy could not solve inequalities like $x^2 - 1/x > 0$ upon entering the university mathematics courses (Boero, 2001). Their to understand an inequality as a comparison between two sides will probably prevent them from solving unfamiliar inequalities such as $3x^2 - |x| + 2 > 0$ and $(4x^2 - 3x)^2 + 1 < 0$ in a "meaningful" way.

Students' inability to capitalize on "structures". Many students do not capitalize on structure. For example, only 6 out of 33 11th-graders in Israel solved $\left(\frac{1}{4} - \frac{x}{x-1}\right)x = 6 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$ by attending to its structure and subtracting the function $\left(\frac{1}{4} - \frac{x}{x-1}\right)$ from both sides (Hoch & Dreyfus, 2004). Most algebra students are not able to conceive a function as a conceptual entity, a conception essential for recognizing structure. Wagner, Rachlin, and Jensen (1984, cited in Kieran 1989) found that while most 9th graders in their study knew that the solution to $s/8 - 3 = 14$ would not change when the letter s is changed to t, they had to solve the equation $(t + 1)/8 - 3 = 14$ for t when asked to determine the value for $t + 1$. When they were subsequently asked to solve $4(2r + 1) + 7 = 35$ for $2r + 1$, only one student solved it directly for $2r + 1$. Students, especially when they have a procedure, generally do not attend to the structural aspect of an equation; thus, for example, they do not see solving $4(2r + 1) + 7 = 35$ for $2r + 1$ is essentially equivalent to solving $4x + 7 = 35$ for x.

Students' not being goal-oriented. In general, many students do not explicitly set a goal, assess their progress towards it, and/or fail to recognize the attainment of it. English and Halford (1995) refer to this deficiency as "lack of strategic knowledge in solving algebraic tasks" (p. 232). Wenger (1987) observed that students tend to "go around in circles" (p. 219) and seem to choose their next move without having a specific goal in mind. Although students may perform legal transformations, they sometimes end up with a more complex equation. For example, in trying to solve $v\sqrt{u} = 1 + 2v\sqrt{1+u}$ for v, 34 out of 64 college students were compelled to get rid of the square-roots even though the equation is actually linear in v (Wenger, 1987). In this case, the students did not assess whether a particular step would get them closer to their goal. Matz (1982) has observed that students tend to perform steps that are applicable but not necessarily productive. Matz (1980) observes that "when students do not assess whether [an action] gets them closer towards a goal, they execute steps that are obviously *applicable*, but not *productive*; their work has an aimless character" (p. 148). For example, students may pull out a common factor in one step and undo it in another step by multiplying out the newly factored expressions. When asked to solve $(q-1)(q+4)=0$ for q, some students in my pilot study multiplied out the factors to obtain the standard form, which they then solved either by factoring (i.e., undid what they had done) or using the quadratic formula. These behaviors may be described as *do what you can and see what happens next*.

To attain a goal, one must work within the constraints of the situation. In the context of symbol transforming, one must attend to *invariance*—the preservation of the denotation of an expression when its form is changed. In the case of simplifying an inequality or equation, one must preserve its solution set. However, one must recognize that expressions are transformed purposefully towards a certain desired form and not aimlessly. One cognitive objective for instruction should be to help algebra students to develop *algebraic invariance*, which Harel (in press c) defines as "the way of thinking by which one recognizes that algebraic expressions are manipulated not haphazardly but with the purpose of forming a desired structure and maintaining certain properties of the expression invariant" (p. 14). Harel points out that the *completing the square* algorithm can be taught in a manner that promotes this way of thinking. If students can solve $(x + T)^2 = L$ readily, then they can be challenged, in phases, to manipulate

$ax^2 + bx + c = 0$ with the goal of attaining the desired from of $(x + T)^2 = L$, while ensuring that the solution set is preserved. With an algebraic invariance way of thinking, goal-directed operations become learnable for students. Without it, on the other hand, “symbol manipulation is largely a mysterious activity for students—an activity they carry out according to prescribed rules but without a goal in sight” (Harel, in press c).

Desirable Ways of Understanding and Ways of Thinking for Algebraic Instruction

In an earlier part of this discussion, I mentioned a few ways of understanding and ways of thinking that could be incorporated as cognitive objectives for instruction. Desirable ways of understanding include understanding that the referent of an equation is its solution set, understanding equivalent equations as having same solution set, and understanding that variables are both *definite* and *unspecified*. Desirable ways of thinking include referential symbolic reasoning, capitalizing on structure, acting it out operationally to make sense of something abstract, treating a collection of objects as one entity, accepting lack of closure, and algebraic invariance.

Developing these ways of understanding and ways of thinking would require students to reason repeatedly with algebraic inequalities and equations far beyond the traditional practice of applying standard procedures to solve different families of equations and inequalities. For example, students should experience some numerical substitutions that make an equation/inequality true and others that make it false. Finding all the numbers or ordered pairs that make an equation/inequality true is presumably an intrinsic task that students can appreciate. Comparing different solution strategies may promote a *striving-for-efficiency* way of thinking and may encourage reflection. According to Davis (1986), as students reflect on the process of solving and notice structures/patterns, they may “discover” rules for rewriting equations/inequalities. With a repertoire of equation-rewriting rules, students can begin analyzing each new equation to determine a sequence of equation-transformations towards the solution (Davis, 1986). Such analysis and reasoning helps students cultivate the algebraic invariance way of thinking.

2.6 A Summary of the Theoretical Framework and Research Objectives

The theoretical perspective in this research is based on Piaget’s (1967/1971) notion of anticipation, von Glasersfeld’s (1998) three general types of anticipation, and Cobb’s (1985) hierarchical levels of anticipation. Two acts of anticipating are identified: foreseeing an action and predicting a result.

Harel’s (in press c) MA-WoU-WoT framework is used to analyze students’ acts of foreseeing and predicting. A way of understanding refers to the result/action a student actually predicts/foresees, and a way of thinking characterizes the manner in which the student predicts/foresees.

The research has three objectives: (a) to categorize students’ ways of thinking associated with the mental acts of predicting and foreseeing, (b) to identify the relationship between these ways of thinking and students’ ways of understanding algebraic inequalities/equations, and (c) to explore the potential for advancing students’ ways of thinking through a short-term instructional intervention that is guided by Harel’s DNR-Based Instruction (2001, in press a). The next chapter presents the research methods employed in this study to achieve these objectives.

CHAPTER 3: RESEARCH METHODOLOGY

The research design and the methods for studying students' mental act of anticipating (foreseeing and predicting) are discussed in the first section. Information on data collection is presented in the second section. The research instruments are discussed in the third section. The data analysis process is described in the fourth section.

3.1 Research Methods

This study has three parts. The preliminary part consisted of administering a written assessment to four classes of 11^{th} graders. In Part 1, clinical interviews (Ginsburg, 1997) were conducted with 14 eleventh-graders. In Part 2, four learners received one-on-one teaching interventions.

Preliminary Part: Written Assessment

One purpose of administering the written assessment was to give the students a feel for the nature of this research and to let them know how their participation in this research could contribute to the improvement of mathematics education. A second purpose was to use their written responses to select participants for Part 1 of the study. A third purpose was for me to gain a general sense of 11^{th} graders' algebraic knowledge pertaining to inequalities/equations at that particular high school.

Before administering the written assessment, I used an activity (see Appendix A) to get the students acquainted with the format of the written assessment as well as to get them interested in this research. During the activity, I emphasized that the focus of the research was on their reasoning and not on their memory of how to do stuff.

The written assessment consisted of 5 items (see Appendix A). The students were asked to write down their initial response to the problem and their subsequent thoughts. After the written responses were collected, I introduced the research project and encouraged them to participate in both parts of it. I explained to them that an understanding of the way students think as they do mathematics can help future teachers to teach in a manner that is consistent with the way students think.

Part 1: Semi-structured Clinical Interviews

Eleventh graders were chosen because most eleventh graders at this school had completed two years of algebra. Since the purpose of the interview was to elicit students' mental acts of foreseeing and predicting, the tasks were designed to see how students would apply their algebraic knowledge to solve unfamiliar problems. This requires that the interviewees have some experience in working with algebraic equations and inequalities.

The objective of the clinical interviews was to elicit students' foresights and predictions. The interviews were *semi-structured*: they were structured in the sense that a standard protocol (see Appendix B) and tasks from a fixed set of items (see Appendix C) were used. The interviews were unstructured in the sense that I, as the interviewer, had freedom to pursue any direction that I deemed promising in eliciting anticipatory behaviors based on the interviewee's particular ways of understanding.

I adopted a two-phase approach for each task in the interviews: a non-interactive phase and an interactive phase. In the non-interactive phase, the interviewee solved a problem with minimal intervention other than a prompt requesting that the interviewee think out loud. I avoided asking subjects to reflect on their thought process during this phase because introspection would disrupt the interviewee's flow of thoughts (Ericsson & Simon, 1993). However, not all interviewees were comfortable with thinking out loud. In that case, the interviewee was asked intermittently, "can you tell me what you are thinking?" During this stage, probes were withheld until the interviewee had arrived at an answer, a prediction, or an impasse. In the interactive phase, the interviewee might be asked to share her or his reasons for certain actions, her or his interpretation of the problem statement, the meaning she or he had for certain symbols, etcetera. Whenever the interviewee arrived at a conclusion, she or he would be asked, "On a scale of 1 to 10, how confident are you that your answer is correct?" followed by "Why are you ___ (the number) confident?" If the student was not "10" confident, the student would be asked "how can you make it a 10?" From that point onward, the interaction was no longer structured in the sense that I would "follow where the child's thought leads" (Ginsburg, 1997, p. 49). On a few occasions, impromptu questions or tasks were also posed to test certain hypotheses about the interviewee's ways of understanding and/or ways of thinking.

My role as interviewer involved (a) putting the interviewee at ease in the interview; (b) prompting the interviewee to think out loud and share her (or his) thinking; (c) probing the interviewee's thinking by getting her to report what she had been thinking, to explain her reasons for certain actions, to explain why she thought a certain thing, or to rate her level of confidence; (d) ensuring that I understood the interviewee's interpretation of the tasks and subsequently, if necessary, helped her to interpret the task as intended; and (e) managing the flow of the interview, for instance, posing follow-up questions to pursue certain interesting avenues associated with interviewee's responses and negotiating the transition from one task to another.

In the introductory statements during the interview (see Appendix B), I emphasized the intent of the interview—to understand how students think and not to test whether they know how to do certain things correctly. This emphasis aimed to minimize the tendency of interviewees to engage the interview with the "school mathematics" mentality (Nunes et al., 1993), in which she (or he) might feel obliged to apply procedures taught to them. The emphasis on thinking instead of performing was reinforced throughout the interview by focusing on the interviewee's level of confidence of her answer instead of the correctness of her answer. Once the interviewee had indicated a confidence level of 10 (most confident) and had communicated her reason for her confidence, we would proceed to the next task, or conclude the interview if that was the last task. Likewise, if the interviewee indicated that she could not increase her confidence level any higher, we would also move on to the next task.

Part 2: Teaching Interventions

Four learners participated in the second part of the study. Each learner went through a series of five 60-minute problem-solving sessions followed by a post-intervention interview. I chose to work with one learner at a time because I found, in my pilot studies, that I was not able to keep track of two or more students' thought processes simultaneously, especially when they were taking different lines of reasoning. The nature of the teaching intervention was also a consideration. If the setting involved a common milieu where students worked together on something such as a computer simulation or a physical instrument, then having two or more students would be appropriate because the individual lines of reasoning would revolve around the “public” line of reasoning. However, the setting for this teaching intervention was paper-and-pencil problem-solving, which is inherently a private endeavor rather than a team effort.

One purpose for this part of the study was to explore the potential in advancing students’ ways of thinking. Another purpose was to have additional data for developing the categories for ways of thinking associated with anticipating. This data, in contrast to the interview data, provided a better sense of learners’ robustness in their ways of thinking and the influence of problem situations on their ways of thinking.

For this interventions, I focused on “breath” rather than “depth” because I wanted to maximize the opportunity for learners to solve problems. My strategy was to get them to engage with a variety of problems so as to help them to improve their ways of thinking associated with problem-solving.

The teaching interventions in this research are not considered *teaching experiments* (Steffe & Thompson, 2000), which generally involve creating theoretical models for students’ development of mathematical concepts such as rate of change, derivative, and equivalent inequalities. I found that creating models to represent the change in learners’ ways of thinking associated with foreseeing/predicting was not possible in this study because change in ways of thinking does not occur in phases and is usually not linear.

Nevertheless, I found many features of Steffe and Thompson’s (2000) teaching experiment model to be appropriate for my teaching interventions. One such feature is the coupling of the teaching component with the research component: while a teaching experiment strives to enhance students’ ways of understanding of equations and inequalities and ways of thinking associated with anticipating, hypotheses are generated during and after each teaching episode (i.e., problem-solving session). After testing hypotheses about students' ways of understanding and ways of thinking during a teaching episode, the instructor uses his findings to plan the teaching actions for the next session. In terms of research, the features of a teaching experiment offered me: (a) the flexibility to pose questions and tasks so as to increase the chance of observing the learner's foresights/predictions; (b) the opportunity to construct and test hypotheses about the learner's ways of understanding and ways of thinking associated with foreseeing/predicting; and (c) the flexibility in tailoring a lesson to the learner so as to work at the boundaries of her or his mathematical knowledge.

My interactions with the learner were guided by Piaget's (1970) theory of learning and Vygotsky's (1978) zone of proximal development. I consider learning to be the process by which learners construct new knowledge from their existing knowledge as they interpret and make sense of a situation. As a teacher, my role is to create opportunities for learning, to provoke intellectual need, and to pose questions that shift the learner's attention. To advance a learner's ways of understanding and ways of thinking, non-cognitive factors that facilitate learning were incorporated into the teaching intervention. I strove to create a positive, non-threatening learning environment and to arouse the learner's interest in the lessons. To achieve this, I tried to help the learner feel challenged and in control of her or his learning, as well as having a sense of success in "creating" mathematics and gaining insights.

The planning and the implementation of a problem-solving session were in accordance with Simon's (1995) *hypothetical learning trajectory*. In designing the entire intervention for a learner, I identified learning objectives, created activities, and anticipated a hypothetical learning process—"a prediction of how the students' thinking and learning will evolve in the context of the learning activities" (Simon, 1995, p. 136). The hypothetical learning trajectory was constantly revised because the actual learning trajectory tended to be different.

The designing, sequencing, and assigning of tasks were based on Harel's DNR-based instruction (in press a). In accordance with the Duality Principle, tasks were tailored to the individual learner's ways of understanding and ways of thinking simultaneously. In accordance with the Necessity Principle, tasks were designed to necessitate, and to help the learner to acquire, the target ways of understanding and ways of thinking. The tasks were supposed to be challenging yet within the learner's *zone of proximal development* (Vygotsky, 1978). During the problem-solving sessions, I posed questions to guide the learner towards certain desirable ways of thinking in a non-intrusive manner. If the learner began to engage in non-referential symbolic reasoning, I posed appropriate questions to get the learner to attend to meaning. In accordance with the Repeated-reasoning Principle, meaningful tasks were used to get the learner to apply, to experience the limitations of, and to refine, her or his existing ways of understanding and ways of thinking. Tasks were sequenced so that the learner could reason repeatedly and thereby internalize and retain certain desirable ways of understanding and ways of thinking. Homework was assigned as a means to foster repeated reasoning, but only two learners completed the homework assignments with a motivation to learn.

All the tasks used in the problem-solving sessions involved only one variable. Thus, the interviewee's responses to the two-variable tasks in the post-interview were valuable for providing information about possible improvement in the learner's ways of understanding and ways of thinking beyond the settings in which they occurred.

Each teaching intervention was concluded with a post-interview. From the learner's perspective, the last session was just another problem-solving session. From my perspective, the purpose was to assess any change in the learner's ways of thinking associated with foreseeing/predicting. Six tasks were used in all the post-interviews; they were chosen from the set of tasks used in the pre-interviews.

Finally, this study did not attribute changes in a student's ways of thinking solely to the teaching intervention. Establishing such a causal relationship was not necessary because this research was not concerned with the effectiveness of the instructional intervention. For this reason, interviewees who did not participate in the teaching intervention did not have a post-interview.

3.2 Data Collection

Site for the Study

The site for this study was a public middle/high school in Southern California that spanned grades 6 through 12. One unique feature of this school is that it is a university-based charter school that practices detracking—one track for all students, although different students within a particular grade level may be taking content courses at different levels. For example, the majority of 11th graders at this school are enrolled in Pre-calculus, with some in Calculus and some in Algebra II.

This school aims to provide an intensive college preparatory education for low-income students who are motivated to become the first generation in their families to graduate from a four-year university. The enrollment in 2004/2005 was approximately 780 students with 59% Hispanic, 22% Asian, 13% African American, and 6% White. Although the school is located in an affluent neighborhood, about 90% of the students come from non-affluent neighborhoods and are bused in.

Selection of Participants

The participants for this study were recruited from the four 11th grade university-preparatory periods[11] in the school (the recruitment script is in Appendix E). Prior to the recruitment, the students were asked to complete a written assessment: 67 out of about 90 students completed the written assessment.

Eight students (Ali, Chela, Jose, Maria, Noel, Talia, Raul, and Vito) indicated interest in participating in both parts of the study (the clinical interviews and the teaching interventions), and another 23 students indicated interest in only Part 1 of the study. All eight students who indicated interest in both parts were included for Part 1. Another six students were selected, based on their written responses and the mathematics class in which they were enrolled, to form a diverse group of interviewees with different levels of competence in mathematics. In total, I interviewed 4 Algebra II students, 4 Pre-calculus students, 5 Calculus students, and 1 student taking Calculus II at a university affiliated with the school.

Out of the eight students who were interested in both parts of the study, four learners (Vito, Ali, Talia, and Chela) were chosen for Part 2. Jose and Pham were not

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[11] University-preparatory periods (commonly known as advisory periods) involve 30 minutes of reading (Kick-Back-and-Read program) and 30 minutes of doing mathematics (Kick-Back-and-Calculate) each week. Other activities were aimed at helping students to develop skills like test-taking, notes-taking, etcetera.

chosen because they demonstrated desirable ways of thinking in their interviews. Maria and Noel were not chosen because they had weak foundation in algebra and arithmetic, and I anticipated that an intervention with them would be at a pre-algebra level. My goal was to select four learners with different ways of thinking but similar levels of competence in algebra.

Data Collection Process

The interviews and the problem-solving sessions were conducted during university-preparatory periods. The participants were pulled out from their classrooms and interviewed either in an adjacent discussion room or in another classroom. All the sessions were videotaped and audiotaped.

The interviews were transcribed during the period of data collection. I conducted an initial analysis of the interview data before starting the teaching interventions. The purpose of this preliminary analysis was to determine an overall plan for the teaching intervention for each learner. The overall plan included (a) cognitive objectives in terms of ways of understanding and ways of thinking, (b) tasks to achieve these objectives, and (c) a *hypothetical learning trajectory* (Simon, 1995). The overall plan for each learner was modified during the course of the teaching intervention.

The four teaching interventions were conducted in two rounds: Vito and Ali participated in the first round, and Talia and Chela were in the second round. I met with each learner on average once a week over a period of five to six weeks. Each problem-solving session was transcribed prior to the next session. Interesting segments were analyzed to determine the cognitive goals and instructional tasks for the next session.

At the end of the teaching intervention, the learners were asked to write a summative report on their participation in the tutoring sessions. Approximately a year later, they were asked to comment on the change in the way they learn mathematics and solve mathematics problems, as well as what they had learned from their participation in the project. Talia's responses are included in Appendix F.

The written assessment and recruitment was completed in February 2005. The interviews were conducted in February-April 2005. The teaching interventions for Vito and Ali were conducted in May-June 2005 and those for Talia and Chela in June-July 2005.

3.3 Research Instruments

The main objective of this research was to study students' mental acts of foreseeing and predicting. The tasks used in the interviews and the problem-solving sessions were critical factors for success. These tasks were developed and refined during my pilot work, in which I interviewed 13 students and held a total of 17 problem-solving sessions with 3 of them. I also tested out certain items in a written assessment with 84 Algebra 2 students at a different high school. I tested the four tasks listed in Figure 1.1 with a group of 9 calculus students in another high school.

The appropriateness of the tasks used in the clinical interviews is discussed next. Following that, I present rationales for using certain tasks in the problem-solving sessions.

Characteristics of the Interview Tasks

Two types of tasks were used in the interviews: single-variable tasks and two-variable tasks. As mentioned earlier, only single-variable tasks were used in the teaching intervention. This approach allowed me to feel confident that changes in students' responses to two-variable tasks from the pre-interview to the post-interview were not a consequence of having done similar tasks in the problem-solving sessions. Figure 3.1 lists the six items used in the post-interview (see Appendix C for the problem set from which tasks were used in the pre-interview).

S2. Is there a value for x that will make the following statement true?
$(2x - 6)(x - 3) < 0$

S5. Is the following statement always true, sometimes true, or never true?
$(x + 1) + (x + 2) + (x + 3) + \ldots + (x + 99) + (x + 100) < 100x$

T1. Given that $5a = b + 5$, which is larger: a or b?

T2. p and q are odd integers between 20 and 50. For these values,
is $5p - q > 2p + 15$ always true, sometimes true or never true?

T3. Given that m is greater than n, can $m - 14$ ever be equal to $7 - n$?

T4. Consider $(x + 1)(2k - 7) = 3(2k - 7)$.
Is there a value for k that makes this equation true for all values of x?

Figure 3.1: Items used in the post-interview

One important characteristic of the tasks is that they were intended to be non-routine for the students; in other words, these tasks are not typically found in algebra textbooks. Although these tasks were generally unfamiliar to the interviewees, they were "meaningful" for the interviewees in the sense that they seemed to engage the interviewees.

Another characteristic of the tasks is that they were non-directive; they did not direct students to do perform certain actions. Examples of directive tasks, in contrast, include, "solve $(2x - 6)(x - 3) < 0$ for x", "simplify $(x + 1) + (x + 2) + \ldots + (x + 100)$" and "solve for b: $5a = b + 5$". In addition, the tasks were phrased in the form of a question so that interviewees could make a prediction, if they chose to, prior to doing any work.

A third characteristic is that the tasks could be approached in a variety of ways, such as by reasoning with the structure of the inequality/equation, by manipulating symbols, or by plugging in numbers. This characteristic was designed to elicit a greater variety of anticipatory behaviors. In addition, the tasks were structurally different from each other. However, the same format was intentionally used for items S1, S2, S3, and S4 so that the interviewees can focus on the inequalities/equations. In addition, I wanted to reduce the cognitive load required to interpret the first few tasks so as not to overwhelm the interviewee, especially at the beginning of an interview. Psychologically, familiarity is presumably more comforting. Using the same format was a means to minimize the interviewee's stress level.

Each inequality/equation was carefully designed to serve certain purposes. For example, consider $(2x - 6)(x - 3) < 0$. The factored form was chosen to see whether an interviewee would capitalize on its structure, or would immediately expand the factors to obtain the standard form, a move that is counter-productive. The factor $2x - 6$ was chosen to be twice of the factor $x - 3$ so that there is only one critical value, making the inequality false but the corresponding equation $(2x - 6)(x - 3) = 0$ true. The factor $x - 3$ was intentionally made to be simple so that those who attend to structure would notice the relation between $2x - 6$ and $x - 3$. For another example, consider Item T2. The functions in the inequality $5p - q > 2p + 15$ were chosen such that there was only one pair of values for p and q in the interval (20, 50) that will make the inequality false. The extreme values of 21 for p and 49 for q allowed me to determine whether students were goal-oriented in their choices of numbers for substitution. The "greater than" sign was chosen instead of a "less than" sign to see whether students would falsify their claim that the inequality is always true after finding a few pairs of values that made it true. Having p appear on both sides of the inequality allowed me to see whether students would simplify the inequality, say to $3p - q > 15$, to ease their reasoning.

Characteristics of Tasks used in the Problem-solving Sessions

The general characteristics of the items used in the teaching intervention were similar to those used in the interviews. In addition to assessing learners' ways of thinking, the problem-solving sessions were aimed at helping learners learn. Occasionally, a non-directive task was used to focus on certain ways of understanding such as the notion of solution set. For example, "what is the solution set for $x(6x + 8) < 0$?"

The tasks used in the teaching intervention were not designed to prepare the learners to do well in the post-interview (compare the interview tasks listed in Appendix C with problem-solving-session tasks listed in Appendix D). Nevertheless, the tasks were designed to help students develop desirable way of thinking and ways of understanding, which were supposed to improve learners' responses for the tasks in the post-interview.

One of the cognitive objectives for the teaching intervention was for students to reason with structure. Hence, many of the tasks were aimed at promoting the attending-to-structure way of thinking. An example of such a task is Item TE1-$_{V}$N5b[12]: "Suppose we want to make $2x + 2222 < 8x + 88$ never true. Is it possible to change one of the numbers (2, 2222, 8, 88) so as to make it never true?" Another example is Item TE5-$_{T}$N4: "Is the following statement always true, sometimes true, or never true? $(x - 3)^2 + 1 > 0$".

A second objective for all four learners was to help them develop desirable ways of understanding for solution sets and equivalent inequalities. To achieve this objective, I used the problems like the following: "Consider these two inequalities: $5x + 10 > x + 5000$ and $4x > 4990$. Is there a value for x that will make one of them true but will make the other false?" (Item TE2-$_{V}$N2). I also created tasks to test a learner's understanding of equivalent inequalities. An example is "Write an inequality that has the same solution set as, but looks different from $2x - 10 > 50 - x$" (Item TE5-$_{V}$R3).

Tasks used in the problem-solving sessions were individually tailored to each learner. The type of tasks used could be the same, but the inequalities/equations were often different. For example, the pair of equivalent inequalities used for Vito (TE2-$_{V}$N2) and Ali (TE2-$_{A}$N3) was $5x + 10 > x + 5000$ and $4x > 4990$, whereas the pair used for Talia (TE3-$_{T}$N3) and Chela (TE2-$_{C}$N2) was $6x + 15 < 0$ and $8x + 20 < 0$. The functions in the former pair are related additively, whereas those in the latter pair are related multiplicatively.

Certain tasks were created specifically to help a learner deal with certain ways of understanding. An example is the follow-up to Item TE3-$_{T}$N3, in which Talia did not notice the multiplicative relation between the two functions (Chela did). To help Talia

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[12] The code TE1-$_{V}$N5b stands for Teaching Episode 1 for Vito, New item, Fifth item in the session, Part b of the task. The code TE5-$_{T}$R3 stands for Teaching Episode 5 for Talia, Reuse-of-a-homework-task, Third item in the session.

extend her way of thinking from comparing functions additively to comparing functions multiplicatively, I created tasks to help her develop the way of understanding that $8x + 20$ is a multiple of $6x + 15$. I assigned these two items for homework: "Consider these two functions $y_1 = 6x + 15$ and $y_2 = 8x + 20$. Can you find the relationship between these two functions?" and "Try to use algebra to show why the ratio between the two functions is always 4/3". In her homework, she observed the multiplicative relation from a table of numerical values, but was not able to show algebraically why the ratio is always 4/3. So I created Item TE5-$_{\mathrm{T}}$R3: "(a) Find the function $\frac{y_2}{y_1}$. (b) What do you expect to get if you were to solve $\frac{8x+20}{6x+15} = \frac{4}{3}$?"

There were some differences between the teaching interventions in the first round for Vito and Ali and those in the second round for Talia and Chela. After finding that their developing a structural understanding for the linear inequality of the form $Ax + B < Cx + D$ did not contribute much to improving Vito's and Ali's ways of thinking, I focused on quadratic inequalities in factored form for Talia and Chela.

In the second round of teaching interventions, I included tasks that could promote an awareness of the danger of "blindly" applying a newly learned idea. For example, I sequenced Item TE2-$_{\mathrm{T}}$R2 ("Is there a value of y that makes $2y + (4y – 9) \leq 0$ true?") after Talia had learned the critical point method in solving Item TE1-$_{\mathrm{T}}$N2b ("What is the solution set for $x(6x + 8) < 0$?").

Another difference between the two rounds of interventions is that tasks that foster prediction based on the structural property of inequality were used in problem-solving sessions for Talia and Chela, but not in those for Vito and Ali. An example is Item TE3-$_T$N5a: “When I plug $x = 61$ into $5(8x – 20) < 10(8x – 20)$, the output value for the function on its LHS is 2340. What is the output value for the function on its RHS?”

In this section, I have discussed how the characteristics of the tasks were designed in accordance with the research objectives and tailored to individual learners. I have also highlighted some differences between the two rounds of teaching interventions.

3.4 Data Analysis

There were two rounds of analyses. The first round was conducted during the data collection phase, in the transition period between Phase 1 (clinical interviews) and Phase 2 (teaching interventions) of the study. Having selected four learners for the interventions, I analyzed their responses to the items in the pre-interview. I identified their general ways of thinking—general in the sense that these ways of thinking were associated with problem-solving, rather than specific mental acts such as foreseeing, predicting, inferring, and interpreting. Examples of these ways of thinking included non-referential symbolic reasoning, association-based reasoning, inductive reasoning, considering-for-falsity, being-goal-oriented, and being-algorithm-oriented. I also identified deficient ways of understanding such as interpreting-an-inequality-as-an-equation and conflating-quadratic-formula-with-quadratic-function. I analyzed the responses of interviewees who performed well in order to generate a list of desirable ways of thinking that could be set as learning objectives for the four participants. Examples of these desirable ways of thinking included reasoning-with-structure, reasoning-with-generality-in-mind, algebraic invariance, and coordinating-goal-condition-action.

The second round of analysis was comprehensive and was conduced after all data had been collected. The data was analyzed in the following order: (a) Talia's pre-interview and post-interview, (b) Talia's teaching intervention, (c) pre-interview and post-interview for the other three learners, (d) interviews for the remaining 9 learners who did not participate in Part 2 of the study (Noel's interview was discarded because of her weakness in arithmetic), and (e) teaching intervention for Chela. I began with one learner because I wanted to identify the change in Talia's ways of thinking and consider how that change could be characterized. Another reason for focusing on one particular learner was to minimize the variance so that I could more quickly get a handle on how to infer the learner's mental act of anticipating from the learner's actions and statements. In this sequence of analysis, pre- and post-interview data from the 4 learners were used to create the categories for ways of thinking associated with foreseeing/predicting. Interview data from the 9 interviewees (Part 1 only) was used to test and refine those categories.

I analyzed the data in three phases that correspond to the three research questions. The results of the analysis are reported in Chapters 4, 5, and 6, respectively.

Phase 1: Developing Categories

I began the analysis by identifying *observation categories* (Clement, 2000) for students' ways of thinking associated with foreseeing/predicting, and students' ways of understanding inequalities/equations. These categories were derived from the data using a *constant comparative approach* (Glaser & Strauss, 1967) in the sense that existing categories were subject to modification as incoming data were analyzed against them. The analysis involved, (a) identifying instances of the mental acts of foreseeing and predicting (inferred from student's actions and statements); (b) generating, comparing, and refining categories for ways of understanding and ways of thinking; and (c) consolidating and collapsing some of the categories. The consolidated categories were further revised and refined in light of new information generated in subsequent phases of the analysis.

Phase 2: Identifying Relations between Ways of Thinking and Ways of Understanding

In the second phase of the analysis, a portion of the data was coded using the consolidated categories of observation categories. Tables of codes were created for interviewees' responses to two pre-interview items, namely Item Pre-T1 and Item Pre-S2. The purpose was to help me notice patterns of relations among ways of thinking associated with foreseeing/predicting, ways of understanding inequalities/equations, quality/correctness of solutions, and the *sub-context* (Cobb, 1985) in which students were operating. The data were re-analyzed, this time to establish the nature of the relations, which were formulated as *theoretical concepts* (Clement, 2000). The ultimate goal was to integrate these theoretical concepts as hypotheses in a theoretical model. For the purposes of this research, I did not attempt to construct such a model because it goes beyond the scope of this particular study.

Phase 3: Accounting for Change

A table of codes was created for each learner to make noticeable the change from the pre-interview to the post-interview in the learner's ways of thinking associated with foreseeing/predicting, ways of understanding inequalities/equations, quality/correctness of solutions, and *sub-context* in which the learner was operating. Episodes of all five problem-solving sessions for Talia were analyzed to gain a general sense of her ways of thinking and ways of understanding. I later re-visited the data to account for significant transitions as well as to account for the change in her ways of thinking and ways of understanding. A similar analysis was conducted for Chela. As for Vito and Ali, the analysis was mainly on their pre-interview and post-interview and not on their problem-solving sessions because their pre-to-post improvements were marginal. These results are discussed in detail in Chapter 6.

CHATPER 4: STUDENTS' WAYS OF THINKING ASSOCIATED WITH FORESEEING/PREDICTING

There are three chapters on results and discussions. They correspond to the three research questions: (a) What are students' ways of thinking associated with the mental acts of foreseeing and predicting when they solve problems involving algebraic inequalities/equations? Are these ways of thinking related to the quality of their solutions? (b) What relationships exist between these ways of thinking and their ways of understanding inequalities and equations? (c) What is the potential for advancing students' ways of thinking through an instructional intervention informed by DNR-based instruction?

This chapter addresses the first question and begins by contrasting two students' responses to highlight the need for attending to ways of thinking associated with anticipating. This is followed by a description of five ways of thinking associated with anticipating and three ways of thinking associated with predicting that were identified in this study. 13 interviewees' responses to two interview tasks were analyzed to see if there was a relationship between their ways of thinking and the quality of solution. I conclude with a discussion on why these ways of thinking are important in mathematics education.

4.1 A Comparison of Two Students' Responses

Two interviewees' responses are used to compare the way they approach the first item in the interview, namely Item Pre-S1[13]: "Is there a value for x that will make the following statement true? $(6x – 8 – 15x) + 12 > (6x – 8 – 15x) + 6$" Both interviewees, Talia and Pham, were 11th graders enrolled in Calculus.

For the excerpts used throughout the result chapters, I used the following conventions. Three spaced ellipsis points, …, within a sentence denotes omission of phrases. A pair of parentheses like, ( ), denotes a comment. A pair of square brackets, [ ], denotes an additional phrase.

Excerpt 01: Talia's initial response

Talia: Is there a value for x that will make the following statement true? Of course there is. Let see, umm.
Lim: Why did you say "of course, there is"?
Talia: Because, well, I figure there should be an answer to this problem, and, um, let's see, I was taught to combine like terms. I was taught this (>) is actually an equal sign.
Lim: OK.
Talia: To solve it like I would solve an equation. … (She obtained $-9x + 6 = -9x$ and then wrote $6 > 0$.) Umm, that doesn't [seem] right, because x has canceled out. What did I do wrong? … OK. Is there a value for x that will make the following statement true? Maybe there isn't.

Excerpt 02: Pham's initial response

Pham: OK. Let's see. The stuffs in the parentheses are the same. Umm, OK, first I guess I would combine all like terms. … (He got $-9x + 4 > -9x – 2$). Umm, now it's asking is there a value for x that will make the following statement true. Umm, let me see, I think 4 and -2, so you have a common term (i.e. $-9x$).

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[13] The code Pre-S1 stands for Pre-interview, Single-task Item 1. The code Post-T1 stands for Post-interview, Two-variable Item 1. The code TE3-$_C$N4 stands for Teaching Episode 3 for Chela, New item, Fourth item in the session. The code TE2-$_T$R1 stands for Teaching Episode 2 for Talia, Reuse-of-a-homework-task, First item in the session.

> OK, so it's, you have a -9, so anything [positive] that you multiply will [make it] a negative number, and this (+4) is positive. Let's see, yes, there is a value because… this, this [left] side will be greater. I guess, if it (-9*x*) was positive then, so is this side (-9*x*). So any negative number would make the statement true. … Umm, I think all numbers would make the statement true.

One difference between these two responses is that Pham arrived at the correct answer but not Talia. Another difference is that Talia's way of understanding of inequality is weaker than Pham's. Talia interpreted the inequality as a task to isolate x and treated it as an "equation", whereas Pham treated the inequality as a comparison of two algebraic expressions. A third difference is the manner in which they approach the problem. How can we characterize the thinking[14] that underlies the actions these two students took to solve this problem?

Both Talia and Pham combined like terms. Recall that within a Piagetian perspective, action presupposes anticipation. Therefore, we can assume that Talia and Pham had anticipated, or foresaw, the expediency of combining like terms. As previously stated, a way of understanding associated with foreseeing refers to the action one actually anticipates. Hence, both Talia and Pham are said to have the same way of understanding: combining like terms. Both of them were spontaneous in their foresight of combining like terms. However, the spontaneity in Talia's anticipation was characteristically different from that in Pham's. Upon seeing the problem, Talia immediately thought of what she could do to the inequality, rather than thinking about what the question was asking. Her act of anticipating had an element of impulsiveness, impulsive in the sense that she had

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[14] The thinking that underlies their actions is related to their understanding of inequality, and this relation is discussed in Chapter 5.

routinized a particular way of understanding (i.e., combining like terms is a routine for her to solve certain inequalities/equations). I categorized her way of thinking associated with foreseeing as *impulsive*[15] *anticipation*[16]. This way of thinking is inferred when a student immediately applies a procedure without considering its appropriateness.

Pham, on the other hand, noticed that “the stuffs in the parentheses are the same” and combined like terms with the probable intent of obtaining a simpler form. He might have predicted in his mind that the left side was always larger than the right side and was confirming his prediction. He seemed to have interiorized the usefulness of combining like terms and was capitalizing on his understanding that it would be easier to reason with simpler expressions. Thus his way of thinking was coded as *interiorized anticipation*. This way of thinking is inferred when a student spontaneously applies her or his interiorized way of understanding to a problem situation that is familiar to her or him. In order to have interiorized one’s way of understanding of a concept, one must have abstracted the way of understanding to the next level of understanding by reorganizing

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[15] I considered using the term *routinized anticipation*, which is a good contrast for *interiorized anticipation*. On the other hand, *impulsive anticipation* is a good contrast for *analytic anticipation*. I chose impulsive anticipation because impulsiveness is a characteristic that is more readily observed in students’ actions and statements. Moreover, being impulsive implies having algorithmatized a particular way of understanding, but the converse may not be true.

[16] I considered using the phrase *impulsive anticipative scheme*. From a Piagetian perspective, associated to a mental act is the activation of scheme(s). Since the anticipative scheme governing Talia’s act of anticipating results in an impulsive response, I could use the term impulsive anticipative scheme to describe her way of thinking associated with her anticipating. However, I chose to use impulsive anticipation to characterize her act of anticipating because it describes her anticipatory behavior and does not imply her possession of an impulsive anticipative scheme, which can be viewed as a conceptual tool. The analysis required for inferring one’s anticipative scheme is more demanding than for categorizing an act of anticipating. This is because one’s anticipative scheme cannot be inferred from a single act of anticipating. The analysis conducted in this study was insufficient to infer the scheme underlying the students’ mental acts. A direction for future research is to identify and characterize students’ anticipatory/predictive schemes.

one's conceptual structures, which one can then autonomously, but not necessarily spontaneously, apply to new situations.[17]

With respect to the mental act of predicting, Talia's impulsiveness is reflected in her prediction. Upon seeing the problem, she predicted "of course there is" because she figured that "there should be an answer to this problem." She seemed to have associated her having a procedure for isolating x with the inequality having a solution.[18] Because of this, I categorized her way of thinking characterizing her prediction as *associated-based prediction*. This way of thinking is inferred when a student predicts by merely associating two ideas without establishing the basis for making such an association. Talia's prediction of "maybe there isn't" upon observing the disappearance of x from the inequality is also considered association-based because she associated the disappearance of x with the nonexistence of a value for x that would make the inequality true.

Pham, on the other hand, did not explicitly make a prediction, so I cannot comment on his way of thinking associated with predicting. However, when anticipating, he reasoned with $-9x + 4 > -9x - 2$. His way of thinking associated with foreseeing is

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[17] If a student has internalized, but not interiorized, a particular way of understanding, then the student is able to autonomously and spontaneously apply a particular way of understanding in familiar situations but not in new situations because the student has not abstracted it to a higher level. For example, one who has internalized the quadratic equation in factored form can autonomously and spontaneously solve problems of the form $(ax + b)(cx + d) = 0$, whereas a student who has interiorized it would be able to that as well as to apply it to solve autonomously, but not necessarily spontaneously, novel equations such as $(3x + 8)(x - 2) = 5(3x + 8)$ and $(2x^2 - 8)(4 - y)^2 = 0$. I chose the term *interiorized anticipation* over *internalized anticipation* to emphasize that the internalization must be beyond being able to autonomously apply an algorithm or a routine.

[18] This association may be a consequence of her perceiving the didactical situation in the interview to be similar to that in her regular mathematics classroom. She might be thinking "there should be an answer to this problem otherwise you wouldn't have asked me." However, her engagement in the tasks and her verbalizing of her thoughts suggest that we had established a didactical contract by which she was supposed to put her best effort to solve the problem. So her responding to the problem situation in a manner that is similar to how she would respond in a regular classroom situation indicates her beliefs about learning mathematics.

considered *analytic anticipation* because he identified the goal of determining whether there is a value of *x* that will make the new inequality true, and anticipated the usefulness of reasoning with the common term -9*x*.

By comparing Talia's and Pham's initial responses to Item Pre-S1, I have introduced four ways of thinking: impulsive anticipation, interiorized anticipation, analytic anticipation, and association-based prediction. A total of five W_soT associated with foreseeing and three W_soT associated with predicting emerged from the data.

4.2 Ways of Thinking Associated with Foreseeing

The five ways of thinking associated with the act of foreseeing include impulsive anticipation, interiorized anticipation, analytic anticipation, explorative anticipation, and tenacious anticipation. Definitions and examples will be provided for each of theses ways of thinking.

Impulsive Anticipation

Impulsive anticipation is defined as the way of thinking in which one spontaneously proceeds with an idea that comes to mind, without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation. This way of thinking is inferred when a student rushes into a procedure or acts out the first idea that comes to mind. As discussed in the previous section, Talia's foresight of combining like terms is characterized as impulsive anticipation because she jumped into the equation-solving procedure.

When a student exhibits impulsive anticipation, the student does not seem to be "anticipating" because the student is merely acting out what comes to mind. However, according to Piaget, anticipation is presupposed in all actions. We may view impulsive anticipation as a case where the element of anticipation is negligible, just like an empty set is a set and a zero vector is a vector.

Interiorized Anticipation

Interiorized anticipation is defined as the way of thinking in which one spontaneously proceeds with an idea without having to analyze the problem situation because one has interiorized the relevance of the anticipated action to the situation at hand. Notice that both interiorized anticipation and impulsive anticipation are spontaneous in nature. The difference is that interiorized anticipation capitalizes on interiorized ways of understanding that are appropriate for the problem situation. For example, Pham's foresight of combining like terms was based on his understanding of the effectiveness of combining like terms, which facilitated his comparison of the two sides of the inequality.

Analytic Anticipation

Analytic anticipation is defined as the way of thinking in which one analyzes the problem situation and establishes a goal or a criterion to guide one's actions. It is inferred when a student attempts to understand the problem statement, studies the constraints, identifies a goal, imagines what-if scenarios, and/or considers alternatives. Pham's foresight of reasoning with the common term $-9x$ is characterized as analytic anticipation because it was goal-oriented in that it was aimed at comparing the functions, $-9x + 4$ and $-9x - 2$.

Explorative Anticipation

Explorative anticipation is defined as the way of thinking in which one explores an idea to gain a better understanding of the problem situation. It is inferred when a student performs an action to get a sense of the mathematical terrain of the problem situation, to test the usefulness of an idea to the situation at hand, to test one's prediction, or to explore different cases or numbers. For example, consider Raul's response to Item Pre-S1. Raul was an 11th grader taking Pre-calculus. Like Talia and Pham, Raul also anticipated combining like terms.

> Lim: Or, what are you thinking now?
> Raul: How to, how to solve this problem.
> Lim: Yeah, OK.
> Raul: If I, if I should combine the *x*'s first. I think I should. … (He eventually obtained $-9x > -9x + 10$.) ... You're supposed to add the 9 to this side, but it would cancel the *x*.

Unlike Talia and Pham, Raul did not spontaneously anticipate combining like terms. He was exploring the usefulness of combining like terms. As such, his foresight of combining like terms is categorized as explorative anticipation.

Tenacious Anticipation

Tenacious anticipation is defined as the way of thinking in which one maintains and does not re-evaluate her or his way of understanding of the problem situation in light of new information. The way of understanding in this case could be a prediction, a problem-solving approach, a claim, or a conclusion. Tenacious anticipation is inferred when a student encounters, or is presented with, new information but hold on to her or his way of understanding of the problem situation without considering alternative approaches and without considering that her or his claim might be false.

As an example of tenacious anticipation, consider Vito's response to Item Pre-S5: "Is the following statement always true, sometimes true, or never true? $(x + 1) + (x + 2) + (x + 3) + \ldots + (x + 99) + (x + 100) < 100x$". Vito was an 11th grader taking Pre-calculus. He predicted it was never true. He plugged in 2 for x to obtain $3 + 4 + 5 + \ldots + 101 + 102 < 200$ and commented "if you add these two (101 and 102), that's already 203. So it's always going to be greater." When challenged "you only consider one number though, x equals 2", he plugged in -2 for x. Focusing on the last two terms he got $195 < -200$. He predicted that any negative number would make it true and confirmed by plugging in -5 for x. When I changed the inequality to $(x + 98) + (x + 99) + (x + 100) < 100x$, Vito maintained his way of understanding that the last two terms sufficed to be greater than $100x$ and predicted "never true because this is the same as this. You're just adding right here, … just the last 2, and it would be, umm, greater than the right hand side." When challenged, he plugged in 0.5 for x, and that confirmed his prediction. Vito was so tenacious in his way of understanding that he seemed to choose numbers to support his prediction rather than to falsify it.

In this section, five ways of thinking associated with foreseeing have been presented. These ways of thinking can be divided into two groups based on their desirability from the perspective of mathematics education. Impulsive anticipation is undesirable because it minimizes one's engagement with the problem situation. Tenacious anticipation is undesirable because it tends to confine one's way of understanding. The remaining three ways of thinking are considered more desirable. Explorative anticipation and analytic anticipation require active engagement with the problem situation on the part of the problem-solver. Interiorized anticipation speeds up one's problem solving by capitalizing on one's interiorized ways of understanding. Identifying these ways of thinking provides a means for mathematics educators to characterize and communicate students' problem-solving.

4.3 Ways of Thinking Associated with Predicting

As discussed in Chapter 1, the terms *foreseeing* and *predictin*g are used to distinguish the two aspects of anticipation; namely the foresight of an action to accomplish something, and the prediction of a result prior to performing an action. In general, these two acts do not occur independently of one another. Foreseeing an action to accomplish something inevitably invokes an expectation on its accompanying result, and predicting a result necessarily involves mentally carrying out some operations to arrive at the result. Nevertheless, one of these aspects may dominate a particular behavior, and the aspect I examine depends on whether a person's primary goal is to come up with an action or to predict something.

Recall that what the student actually predicts is referred to as the way of understanding associated with the mental act of predicting, and the character underlying her or his prediction is referred to as her or his way of thinking. Three ways of thinking associated with predicting were identified from the data in this research: association-based prediction, coordination-based prediction, and comparison-based prediction. Definitions and examples will be provided for each of theses ways of thinking.

Association-based prediction

Association-based prediction is defined as the way of thinking in which one predicts a result or an answer by associating two ideas without establishing the basis for making such an association. It is inferred when a student's prediction is based on a mere association between the situation at hand and an idea that comes to mind without thinking about the appropriateness of the association. An example of this way of thinking is Talia's prediction that "of course there is" a value of x that would make $(6x – 8 – 15x) + 12 > (6x – 8 – 15x) + 6$ true. Her prediction was based on her association between her having a procedure and the inequality having a solution.

Coordination-based Prediction

Coordination-based prediction is defined as the way of thinking in which one predicts by coordinating quantities or attending to relationships among quantities. It is inferred when a student considers how the change in one entity affects that in another. Consider Pham's response to Item Pre-S3: "Is there a value for x that will make the following statement true? $1.2x + 3456 < 7 + 8.9x$". Pham predicted "yes, eventually" and explained "8.9 is so much bigger than [1.2], as long as it's bigger than the multiple (i.e., coefficient) of this ($1.2x$). The right side has a bigger multiple (coefficient) than the left side. So, eventually some numbers will make it larger than the left side." Pham's prediction was based on a dynamic comparison of the two functions where he coordinated the input value of x with the output values of $1.2x$ and $8.9x$.

The next example highlights an essential difference between coordination-based prediction and the third type of prediction, namely comparison-based prediction. Consider Ida's initial response to Item Pre-S1: "Is there a value for x that will make the following statement true? $(6x – 8 – 15x) + 12 > (6x – 8 – 15x) + 6$".

> Ida: Since they are both the same right here, I'm guessing that just because this has … plus 12 and this has a plus 6, no matter

> what value of x I put in, this will always be true because it has a greater [sign].

Ida's prediction was most likely based on her understanding that the output value of the function $6x - 8 - 15x$ changes as the input value of x changes. If that is the case, her prediction would involve coordination among the input-value of x, the output value of $6x - 8 - 15x$, the output value of the left side, and the output value of the right side. Hypothetically, if her response did not include the phrase "no matter what value of x I put in," and if she were interpreting $6x - 8 - 15x$ as a non-quantitative object rather than as a function, then her prediction of "always true" would be characterized as comparison-based prediction.

Comparison-based Prediction

Comparison-based prediction is defined as the way of thinking in which one predicts by comparing two elements or situations in a static manner. It is inferred when a student compares the values of two quantities without considering change and without coordinating them with other conditions or quantities. For example, Vito predicted that there was no value for x that will make $1.2x + 3456 < 7 + 8.9x$ true and explained "you are always going to add 3456, and this is higher than this one, than 7." Unlike Pham, he did not coordinate the change in value of x with the change in the value of $1.2x$ and that of $8.9x$. His prediction was based on a comparison of the two constant terms, so his prediction is characterized as comparison-based prediction.

The intricacy in Karen's prediction in Item Pre-S1 is worth discussing.

> Karen: (After failing to isolate x from the equation that she derived: $\frac{(-9x - 8) + 14}{-9} = \frac{-9x}{-9}$) … I have no idea.

Lim: Or can you predict just by looking at this [$(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$]?
Karen: No, it won't.
Lim: What do you mean it won't?
Karen: There won't be, I mean, this, umm, well this plus 12 and this plus 6, and they are the same here. So I, I'm guessing that this side will always be more than this side.
Lim: M-hmm.
Karen: So that this, but I don't know what, but I forgot how to make it true. But I think this side is always going to be more because there is 12 there, and I've 6 there. …
Lim: So you, based on that would still skip, or would choose an answer?
Karen: I would skip it because you need to know what value for it, makes this true … find the value of x that makes this true.

Karen's guess of the left side being greater than the right side was based on a static comparison between the two sides of the inequality. Unlike Ida, she treated the common expression $6x - 8 - 15x$ as a non-quantitative object. Her prediction that there would not be a value of x that would make the inequality true might be due to her interpreting the inequality as an equation. Her prediction seemed to be based on an association between the inequality having a solution and her getting "x is on one side and a number on the other side." If so, her prediction is coded as association-based prediction despite the fact that she performed a static comparison. Nevertheless, a code was not assigned to her prediction because she would not have made a prediction on her own accord.

In this section, three ways of thinking associated with predicting were presented. Association-based prediction is considered undesirable because making associations without considering their basis prevents students from making connections among, and gaining a deeper understanding of, the mathematical ideas involved. Comparison-based prediction seems undesirable because making static comparisons does not offer students the opportunity to practice mental coordination of quantities. Coordination-based prediction is considered desirable because students have the opportunity to practice mental coordination of entities such as relating the change of one quantity to that of another quantity. These three ways of thinking offer educators a means to communicate the sophistication in students' mathematical thinking.

Table 4.1 summarizes the eight categories that were developed from the data in this research. These categories may not be robust because they emerged from the analysis of a very small sample of students. Moreover, they are based on a single domain of mathematics, namely the domain of algebraic inequalities and equations. Further research is needed to refine and expand these categories.

Table 4.1: Definition for Ways of Thinking Associated with Foreseeing/Predicting

Category	**Definition**
Interiorized anticipation	Spontaneously proceeds with an idea without having to analyze the problem situation because one has interiorized the relevance of the anticipated action to the situation at hand
Analytic Anticipation	Analyzes the problem situation and establishes a goal or a criterion to guide one's actions
Explorative anticipation	Explores an idea to gain a better understanding of the problem situation
Tenacious anticipation	Maintains and does not re-evaluate one's way of understanding (prediction, problem-solving approach, claim, or conclusion) of the problem situation in light of new information
Impulsive anticipation	Spontaneously proceeds with an idea that comes to mind without analyzing the problem situation and without considering the relevance of the idea to the problem situation
Coordination-based prediction	Predicts by coordinating quantities or attending to relationships among quantities
Comparison-based prediction	Predicts by comparing two elements or situations in a static manner
Association-based prediction	Predicts by associating two ideas without establishing the basis for making such an association

4.4 Relation between Students' Ways of Thinking and Their Quality of Solution

The second part of the first research question states "Are students' ways of thinking related to the quality of their solutions?" To answer this question, 13 students' responses to two interview tasks were analyzed in terms of the ways of thinking that I have just described.

The two items used were Item Pre-T1 and Item Pre-S2. The problem statement for Item Pre-T1[19] is, "Given that $5a = b + 5$, which is larger: a or b?" There is a follow-up question to this item: "Given that $5a = b + 5$, can a and b be equal to each other?" The problem statement for Item Pre-S2 is, "Is there a value for x that will make the following statement true? $(2x - 6)(x - 3) < 0$". One reason for choosing these two items is the greater variety in students' responses for these two items as compared to the other items. A second reason is that they are substantially different tasks. Item Pre-T1 is a two-variable task, involves an equation, and requires students to compare the relative magnitude of the two variables. Item Pre-S2 is a one-variable task, involves an inequality, and requires students to realize that no values of x would make the inequality true.

Table 4.2 and Table 4.3 compare the 13 interviewees' responses to Item Pre-T1 and Item Pre-S2 respectively. Their ways of thinking associated with foreseeing, their ways of thinking associated with predicting, and their correctness of solution (Item Pre-T1) or quality of explanation (Item Pre-S2) were coded. A binary code is used for entries pertaining to ways of thinking. A "1" indicates that a particular way of thinking is present, and an empty box indicates its absence.

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[19] The phrasing of the problem—which is larger, *a* or *b*—does not suggest the possibility that either variable could be larger. So, interviewees who did not consider that possibility were asked if the other variable could be larger.

The 13 interviewees in tables 4.2 and 4.3 are listed in a certain order (with students exhibiting more sophisticated ways of thinking listed first) to facilitate the comparison between students' ways of thinking and the quality of their solutions/explanations. The list of interviewees in Table 4.2 include Pham, Raul, Jose, Quy, Chela, Ali, Talia, Ida, Bella, Maria, Elsa, Karen, and Vito. In the table, they are represented by the first letter of their name. The ordered list of interviewees in Table 4.3 is slightly different: Pham, Quy, Raul, Jose, Chela, Vito, Karen, Ali, Elsa, Bella, Talia, Ida, and Maria.

Table 4.2: Comparison Among 13 Interviewees Based on Their Response to Item Pre-T1

	Interviewee	**P**	**R**	**J**	**Q**	**C**	**A**	**T**	**I**	**B**	**M**	**E**	**K**	**V**
	Mathematics Course	C	C	C	C2	A2	PC	C	A2	C	A2	PC	PC	PC
	Grade in Mathematics Course	B	A	B	A	B	B	B	C	C	A	C	B	A
WoT Foreseeing	Interiorized anticipation	1	1	1	1									
	Analytic anticipation			1		1	1	1	1	1	1			
	Explorative anticipation	1	1	1	1	1	1	1	1	1	1	1	1	1
	Tenacious anticipation												1	1
	Impulsive anticipation													1
WoT Predicting	Coordination-based prediction	1		1		1	1			1				
	Comparison-based prediction			1	1	1	1	1	1					1
	Association-based prediction									1	1	1		1
	Correctness of solutions	2	2	1	1	1	0	1	1	1	0	0	0	0

A ternary code is used for "correctness of solution" category in Table 4.2. A "2" indicates that the interviewee gave correct answers to both the original question and the follow-up question, a “1” indicates a correct answer for the original question but an incorrect answer for the follow-up question, and a “0” indicates incorrect answers for both questions.

Table 4.2 suggests that the desirability of students’ ways of thinking is related to the correctness of their solutions. The two students who answered both parts of the problem correctly exhibited interiorized anticipation. Except for Ali and Maria, all the students who exhibited analytic anticipation answered the original question correctly. Except for Ali, all the students who exhibited coordination-based prediction answered the original question correctly. Except for Bella, all the students who exhibited association-based prediction answered both parts incorrectly. Two students exhibited tenacious anticipation and/or impulsive anticipation, and they answered both parts incorrectly. In summary, students who exhibited more desirable ways of thinking such as interiorized anticipation, analytic anticipation, and coordination-based prediction achieved better scores than students who exhibited less desirable ways of thinking such as tenacious anticipation, impulsive anticipation, and association-based prediction.

Table 4.3: Comparison Among 13 Interviewees Based on Their Response to Item Pre-S2

	Interviewee	P	Q	R	J	C	V	K	A	E	B	T	I	M
	Mathematics Course	C	C2	C	C	A2	PC	PC	PC	PC	C	C	A2	A2
	Grade in Mathematics Course	B	A	A	B	B	A	B	B	C	C	B	C	A
WoT Foreseeing	Interiorized anticipation	1	1	1										
	Analytic anticipation			1	1	1			1	1	1	1		
	Explorative anticipation	1	1	1	1	1	1	1	1	1	1	1	1	1
	Tenacious anticipation													
	Impulsive anticipation											1		
WoT Predicting	Coordination-based prediction			1	1	1	1	1						
	Comparison-based prediction										1			
	Association-based prediction			1								1		
Quality of explanation		2	2	1	1	1	2	1	2	0	0	0	0	0

For "quality of explanation" category in Table 4.3, a "2" indicates that the response sufficiently explains why the inequality is never true. An example is Quy's response: "if *x* is less than 3, then … both of these [factors] would be negative, which makes it positive, and it's still not true. And if *x* is greater than 3, it would make them all positive, and it would still not be true. So, there is no value for *x* that would make it true." A "1" indicates that the explanation is incomplete. An example is Jose's explanation: "[*x* being equal to] 3 is the closest that'll get to anything (the function) below 0, but it gets to 0 [not below 0]," but his assumption that the vertex of the parabola was at (3, 0) was empirically-based. A "0" indicates that no explanation is provided as to why there could be no values of *x* that would make the inequality true other than empirical results.

Table 4.3 suggests the existence of certain relationships between ways of thinking and quality of explanation. All the students who exhibited interiorized anticipation scored a “1” or “2”. Likewise, all the students who exhibited coordination-based prediction also scored a “1” or a “2”. However, only four out of seven students who exhibited analytic anticipation scored a “1” or a “2”. This suggests that analytic anticipation does not necessarily leads to success.

The relations betweens students’ ways of thinking and their quality of solution/explanation are discussed below and are substantiated by students’ responses to these two tasks. These responses are organized in terms of the relations between certain ways of thinking associated with foreseeing/predicting and certain aspects of their problem solving that lead to favorable solution/explanation.

Association-based Prediction is Related to the Non-referential Symbolic Way of Thinking

For Item Pre-T1 (involving $5a = b + 5$), Bella, Maria, Elsa, and Vito exhibited association-based prediction and did not arrive at a correct answer for either question. Bella, Elsa, and Vito initially predicted that a would be bigger than b because multiplication has a greater effect than addition. For example, Vito commented “I would say a because you’re multiplying by 5 and then right here you are just adding.” Their association of the arithmetic operation with the magnitude of the variable indicates that they did not attend to the meaning of the equal sign, which in this situation refers to the equivalence in value between the two sides.

Maria also did not attend to the equivalence characteristic of an equal sign. Upon incorrectly obtaining $a = b/5$, she predicted that “b is going to be bigger than 5 in order to divide." She associated *b* being greater than 5 with *b*/5 not being a proper fraction because she was focusing on the division on the right side instead of the equivalence between the two sides: "because if you do a smaller number, it's going to be a fraction, and *b* is going to be smaller than *a*."

For Item Pre-S2, Raul and Talia demonstrated association-based prediction. When Raul obtained $\frac{12 \pm \sqrt{8}i}{4}$ as the solutions for $(2x - 6)(x - 3) = -1$, he predicted that numbers in the neighborhood of 3 would make the equation true. His association-based prediction was a consequence of his way of understanding imaginary numbers, a pseudostructural conception (Sfard & Linchevski, 1994b) under which he worked with *i* without knowing/remembering the *primary process* that is encapsulated by the object *i*.

Talia's association-based prediction was a consequence of her lack of understanding of the quadratic formula. When Talia obtained 3 from the quadratic formula, she rejected 3 as a solution because $3 < 0$ is false. She associated $\frac{6 \pm \sqrt{6^2 - 4(1)(9)}}{2}$ with $(2x - 6)(x - 3)$; that is, she associated the root of a function with the output-value of the function. Her association-based prediction indicates that she was doing mathematics without attending to the referent or meaning of a symbol, which Harel (1998, in press c) labels the *non-referential symbolic way of thinking*.

Discussion. The data suggests a strong relation between association-based prediction and the non-referential symbolic way of thinking. For students to advance to referential symbolic reasoning, their non-referential symbolic tendency and/or pseudostructural conception must be allowed to surface. Hence, we should provide students opportunities to predict. Upon detecting students' association-based predictions, we can pose questions or offer activities that can help students address their inappropriate associations and thereby cultivate the habit of attending to meaning. The implementation of this instructional strategy is discussed further in Chapter 6.

Impulsive Anticipation is Related to the Forward-Strategy Approach

There were only two instances of impulsive anticipation: Vito for Item Pre-T1 and Talia for Item Pre-S2. This low frequency might be due to the characteristics of the tasks. The tasks were intentionally designed to minimize impulsive behaviors so that other anticipatory behaviors could be observed.

Vito demonstrated impulsive anticipation for Item Pre-T1: "Given that $5a = b + 5$, which is larger: *a* or *b*?"

> Vito: I don't know, because this (*a*) could be any number and this (*b*) one could be any number too. *a* could equal, let's say 4, and then *b* could equal 9. So then that's 20, and that's 9 plus 5 is 14. So then *a* would be larger. And then if I plug in the other way, *a* for 9, and *b* equals 4, you get 45 and then 5 plus 4, 9. So *a* … would be larger.

Vito spontaneously thought of plugging in numbers into the equation without attempting to understand the problem situation. His foresight of plugging in 4 for *a* and 9 for *b* is coded as impulsive anticipation because he acted out the first idea that came to his mind, that is, plugging in numbers to determine the answer.

Talia demonstrated impulsive anticipation for Item Pre-S2: "Is there a value for *x* that will make the following statement true? $(2x – 6)(x – 3) < 0$". Talia immediate response was "Yes. Let's see, I think we have to multiply out first." Her anticipation is considered impulsive because she spontaneously expanded the expression without studying the inequality. She then used the quadratic formula and obtained 3.

Discussion. Both Vito and Talia spontaneously applied a procedure without appearing to study the problem situation. Their impulsive anticipation is related to their having a procedure. Their problem-solving behavior can be characterized as a forward-strategy approach which, according to Sweller (1989), is used when schemas are available. Conversely, if a student engages in means-ends analysis, which is considered a backward-strategy approach, then the student is considered as exhibiting analytic anticipation. Forward-strategy and backward-strategy are considered problem-solving approaches, which are ways of thinking associated with the act of problem-solving.

Tenacious Anticipation is Related to Inflexible Reasoning

Karen demonstrated tenacious anticipation and scored a "0" for Item Pre-T1. She approached the problem by creating a table of values for a and b based on $5a = b + 5$. With these ordered pairs (2, 5), (5, 20), and (10, 45), she inferred that b was larger. When asked if a could be greater than b, she explored "what if a is 100, that ($5a$) would be 500, and that (b) would have to be four hundred and something. No, I don't think a can be larger than b." Her anticipation of plugging in 100 for a is considered explorative. When asked if a could be equal to b, she then considered plugging in 1 for a: "Well, at this point, this ($5a$) is 5, and this (b) has to be zero… Umm, so if this (a) is 3, it'll make that 15. … Nope, a can't be bigger than b." Surprisingly she overlooked that (1, 0) was a counter-example to her statement. Her anticipation of plugging in 3 for a is coded as tenacious anticipation because she used it to support her prediction and ignored the case that falsified her prediction.

On this same item, Vito was tenacious in his way of understanding of the situation: if *a* were larger than *b*, then 5*a* would be greater than *b* + 5. So when asked if *a* could be greater than *b*, he responded "no because you're multiplying, if you're multiply a larger number to 5, it will be a larger answer than, having *b* [plus 5], like what I did right here (where (9, 4) made 5*a* greater than *b* + 5)." His anticipation of using his previous example of plugging in (9, 4) is considered tenacious anticipation because he did not attempt to consider other possibilities. Similarly, when asked if *a* could be equal to *b*, he responded "ngm-mm, cause the same thing" and plugged in 5 for both *a* and *b* to communicate his point.

<u>Discussion</u>. Both Karen and Vito engaged in explorative anticipation. However, their "exploration" was constrained by their tenacious anticipation in that they did not attempt to change their way of understanding of the problem situation. Karen was so tenacious in her way of understanding that she overlooked a counter-example she had computed. Vito was so tenacious in his way of understanding that he provided supporting evidence without considering the possibility of being wrong.

Tenacious anticipation is an indication of students' lack of flexibility in their problem solving. When students engage in tenacious anticipation, they tend to do mathematics without an element of doubt. To counter tenacious anticipation, mathematics teachers should provide learning situations that foster desirable ways of thinking such as considering-alternatives and considering-falsity (i.e., attempting to "falsify" one's way of understanding).

Coordination-based Prediction is Related to Reasoning with Change

For Item Pre-T1, five interviewees demonstrated coordination-based prediction. For the follow-up question, Pham predicted that a could be equal to b because he was interpreting the problem graphically as involving two functions: "what I would do is I would just graph it. Treat this as one function. Treat that as a different, another function."

Jose, Chela, Ali and Bella predicted that b would be larger to compensate for the greater effect of multiplication in $5a$ than the addition in $b + 5$. Consider Ali's response.

> Ali: Um, b is larger, because anytime you multiply something it's going to be [larger], I guess. And you're comparing an addition [to] multiplication … the addition have to be bigger because, in this case, you're having 5 multiplied by a. In this case you have 5 added by b. So if you want to get them to be the same thing, then b will have to be larger.

Ali's prediction of b being larger is characterized as coordination-based prediction because it involves coordination and compensation. He saw that for $b + 5$ to be equal to $5a$, a larger value for b is needed to compensate for the greater effect of multiplication as compared to addition.

For Item Pre-S2, Raul, Jose, Chela, Vito and Karen exhibited coordination-based prediction. I shall use Chela's and Raul's responses to illustrate how their coordination-based prediction led them to an explanation. Chela predicted that the inequality $(2x - 6)(x - 3) < 0$ could not be true because she coordinated the input values of x and the product of the output value of each factor: "whatever you plug in … if this $(2x - 6)$ comes up positive, I think this one $(x - 3)$ will always come up positive. If this one is negative, it will come out negative. And it will still never be, err, less than zero."

Raul predicted that $2x^2 - 12x < -18$ represents a parabola after plugging in 1, 4, and 6 and finding the left side to be -10, -16, and 0, respectively. His prediction of a parabola is considered coordination-based because it was based on his observation that as x changes from 1 to 4 to 6 the output decreased from -10 to -16 and then increased to 0. This prediction led him to create a table for the function $(2x - 6)(x - 3)$ and sketch its graph, from which he then explained "it wouldn't go passed the 0, *x*-axis." These examples illustrate that coordination-based prediction tends to involve reasoning with change.

Discussion. A fundamental aspect of algebra is that it is a study of functions, relations, and joint variation (Kaput, 1999). The notion of function inevitably involves change, yet many algebra students tend to view function as an action involving manipulation of objects, such as plugging in an input value to get an output value, what Dubinsky and Harel (1992) would call the *action* conception of function. Opportunity for students to engage in reasoning that fosters change and coordination provides students with experiences that would help them in developing a *process* conception of function, which involves "imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under his or her control" (Dubinsky & Harel, 1992, p. 20). However, the development from action conception of function to process conception of function is beyond the scope of this research.

Analytic Anticipation Facilitates Problem Solving

Talia and Ida are the only students who arrived at the correct answer for Item Pre-T1 without exhibiting interiorized anticipation or coordination-based prediction. Their success may be attributed to their engagement in analytic anticipation and explorative anticipation. They identified the equation $5a = b + 5$ as a constraint to guide their plugging in of numbers. Talia's response is discussed in Chapter 6. Ida's response is discussed here.

> Ida: So I have to think of 2 variables that will make these two equations equal to themselves. So, if this was 5 (i.e. $b = 5$), 5 plus 5 equals 10, 5 times 2 equals 10. So if these are my variables, I have to say that *b* is larger. I can [try] with another one. 3, yeah. *b* is larger.
> Lim: *b* is larger. Can *a* be larger than *b*?
> Ida: Hmmm? Oh, let's see. (12 seconds elapsed) Yes it can. Like for example, -2 times 5 would give me -10. And then -15 plus 5 would give me -10. And -10 (probably meant -15) is smaller than -2.

Ida's foresight of plugging in 5 for *b* to determine the value of *a* is considered analytic anticipation because it was geared towards her goal of finding a pair of values that would make the equation true and then comparing the values. Her foresight of plugging -2 for *a* is considered explorative anticipation in that she was exploring if negative numbers would change the result. With analysis and exploration, Ida was able solve this problem correctly.

For Item Pre-S2, Raul, Jose, Chela, and Ali exhibited analytic anticipation and provided an explanation for why $(2x – 6)(x – 3) < 0$ has no solution. Their reasoning showed that they acted with purpose. Raul foresaw solving the equation $(2x – 6)(x – 3) = -1$ as a means to check if the parabola would go below the *x*-axis. Jose anticipated plugging in 1 for *x* to falsify his original prediction that "there is no value of *x* that will make the statement (x^2 - $x = \frac{19}{24}$, which is incorrectly derived from $(2x – 6)(x – 3) = -1$) true." Ali's foresight of plugging in a number smaller than 3 was aimed at making both factors negative, the product of which he mistakenly thought would be negative. Chela anticipated "a way where one of these has to be a negative, and one [positive]. I [want to] get a negative out of here ($2x - 6$) and a positive number out of here ($x - 3$)."

Discussion. Analytic anticipation facilitates problem solving in that it provides students with a sense of direction in their exploration. Analytic anticipation complements explorative anticipation in that it helps to guide one's exploratory actions.

Analytic Anticipation Does Not Ensure Success

For Item Pre-T1, Maria and Ali exhibited analytic anticipation but scored a "0". Maria provides an interesting case because she demonstrated analytic anticipation but scored "0" for both tasks. She analyzed the problem situations, but her weak foundation in algebra prevented her from making progress. Her response to Item Pre-T1 is provided below:

> Maria: Which is larger? *a* or *b*? OK. Well, there are two distinctive differences. I'm just thinking, errr, this is multiplying 5, this is adding 5. So, what number, two different numbers, how would you know that? It would depend on which number you put it, … but umm, I don't think I should simplify this one because, you can't sim-, you can but, like you can divide the 5, but then what would be the purpose of that? I mean, oh… they both equal to each other (got $a = b$ by canceling the two 5's in $\frac{b+5}{5}$) … if I simplify it, cause this goes into that (she probably meant one 5 goes into the other 5).

Maria studied the equation, noticed the different operations, and was aware of the variability in *a* and *b*. So her anticipating, and rejecting, the possibility of dividing both sides by 5 is considered analytic anticipation. However, her weakness in algebraic manipulation led her to consider canceling the two 5's and obtain $a = b$. Realizing her mistake, she went back to her original foresight of dividing both sides by 5, but she still did not preserve the solution set.

> Maria: Oh, no, no, no, it's not, I'm sorry, it will be that, cause that's divide by everything (got $a = b/5$ from $\frac{5a}{5} = \frac{b+5}{5}$ because she canceled the 5 in numerator $b + 5$ with the 5 in the denominator). So which one would be larger? Well, since this number was dividing. Let me rewrite it ($5a = b + 5$) so it's equal to b, or it could be that ($b = 5a - 5$), I'm just writing a different format.
>
> Lim: M-hm.
>
> Maria: Maybe to spark something in my head or something, comes back, so minus 5, it will be $5a$ minus 5 (wrote $b = 5a - 5$). So this one ($a = b/5$) we see division going on.

She then explored the fruitfulness of solving for b. Upon obtaining $b = 5a - 5$, she went back to reason with $a = b/5$, probably because the former involves two operations whereas the latter involves only division. Maria demonstrated analytic anticipation and explorative anticipation, but she was not successful because she manipulated symbols without attending to algebraic invariance. She was engaging in the non-referential symbolic way of thinking.

Ali also demonstrated analytic anticipation and scored a "0" for Item Pre-T1, even though he engaged in referential symbolic reasoning and considered falsity.

> Lim: So do you think a can be bigger than b?
>
> Ali: Ummm, let's see, it might, it might be bigger than b. Umm, if I were to do 5 times, errr, so you want, OK, a is to be bigger than b. … Arrr, I don't think [so] because right now I'm thinking … no, it can't.
>
> Lim: No, no, it can't.
>
> Ali: But then, I have a feeling that, you know there might, I, I'm thinking right now if I do something like errr, if I were [do] like 5 times 5, it's like 25. No.
>
> Lim: No.
>
> Ali: It's not going to happen because, you know, b is being added by 5. If 5 were a bigger number like, like 20 or something.

Lim: M-hmm.
Ali: If 5 were like 20 (i.e., $5a = b + 20$), then I could have said like B was 5.
Lim: Hmmn.
Ali: *a* was 5.

In the midst of exploring examples, Ali foresaw changing the equation to $5a = b + 20$ in order to illustrate that *a* could equal b if the constant term were 20 instead of 5. He might be thinking that, if the equation could not be changed, *b* would have to be larger than *a*. That way of understanding most likely stopped him from considering other possibilities further. Although Ali was analytic and goal-oriented in his reasoning, his conviction in his way of understanding prevented him from succeeding in this problem.

For Item Pre-S2, Elsa was one of the three students who exhibited analytic anticipation but scored a “0”. Her analytic anticipation complemented her explorative anticipation. Her choice of plugging 0.1 and -10 was aimed at making $x^2 – 12x + 18$ less than 0. From the results she obtained in her exploration, she explained why she initially thought positive numbers could not work: “squaring the numbers in there would just make it bigger. And adding 18, it would just make it bigger. But now that I see it, the $12x$ could have made it less than 0.” Her foresight that $12x$ could make it smaller led her to explore by plugging in 1, 2, 3, and 4 for *x*. She obtained 8, 2, 0, and 2 respectively as the output values for $x^2 – 12x + 18$. However, she was surprised by the directional change in the output value when she plugged in 4 for *x*: “that’s weird, because this number was going down, and then it went back up again.” Her lack of success was due to her lack of structural understanding for quadratic functions. I conjecture that if she were to have reasoned with the factored form, her engagement in analytic anticipation and explorative anticipation would have led her to an explanation for the non-existence of a solution.

Discussion. Analytic anticipation is a desirable way of thinking because it involves reasoning with the problem situation, studying the constraints, identifying a goal, thinking ahead, making connections, and/or considering hypothetical situations. However, it does not necessarily lead to success in problem solving if a student reasons in a non-referential symbolic manner. Also, ways of understanding that are necessary for solving the problem may not emerge from students' anticipations alone. Collaborative group work or teacher intervention may be necessary to facilitate the emergence of certain ways of understanding. This instructional principle is related to Vygotsky's (1978) notion of *zone of proximal development*, which stipulates that a student can attain certain ways of understanding under the guidance of a teacher, or in collaboration with peers, that could not be achieved alone.

Interiorized Anticipation Provides Efficiency in Problem Solving

Pham and Raul answered the follow-up question in Item Pre-T1 correctly and exhibited interiorized anticipation. Both of them spontaneously foresaw the use of a graphical approach. Consider Pham's response to the follow-up question: "Given that $5a = b + 5$, can a and b be equal to each other?"

> Pham: Yeah, I would think so because, um, really what I would do is I would just graph it. Treat this as one function.
> Lim: M-hmm.
> Pham: Treat that as a diff-, another function and …
> Lim: How would you graph, you know, if you want to share with [me].
> Pham: *y* equals 5, let's just say *y* equals 5, 5*x*. This one, *b* plus 5. So *x* plus 5.
> Lim: Ar-huh.
> Pham: Um, … we want to know when these two ($5x$ and $x + 5$) are equal. Wait, we are saying it equal already. Oh, OK, yeah, OK. Um, *a* and *b*, right here when you are talking about *a*, in your question, *a* and *b* are like different numbers right? Or they can

be [the same], so I'm just going to make it a common variable 5*x* equals 5 plus *x*.

Pham spontaneously foresaw graphing the two functions as $y = 5x$ and $y = x + 5$. For a moment, he was not sure if he could assign the same letter x to both variables a and b. He soon realized that he wanted to make them equal to each other, and foresaw solving $5x = x + 5$ and obtained 1.25 for x. His foresight of graphing the two functions and of solving $5x = x + 5$ for x are both considered interiorized anticipation because he had reified the primary processes of graphing into graphs, and of equating to two functions into equation. These *structural conceptions* (Sfard, 1991), which are necessarily interiorized, helped make his problem solving efficient.

For Item Pre-S2, Pham, Raul, and Quy exhibited interiorized anticipation. Quy's understanding of factored form enabled her to anticipate an explanation: "if *x* is less than 3, then … both of these [factors] would be negative, which makes it positive, and it's still not true. And if *x* is greater than 3, it would make them all positive, and it would still not be true. So, there is no value for *x* that would make it true." Students who lack such understanding of factored form would have to engage in explorative anticipation, analytic anticipation, and/or coordination-based prediction. For example, Karen explored by plugging 1 and -2 for *x*, and then predicted that no values of *x* would make the inequality true. She coordinated the negative input values of *x*, the output value of each factors, and the product.

Karen: If *x* is a negative number, you are subtracting, so, the [result] of the first one will be negative, and the [result] of the second one would be negative, and a negative times a negative would be positive which is going to be greater than 0. So, that can't be true.

She foresaw that the same reasoning would hold for a positive small number. Unlike Quy, Karen's understanding of the structure of the function had to emerge from analyzing the results that she obtained from her exploration by plugging in numbers.

Discussion. Interiorized anticipation is an indication of mastery of a particular concept. A "problem" ceases to be a problem for someone who has mastered the mathematics related to the problem. On the other hand, one may exhibit interiorized anticipation for certain parts of the task but not all. In that case, interiorized anticipation makes problem solving more efficient without trivializing it.

Although interiorized anticipation is a desirable goal for students, it should not be an immediate cognitive objective for instruction. Since interiorized anticipation is something that students will automatically exhibit once they have interiorized the ways of understanding for solving a certain class of problems, we should help students cultivate desirable ways of thinking, such as analytic anticipation, coordination-based prediction and considering falsity, that can empower them to develop those ways of understanding through a variety of problem-solving situations.

Explorative Anticipation is a Part of Problem Solving

Interestingly, all the interviewees exhibited explorative anticipation in both items. I have three hypotheses for this phenomenon. One hypothesis is that the interviewees participated in this research with an understanding that they were expected to share their reasoning as they were solving a problem. This might influence them to engage more with the tasks in the interview than they normally would in classroom situations. Another hypothesis is that the interview problems were not typical textbook tasks and were phrased in a manner that made sense to them, which allowed them to explore their ways of understanding of the problem situation.

The third hypothesis is that explorative anticipation is an inevitable part of problem-solving. If that is the case then explorative anticipation seems to be a trivial category. However, explorative anticipation may be absent in situations where the task is no longer a "problem" for the student. For example, Quy could solve $1.2x + 3456 < 7 + 8.9x$ (Item Pre-S3) by isolating x so efficiently that she was not interested in predicting the answer, even when asked. Also, explorative anticipation may be absent in situations where the student is not engaged with the problem. In this case, the category of explorative anticipation may be useful in classroom situations to differentiate students who are engaged in a problem from those who are not.

As a category of ways of thinking associated with foreseeing, explorative anticipation does not convey information about the quality of students' problem solving. To characterize students' problem solving, we will have to study students' mental act of exploring and focus on their ways of understanding and ways of thinking associated with exploring. That is beyond the scope of this research.

4.5 The Relevance of These Ways of Thinking to Mathematics Education

The primary objective of this research is to develop categories of ways of thinking associated with the mental acts of foreseeing and predicting. Ways of thinking associated with foreseeing provide mathematics educators with the vocabulary to communicate the way students approach a problem: whether they (a) hastily apply a procedure, (b) are tenacious in their way of understanding, (c) explore different ideas, (d) analyze the problem situation and identify a goal, and (e) spontaneously apply their ways of understanding that are pertinent to the problem situation. These descriptions correspond to impulsive anticipation, tenacious anticipation, explorative anticipation, analytic anticipation, and interiorized anticipation. An awareness of these categories can help mathematics teachers to be more explicit about their goal of advancing students from being impulsive and tenacious to being analytic and explorative.

Instruction that leads students to predict can counteract students' tendency of rushing to apply procedures when they are assigned a problem. Having explicit terms to characterize the ways students predict allows teachers to differentiate desirable ways of thinking associated with predicting from less desirable ones. I have suggested that coordination-based prediction is desirable because it promotes reasoning that involves change and coordination, while association-based prediction is undesirable because it tends to foster the non-referential symbolic way of thinking. Having made these distinctions explicit, mathematics educators can design and implement instructional activities that aim to help students progress from association-based prediction to coordination-based prediction.

CHAPTER 5: RELATING STUDENTS' WAYS OF THINKING ASSOCIATED WITH FORESEEING/PREDICTING WITH THEIR WAYS OF UNDERSTANDING INEQUALITIES/EQUATIONS

How a student understands a problem situation affects the action the student foresees and/or the result the student predicts. This may subsequently modify the student's understanding of the problem situation. Therefore, the mental acts of foreseeing and predicting are related to the mental act of interpreting. The second research question for this study focuses on the relationships between ways of thinking associated with the mental act of foreseeing/predicting and ways of understanding associated with the mental act of interpreting, particularly students' interpretations of inequalities and equations.

This chapter begins by revisiting the comparison of Talia's and Pham's responses for Item Pre-S1 in order to highlight the relations between their ways of thinking associated with foreseeing/predicting and their ways of understanding inequalities/equations. This is followed by definitions for, and examples representing, the five ways of understanding inequalities/equations that were identified in this study. These categories are then used to extend the previous discussion of the 13 interviewees' responses to Item Pre-T1 and Item Pre-S2. Finally, the relationships between their ways of thinking associated with foreseeing/predicting and their ways of understanding inequalities/equation are explored.

5.1 Revisiting the Comparison of Two Students' Responses

In Chapter 4, Talia's and Pham's responses to Item Pre-S1 were compared. The problem statement for Item Pre-S1 is, "Is there a value for x that will make the following statement true? $(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$".

Recall that Talia responded "Of course there is … there should be an answer to this problem, and, um, let's see, I was taught to combine like terms. I was taught this (>) is actually an equal sign. To solve it like I would solve an equation." As discussed previously, Talia's foresight of combining like terms is characterized as impulsive anticipation and her prediction of "of course there is" is characterized as association-based prediction. She seemed to interpret the inequality as a signal to isolate x and treated it as an "equation". Her impulsive anticipation and association-based prediction appeared to be consequences of her interpreting the inequality as a signal to do something.

Pham, on the other hand, noticed that "the stuffs in the parentheses are the same," combined like terms with the intent of obtaining a simpler form, obtained $-9x + 4 > -9x - 2$, noticed the constant terms, and focused on the common term $-9x$.

> Pham: Umm, let me see, I think 4 and -2, so you have a common term (i.e., $-9x$). OK, so it's, you have a -9, so anything [positive] that you multiply will [make it] a negative number, and this (+4) is positive … this, this [left] side will be greater. I guess, if it ($-9x$) was positive then, so is this side ($-9x$). So any negative number would make the statement true.

Pham's reasoning with $-9x$ suggests that he was interpreting it as a function[20] whose output depends on the input variable x. It is likely that his analytic anticipation of

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[20] Pham's interpretation of -9x as a function was supported by his response to another item where he could sketch the graph by reasoning with the structural attributes of $(2x - 6)(x - 3)$ and reason with its graph: "[it can't] go below the x-intercept because … like comes down, hit the x-axis and comes right back up."

reasoning with the common term $-9x$ was supported by his interpreting the inequality as a comparison between two functions.

The above discussion suggests that students' ways of thinking associated with foreseeing/predicting are related to their ways of understanding inequalities/equations. The categories of ways of thinking associated with foreseeing/predicting were presented in Chapter 4. The categories for ways of understanding inequalities/equations are presented below.

5.2 Ways of Understanding Inequalities/Equations

Based on my analysis of the data, I identified a total of five ways of understanding inequalities/equations (I/E). They are I/E-as-a-signal-for-a-procedure, I/E-as-a-constraint, I/E-as-a-proposition, I/E-as-a-static-comparison, and I/E-as-a-comparison-of-functions. Definitions and examples are provided for each of these ways of understanding.

I/E-as-a-signal-for-a-procedure Interpretation

The *inequality/equation-as-a-signal-for-a-procedure* interpretation is defined as a way of understanding in which one interprets an inequality/equation as a signal to do something such as isolating the variable, plugging in numbers, or applying a procedure. This interpretation is inferred when a student treats the inequality/equation as an object to be worked on. For example, the inequality $(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$ was a signal for Talia to isolate x.

I/E-as-a-constraint Interpretation

The *inequality/equation-as-a-constraint* interpretation is defined as a way of understanding in which one interprets an inequality/equation as a condition that constrains the values the variable(s) can take. This interpretation is inferred when a student plugs in certain numbers to make the inequality/equation true or false. For instance, compare Chela's and Vito's responses to Item Pre-T2: "p and q are odd integers between 20 and 50. For these values, is $5p - q > 2p + 15$ always true, sometimes true or never true?" Chela's choice of 21 for q was aimed at making the inequality true: "I'm trying to minus a number that's less than what I'm going to get here ($5p$). So I don't take so much from this, so it can be bigger than that." Vito, on the other hand, plugged in numbers to test three possible cases: (25, 35) for $p < q$, (45, 25) for $p >$ q, and (33, 33) for $p = q$. He did not capitalize on the structure of the inequality to guide his choice of numbers. Vito did not interpret the inequality as a constraint, whereas Chela did.

The I/E-as-a-constraint interpretation is also inferred when a student constructs an inequality or an equation to constrain the variability of variable(s). Consider Raul's response to Item Pre-T3: "Given that m is greater than n, can $m - 14$ ever be equal to $7 - n$?"

> Raul: So I'll just set up an equality, *m* minus 14 equals 7 minus *n*. … I was trying to make both of them equal to 0. If *m* was 14, it would be zero, and, if *n* was 7, it would be 0 as well. Alright, so, given that *m* is greater than *n*, can *m* minus 14, yes. I think yes.

From $m - 14 = 7 - n$, Raul thought of equating both sides to 0 and found a pair of values for m and n, namely (14, 7) to work. Using the same strategy, he obtained another two pairs, (15, 6) and (16, 5), that would work. He also found that (10, 11) would work for the condition “given that m is less than n.” When the condition was changed to “given that m equals n,” he spontaneously predicted that is “maybe, in between, 10.5 and 10.5”. Raul’s use of the equation to constrain the values of m and n is an indication of his I/E-as-a-constraint interpretation.

I/E-as-a-proposition Interpretation

The *inequality/equation-as-a-proposition* interpretation is defined as a way of understanding in which one conceives an inequality/equation as a proposition whose truth value depends on the input value(s) of its variable(s). For example, Chela’s response, “I would try to plug in numbers to see if it works,” indicates that she was interpreting the inequality $(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$ as a proposition. This interpretation was observed most frequently in Item Pre-S5 and Item Pre-T2 because these items require students to determine whether the inequality is always true, sometimes true, or never true.

I/E-as-a-comparison-of-functions Interpretation

The *inequality/equation-as-a-comparison-of-functions* interpretation is defined as a way of understanding in which one interprets an inequality/equation as a comparison of expressions that are conceived as either input-output processes that involve dynamic transformation of quantities, or as *reified* objects (Sfard, 1991) that encapsulate the input-output process. These two descriptions correspond to Dubinsky and Harel’s (1992) *process conception* of function and *object conception* of function respectively.

For example, consider Jose’s response to Item Pre-S3: “Is there a value for x that will make the following statement true? $1.2x + 3456 < 7 + 8.9x$”.

Jose: Oh. Um, I would think the statement is true, because this (1.2x) is, um, increasing by a very small amount. … At a certain point, this (8.9x), this is much bigger increase in value.
Lim: M-hmm.
Jose: Or a bigger slope. So, the x will be, [8.9x] increase much more rapidly. So at [some] point, it will cross this equation (7 + 8.9x).
Lim: Mmm, mmm.
Jose: If I draw these graphs, this equation (1.2x + 3456) will be going at a constant slope. And this, and this equation (7 + 8.9x) will be going, shooting up like that (he drew a steeper line).

Based on Jose's comment—"[1.2x] is increasing by a small amount … At a certain point, this (8.9x), this is much bigger increase in value"—his way of understanding $1.2x + 3456 < 7 + 8.9x$ is coded as I/E-as-a-comparison-of-two-functions. Likewise, Pham's way of understanding the inequality $-9x + 4 > -9x – 2$ is also considered an inequality-as-a-comparison-of-functions interpretation.

I/E-as-a-static-comparison Interpretation

The *inequality/equation-as-a-static-comparison* interpretation is defined as a way of understanding in which one interprets an inequality/equation as a comparison of non-varying entities such as numerical values or arithmetic operations. This interpretation is inferred when a student is reasoning with particular instances that involve specific input values. For example, Vito compared the two sides of $5p – q > 2p + 15$ by plugging in pairs of numbers, such as (25, 35), (45, 25), and (33, 33). His comparisons are considered static because he did not attend to the change in the output values in relation to the change in input values.

Although arithmetic operations are mathematical functions, a comparison of two arithmetic operations is considered static rather than dynamic when the operations are treated as actions to be performed on numbers rather than as input-output processes.

Consider Bella's prediction to the problem, "Given that $5a = b + 5$, which is larger: a or b?" Her prediction that "a is larger because you're multiplying" is considered I/E-as-a-static-comparison because she was merely comparing the multiplication in $5a$ with the addition in $b + 5$ without interpreting $5a$ and $b + 5$ as functions.

Determining whether a student's interpretation is I/E-as-a-comparison-of-functions or I/E-as-a-static-comparison may be challenging at times. For instance, consider Karen's reasoning for Item Pre-S2. She first explored by plugging 1 and -2 into $(2x - 6)(x - 3) < 0$. She then predicted that no values of x would make the inequality true. Her prediction is considered coordination-based prediction because she coordinated the negative input values of x, the output value of each factor, and the product.

> Karen: If x is a negative number, you are subtracting here, so the product (result) of the first one (factor) will be negative, and the product (result) of the second one (factor) would be negative, and a negative times a negative would be positive which is going to be greater than 0.

From her response alone, it is unclear whether her "negative number" was based on the single instance of $x = -2$ or a non-specific negative number. When she extended her reasoning to a "positive number", she seemed to be thinking about the specific instance of $x = 1$. If that is the case, her way of understanding the equation $(2x - 6)(x - 3) < 0$ would be coded as equation-as-a-static-comparison.

> Karen: And if you put a positive number in here, the same thing is going to happen, you're going to get, cause you are subtracting, you are going to get two negative numbers multiplied together, and that's going to be positive, which is greater than 0. And if you put like 3 in, for the second part $(x - 3)$, you get 0, and anything times 0 is 0. That's equal 0 but it's not less than 0.

It is unlikely that her reasoning was based on a generic positive number whose value may vary in the interval (0, 3) because if she did she would have noticed that $x = 3$ would also make the factor $2x - 6$ zero. This conjecture was confirmed by her subsequent lack of conviction: "I'm thinking that there might be one number because I've only tried a couple numbers." This example highlights the intricacy in differentiating between comparison of functions and static comparison. This example also highlights that coordination-based prediction is not tightly coupled with the I/E-as-a-comparison-of-functions interpretation, that is, it can also be based on the I/E-as-a-static-comparison interpretation.

Table 5.1 summarizes the five categories of ways of understanding inequalities/equations that were developed from the data in this research. They are listed in order from most sophisticated to least sophisticated. The next section addresses the relation between the sophistication in students' ways of understanding inequalities/equations and the desirability in their ways of thinking associated with foreseeing/predicting.

Table 5.1: Definitions for Ways of Understanding Inequalities/equations

Category	**Definition**
I/E-as-a-comparison-of-functions	As a comparison between its two sides with a process conception of function.
I/E-as-a-constraint	As a condition that constrains the values the variable(s) can take
I/E-as-a-proposition	As a proposition whose truth value depends on the input value(s) of its variable(s)
I/E-as-a-static-comparison	As a comparison between its two sides in an unchanging manner
I/E-as-a-signal-for-a-procedure	As a signal to do something such as to isolate the variable, to plug in numbers, or to apply a procedure

5.3 Results on Interviewees' Ways of Understanding Inequalities/Equations

Students' ways of thinking associated with foreseeing and predicting for two interview items, namely Pre-T1 and Pre-S2, were discussed in Chapter 4. Table 5.2 and Table 5.3 are extensions of Table 4.2 and Table 4.3, into which ways of understanding inequalities/equations are inserted.

Table 5.2: Comparing Interviewees' Response to Item Pre-T1 in Terms of Ways of Thinking (WoT) and Ways of Understanding (WoU)

	Interviewee	**P**	**R**	**J**	**Q**	**C**	**A**	**T**	**I**	**B**	**M**	**E**	**K**	**V**
	Mathematics Course	C	C	C	C2	A2	PC	C	A2	C	A2	PC	PC	PC
	Grade in Mathematics Course	B	A	B	A	B	B	B	C	C	A	C	B	A
WoT Foreseeing	Interiorized anticipation	1	1	1	1									
	Analytic anticipation			1		1	1	1	1	1	1			
	Explorative anticipation	1	1	1	1	1	1	1	1	1	1	1	1	1
	Tenacious anticipation												1	1
	Impulsive anticipation													1
WoT Predicting	Coordination-based prediction	1		1		1	1			1				
	Comparison-based prediction			1	1	1	1	1	1					1
	Association-based prediction									1	1	1		1
WoU Inequalities/Equations	I/E-as-a-comparison-of-functions	1	1	1										
	I/E-as-a-constraint	1	1	1	1	1	1	1	1	1	1		1	1
	I/E-as-a-proposition											1		
	I/E-as-a-static-comparison					1	1	1	1	1	1	1		1
	I/E-as-a-signal-for-a-procedure										1			1
Correctness of solutions		2	2	1	1	1	0	1	1	1	0	0	0	0

Table 5.3: Comparing Interviewees' Response to Item Pre-S2 in Terms of Ways of Thinking (WoT) and Ways of Understanding (WoU)

	Interviewee	**P**	**Q**	**R**	**J**	**C**	**V**	**K**	**A**	**E**	**B**	**T**	**I**	**M**
	Mathematics Course	C	C2	C	C	A2	PC	PC	PC	PC	C	C	A2	A2
	Grade in Mathematics Course	B	A	A	B	B	A	B	B	C	C	B	C	A
WoT Foreseeing	Interiorized anticipation	1	1	1										
	Analytic anticipation			1	1	1			1	1	1	1		
	Explorative anticipation	1	1	1	1	1	1	1	1	1	1	1	1	1
	Tenacious anticipation													
	Impulsive anticipation											1		
WoT Predicting	Coordination-based prediction			1	1	1	1	1						
	Comparison-based prediction										1			
	Association-based prediction			1								1		
WoU Inequalities/Equations	I/E-as-a-comparison-of-functions	1	1	1	1				1	1				
	I/E-as-a-constraint		1	1	1	1			1	1	1	1	1	
	I/E-as-a-proposition													
	I/E-as-a-static-comparison					1	1	1			1		1	1
	I/E-as-a-signal-for-a-procedure				1		1					1		
Quality of explanation		2	2	1	1	1	2	1	2	0	0	0	0	0

One difference between the two tables is that there are more instances of I/E-as-a-constraint in Table 5.2 than in Table 5.3, while there are more instances of I/E-as-a-comparison-of-functions in Table 5.3 than in Table 5.2. I/E-as-a-constraint seems to be more pertinent for Item Pre-T1, whereas I/E-as-a-comparison-of-functions seems to be more pertinent for Item Pre-S2. This suggests that the ways of understanding that students enact are influenced by the nature and characteristics of the task.

It is interesting to study the consistency/inconsistency, for each student, between the two items in each category. For example, Pham exhibited coordination-based prediction in Item Pre-T1 but not in Item Pre-S2, and exhibited I/E-as-a-constraint in Item Pre-S2 but not in Item Pre-T1. Repeating the same procedure for all 13 interviewees and counting the number of mismatches for each category, the following results were found: 1 for interiorized anticipation (i.e., 1 mismatch out of 13), 4 for analytic anticipation, 0 for explorative anticipation, 2 for tenacious anticipation, and 2 for impulsive anticipation; 6 for coordination-based prediction, 8 for comparison-based prediction, and 6 for association-based prediction; 3 for I/E-as-a-comparison-of-functions, 5 for I/E-as-a-constraint, 1 for I/E-as-a-proposition, 4 for I/E-as-a-static-comparison, and 3 for I/E-as-a-signal-for-a-procedure. On average, there are 1.8 mismatches (14%) for ways of thinking associated with foreseeing, 6.7 mismatches (51%) for ways of thinking associated with predicting, and 3.2 mismatches (25%) for ways of understanding inequalities/equations. These figures suggest that ways of thinking associated with foreseeing are usually more “stable” than ways of thinking associated with predicting.

Surprisingly, few relations between the sophistication of ways of understanding and the quality of solutions can be found. In both tables, students who exhibited I/E-as-a-comparison-of-functions, with the exception of Elsa, scored at least a “1”. In Table 5.2, both students who exhibited I/E-as-a-signal-for-a-procedure scored a “0”, but in Table 5.3, only one out of the three students who exhibited I/E-as-a-signal-for-a-procedure scored a “0”. These observations suggest that the quality of solutions is more related to ways of thinking associated with foreseeing/predicting than ways of understanding inequalities/equations. This hypothesis is not surprising because a desirable way of understanding inequalities/equations without effective means to explore and analyze a problem situation is unlikely to lead to success. On the other hand, possessing effective means to explore and analyze a problem situation can help one to replace undesirable way of understanding inequalities/equations with a more appropriate one.

5.4 Relation between Students' Ways of Thinking Associated with Foreseeing/ Predicting and Their Ways of Understanding Inequalities/Equations

To determine whether students' ways of understanding inequalities/equations are related to their ways of thinking associated with foreseeing/predicting, each occurrence of students' way of understanding were examined to investigate possible relations with ways of thinking. Table 5.4 highlights the relations between the sophistication in students' ways of understanding inequalities/equations and the desirability in their ways of thinking associated with foreseeing/predicting. Relations to somewhat neutral categories, such as explorative anticipation, comparison-based association, and inequality-as-a-proposition, were not included in Table 5.4.

Table 5.4: Relating Desirability of WoT and Sophistication of WoU

		More Sophisticated WoU				**Less Sophisticated WoU**			
		I/E-as-a-Comparison-of-Functions		I/E-as-a-Constraint		I/E-as-a-Static-Comparison		I/E-as-a-Signal-for-a-Procedure	
		Pre-T1	Pre-S2	Pre-T1	Pre-S2	Pre-T1	Pre-S2	Pre-T1	Pre-S2
More Desirable WoT	Interiorized Anticipation	Jose Pham Raul	Pham Quy	Pham Raul	Quy				
	Analytic Anticipation		Ali Elsa	Ali Chela Ida Maria Talia	Ali Bella Chela Elsa Ida Jose Raul Talia	Maria		Maria	
	Coordination-based Prediction		Jose Raul	Ali Bella Chela			Karen Vito		
Less Desirable WoT	Association-based Prediction					Bella Elsa Vito			Talia
	Tenacious Anticipation			Karen		Vito			
	Impulsive Anticipation							Vito	Talia

Four specific relations were identified: (a) I/E-as-a-signal-for-a-procedure is related to association-based prediction and impulsive anticipation; (b) I/E-as-a-constraint is related to analytic anticipation; (c) I/E-as-a-comparison-of-functions is related to coordination-based prediction; and (d) interiorized anticipation involves I/E-as-a-comparison-of-functions or I/E-as-a-constraint.

I/E-as-a-signal-for-a-procedure is Related to Association-based Prediction and Impulsive Anticipation

Only Talia and Vito exhibited impulsive anticipation, and both of them exhibited the I/E-as-a-signal-for-a-procedure interpretation. Talia's association-based prediction is also related to her I/E-as-a-signal-for-a-procedure interpretation.

Talia approached Item Pre-S2 in the same manner as she had approached Item Pre-S1. Her initial response was, "Is there a value of *x* that will make the following statement true? Yes. Let's see, I think we have to multiply out first." She expanded the factors without considering the structure of the inequality. Her prediction of "yes" and her impulsive anticipation of expanding the factors seemed to be a consequence of her interpreting the inequality $(2x – 6)(x – 3) < 0$ as a signal for her to manipulate symbols. Upon obtaining $x^2 – 6x + 9 < 0$, she used the quadratic formula and obtained $\frac{6 \pm \sqrt{6^2 - 4(1)(9)}}{2}$. She commented, "that reduces to 3, which is less than 0 (wrote $3 < 0$). That's not true." Her rejection of 3 as a solution was based on her associating the root of a function with the output-value of a function; that is, associating $\frac{6 \pm \sqrt{6^2 - 4(1)(9)}}{2}$ with $(2x – 6)(x – 3)$. This association was a consequence of her failure to attend to the meaning of the symbols and her procedural orientation, both of which are related to her I/E-as-a-signal-for-a-procedure interpretation.

Similarly, Vito exhibited I/E-as-a-signal-for-a-procedure interpretation and was not attending to the meaning of the symbols in Item Pre-T1: “Given that $5a = b + 5$, which is larger: *a* or *b*?” Vito interpreted the inequality as an object into which values were plugged: “*a* could be any number and this (*b*) one could be any number too. *a* could equal, let’s say 4, and then *b* could equal 9. So then that’s 20, and that’s 9 plus 5 is 14. So then *a* would be larger.” His foresight of plugging in 4 for *a* and 9 for *b* is coded as impulsive anticipation because he acted out the idea that came to his mind: plugging in numbers to determine the answer. His inference/prediction that “*a* would be larger” is association-based in that he associated the output of the functions with the value of the variable. His conflation of $5a$ and $b + 5$ with *a* and *b* respectively was due to his non-referential symbolic way of thinking, which is generally related to the I/E-as-a-signal-for-a-procedure interpretation.

Discussion. The I/E-as-a-signal-for-a-procedure interpretation tends to result in impulsive anticipation because when a student interprets an inequality or equation as a signal to do something, such as to isolate the variable or to plug in numbers, the student is more likely to rush into actions than to analyze the problem situation. In addition, the I/E-as-a-signal-for-a-procedure interpretation tends to result in association-based prediction. When a student interprets an inequality as task, the student tends to focus on the procedure rather than the meaning, and when the student reasons in a non-referential symbolic manner, the student is more likely to engage in association-based reasoning.

I/E-as-a-constraint is Related to Analytic Anticipation

For Item Pre-S2, eight students (Ali, Bella, Chela, Elsa, Ida, Jose, Raul, Talia) exhibited the I/E-as-a-constraint interpretation and analytic anticipation. This interpretation allowed them to be goal-oriented in their reasoning. For example, it allowed Raul to formulate and solve $(2x - 6)(x - 3) = -1$ as a means to check if the parabola would go below the x-axis. In also allowed Jose to construct the equation $2x^2 - 12x = -19$ from which he thought he could find a value of x that would satisfy $2x^2 - 12x + 18 < 0$.

Chela's reasoning was aimed at making the inequality true: "I'm trying to find a number that will make this one $(2x - 6)$ positive and this one $(x - 3)$ negative." After exploring by plugging in numbers, she deduced "if this comes up positive, I think this one will always come up positive. If this one is negative, it will come out negative." Her goal of trying to make $(2x - 6)(x - 3)$ negative presupposed her interpreting the inequality $(2x - 6)(x - 3) < 0$ as a constraint which values of x must satisfy.

The I/E-as-a-constraint interpretation influenced Talia to plug in 2 for the x in the second factor $(x - 3)$ so as to make $(2x - 6)(x - 3)$ negative. Representing the first factor $2x - 6$ by X, she obtained $(X)(-1)$ and thought that $x = 2$ could make the inequality $(2x - 6)(x - 3) < 0$ true. She overlooked the x in the first factor $2x - 6$ and treated it as a variable instead of assigning to it the same value of 2. Nevertheless, her foresight of plugging in 2 is considered goal-oriented and is characterized as analytic anticipation. While her initial interpretation of inequality as a task, as discussed previously, contributed to her impulsive anticipation, her subsequent interpretation of inequality as a constraint facilitated her analytic anticipation.

For Item Pre-T1, the I/E-as-a-constraint interpretation also contributed to some students' (Ali, Chela, Ida, Maria, and Talia) analytic anticipation. Ida's interpretation of $5a = b + 5$ as a constraint allowed her to reason in a goal-oriented manner: "I have to think of 2 variables (values) that will make these two equations (functions) equal to themselves (each other)." Ida's strategy was to plug in a value for *b* into the equation to constrain the value of *a*, and Maria used this same strategy.

Chela and Talia used a different strategy. Their strategy was to assign a particular value to both $5a$ and $b + 5$ and use the two constraints, say $5a = 10$ and $b + 5 = 10$, to obtain values for *a* and *b*. This strategy presupposed interpreting $5a = b + 5$ as a constraint. In contrast, Elsa's plugging in (4, 5) and (5, 4) was not aimed at satisfying $5a = b + 5$. Failing to make the equation true with those values, she incorrectly inferred that "neither of them has to be larger."

<u>Discussion</u>. The examples above suggest that the I/E-as-a-constraint interpretation facilitates students' goal-orientation in their reasoning, which is an attribute of analytic anticipation. Interpreting an inequality/equation as a constraint tends to focus a student's attention on the inequality/equation, and/or leads the student to foresee actions that would make the inequality/equation true or false. Such foresight is considered analytic anticipation.

I/E-as-a-comparison-of-functions is Related to Coordination-based Prediction

Raul and Jose exhibited the I/E-as-a-comparison-of-functions interpretation and coordination-based prediction while attempting Item Pre-S2. Jose's interpretation of the equation $x^2 - x = \frac{19}{24}$ as a dynamic comparison allowed him to coordinate the change in the input value of *x* with the change in the output value of the function $x^2 - x$. He predicted that "there is no value of x that will make the statement true because … as x increases the x squared increases and, so it (the difference) doesn't go below 1 (1 being greater than 19/24)." He soon realized "oh, when x equals 1, it may work."

The I/E-as-a-comparison-of-functions interpretation allowed Raul to predict the behavior of the function $(2x – 6)(x – 3)$ from the results that he had obtained by plugging numbers into $2x^2 – 12x < -18$: "Maybe this will be a parabola. It doesn't go less than 0." His prediction was considered coordination-based because it was based on his observation that the output value for $2x^2 – 12x$ decreased from -10 to -16 and increased to 0 as x increased from 1 to 4 and then to 6. His conceiving $2x^2 – 12x < -18$ as a dynamic comparison between the function $2x^2 – 12x$ and -18 allowed him to relate its down-then-up behavior with the parabolic behavior of $(2x – 6)(x – 3)$.

Discussion. The I/E-as-a-comparison-of-functions interpretation supports reasoning that involves change and coordination, which are features of coordination-based prediction. However, I/E-as-a-static-comparison may suffice to allow coordination-based prediction that does not involve change. An example of this (seen earlier) is Karen's coordination of the negative input values of x, the output value of each factor, and the product of the two output values in $(2x – 6)(x – 3)$.

Interiorized Anticipation Involves I/E-as-a-comparison-of-functions or I/E-as-a-constraint

Pham and Raul exhibited interiorized anticipation, I/E-as-a-comparison-of-functions, and I/E-as-a-constraint while attempting Item Pre-T1. They conceived the situation as a comparison of two functions, foresaw the use of a graphical approach, and solved $5x = x + 5$ to determine the value when a and b are equal each other. Consider Raul's response:

> Raul: I think if, if this was a graph, both of these would be linear. And, that's just be, say y equals $5x$ and this one is going to be y equals b plus 5, so x plus 5. Um, wherever these two cross, it would be where a is equal to b. … If I can graph these two and see if there is a point where they crossed … (gesturing to use the graphing calculator)
>
> Lim: M-hmm, so you need a calculator. What if we are not allowed to use the calculator, are you able to find that, without using a calculator?
>
> Raul: May be if I set these equal to each other? … (He wrote and solved $5x = x + 5$) … So x is five-fourths.

Raul's interiorized anticipation of comparing the two functions graphically presupposed his conceiving the equation $5a = b + 5$ as a comparison between two functions. His formulation of the equation $5x = x + 5$ presupposed an understanding of equation as a constraint.

For Item Pre-S2, Pham and Quy exhibited interiorized anticipation and the I/E-as-a-comparison-of-functions interpretation. Pham spontaneously commented that "if it [$(2x – 6)(x – 3) < 0$] wasn't true, then the whole graph would be above the axis." Pham's graphical reasoning presupposes the I/E-as-a-comparison-of-functions interpretation. Quy, on the other hand, foresaw reasoning with cases: "If x is less than 3, then … both of these [factors] would be negative, which makes it positive, and it's still not true. And if x is greater than 3, it would make them all positive, and it would still not be true." Her reasoning with an interval rather than specific numbers suggests that she was interpreting $(2x – 6)(x – 3)$ as a function.

Discussion. Interiorized anticipation is generally associated with more sophisticated ways of understanding because interiorized anticipation, by definition, requires one to be able to not only autonomously and spontaneously apply one's ways of understanding in a familiar situation, but also to apply them in novel situation. Less desirable interpretations, such as I/E-as-a-signal-for-a-procedure and I/E-as-a-static-comparison, are usually internalized but not interiorized. Hence, spontaneous foresights associated with such interpretations tend to be impulsive anticipations rather than interiorized anticipations. In general, one's existing ways of understanding determine one's interpretation of the problem situation, which in turn influences one's foresight and/or prediction.

To recapitulate, less sophisticated ways of understanding are related to less desirable ways of thinking associated with foreseeing/predicting. Conversely, more sophisticated ways of understanding are related to more desirable ways of thinking. The I/E-as-a-signal-for-a-procedure interpretation tends to lead to impulsive anticipation, and it tends to foster a non-referential symbolic way of thinking from which association-based prediction results. The I/E-as-a-constraint interpretation facilitates goal-oriented reasoning, which is an attribute of analytic anticipation. The I/E-as-a-comparison-of-functions interpretation supports reasoning that involves change and coordination. Such reasoning tends to promote coordination-based prediction. Interiorized anticipation has to be supported by sophisticated ways of understanding, such as I/E-as-a-constraint and I/E-as-a-comparison-of-functions.

CHAPTER 6: CHANGE IN STUDENTS' WAYS OF THINKING

In the previous two chapters, categories for ways of thinking associated with foreseeing/predicting and ways of understanding inequalities/equations were introduced. In this chapter, I focus on the change in students' ways of thinking associated with foreseeing/predicting. The third research question for this study asks, "What is the potential for advancing students' ways of thinking through an instructional intervention informed by DNR-based instruction?" To answer this question, I will provide an in-depth discussion of one learner. Talia was chosen for this case study because she demonstrated the most improvement.

This chapter is organized into seven sections: (a) Talia's improvement from the pre-interview to the post-interview; (b) Talia's transition from manipulating symbols to reasoning with symbols; (c) probable factors in the teaching intervention that could account for Talia's improvement; (d) difficulties Talia encountered in the teaching intervention; (e) ways of thinking and ways of understanding of three other learners; (f) two interesting phenomena identified from this analysis; and (g) a recapitulation of the main points.

6.1 Talia's Pre-interview and Post-interview Comparison

At the time of this study, Talia was an 11^{th} grade high school student enrolled in Calculus. In the pre-interview, like most of the students in the study, Talia began by working on items S1 and S2. However, she spent 30 minutes on these two items alone. Most of the other students in this study took approximately 60 minutes to complete the following sequence of items: S1, S2, S3, T1, T2, S5, T3, and T4 (see Appendix C for the complete list of items from which interview tasks were selected). To make sure that Talia had an opportunity to work on items T1 and T2 in the remaining 30 minutes of her pre-interview, I deviated from the standard sequence and directed her attention to items T1 and T2. After completing these, she then worked on Item S3. Item S5 was replaced by Item S6 because I anticipated that she would spend too much time trying to apply formulas related to arithmetic progression to solve Item S5 (since she did not attend to meaning when she worked on S1 and S2). Talia spent an additional 15 minutes on Item T3. In summary, for her pre-interview, Talia completed S1, S2, T1, T2, S3, S6 and T3 in approximately 75 minutes.

For the post-interview, all four learners completed the same sequence of tasks—S2, T1, T2, S5, T3, and T4—within 60 minutes. Therefore, there are only four tasks that were common in both interviews for Talia: S2, T1, T2, and T3.

Table 6.1 compares Talia's pre-interview and post-interview results in terms of ways of thinking associated with foreseeing, ways of thinking associated with predicting, ways of understanding inequalities/equations, and quality of solution/explanation. The same coding scheme used in previous tables, such as Table 4.2 and Table 5.2, is used in Table 6.1. For ways of thinking and ways of understanding, a "1" indicates presence and an empty box indicates absence. For quality of solution/explanation, a "2" indicates a correct solution/explanation, a "1" indicates a partially correct solution/explanation, and a "0" indicates an incorrect solution or inappropriate/no explanation.

Table 6.1: Pre-and-post Comparison of Talia’s Response to Interview Items

		Pre-Interview				Post-Interview			
	Interview Item	**S2**	**T1**	**T2**	**T3**	**S2**	**T1**	**T2**	**T3**
WoT Foreseeing	Interiorized anticipation								
	Analytic anticipation	1	1	1	1	1	1	1	1
	Explorative anticipation	1	1	1	1	1	1	1	1
	Tenacious anticipation								
	Impulsive anticipation	1							
WoT Predic ting	Coordination-based prediction					1	1	1	1
	Comparison-based prediction		1				1		1
	Association-based prediction	1							
WoU Inequalities/ Equations	I/E-as-a-comparison-of-functions					1	1		1
	I/E-as-a-constraint	1	1	1	1	1	1	1	1
	I/E-as-a-proposition			1				1	1
	I/E-as-a-static-comparison		1		1				
	I/E-as-a-signal-for-a-procedure	1			1				
	Quality of solution/explanation	0	1	2	1	2	1	2	2

According to Table 6.1, Talia’s appeared to improve in several ways:

(a) impulsive anticipation in the pre-interview, but not in the post-interview;

(b) association-based prediction in the pre-interview, but coordination-based prediction in the post-interview; and (c) the I/E-as-a-signal-for-a-procedure and inequality-as-a-static-comparison interpretations in the pre-interview, but the I/E-as-a-comparison-of-functions interpretation in the post-interview. In terms of the quality of her solutions, Talia demonstrated an improvement for two of the four tasks.

In the discussion that follows, I present Talia’s improvement in terms of (a) ways of thinking associated with foreseeing, (b) ways of thinking associated with predicting, and (c) ways of understanding inequalities/equations. Although the improvements are presented separately, they are interrelated. At the end of this section, Talia’s improvements are discussed in terms of the *sub-contexts* (Cobb, 1985) in which she was operating.

Improvement in Ways of Thinking Associated with Foreseeing

As shown in Table 6.1, Talia exhibited impulsive anticipation in the pre-interview but not in the post-interview. Although she exhibited analytic anticipation in all four items in both interviews, she had a greater disposition to be analytic in the post-interview than in the pre-interview. Her responses for Item S2 and Item T3 are used to illustrate her impulsiveness in the pre-interview and her tendency to be analytic in the post-interview.

Impulsive anticipation in the pre-interview

Talia exhibited impulsive anticipation in the pre-interview, but not in the post-interview, on Item S2: "Is there a value for x that will make the following statement true? $(2x - 6)(x - 3) < 0$" In the pre-interview, Talia was impulsive in that she expanded the expression without studying the inequality, used the quadratic formula, obtained $\frac{6 \pm \sqrt{6^2 - 4(1)(9)}}{2}$, and commented "that reduces to 3, which is less than 0 (wrote $3 < 0$). That's not true."

In the post-interview, Talia analyzed the inequality with the goal of making the function $(2x - 6)(x - 3) < 0$ less than zero, and foresaw the sub-goal of making one factor positive and one factor negative.

> Talia: Um, $2x$ minus 6 times x minus 3 is less than 0. So … this [side] has to give me a negative number. I can get a negative number from here $(2x - 6)$, oh, but there is also a negative times negative is positive. So I have to make one of these negative and one of these positive. In order to get this, so this will be negative if it is less than 6, but then if I want to make this one positive, it has to be greater than 3. So, or I could go the other way around. … This side could be, umm, greater than 6, x could be greater than 6, makes this positive, $2x$, I'm sorry, $2x$ [could be greater than 6]. And x could be less than 3, which will make this negative, and so these two conditions will make this statement true.

The contrast in Talia's initial responses between pre-interview and post-interview, as depicted in Figure 6.1, highlights the substantial improvement in terms of ways of thinking associated with foreseeing. She was impulsive and procedure-oriented in the pre-interview, but was analytic and goal-oriented[21] in the post-interview. This improvement is not considered trivial because this inequality seemed unfamiliar to Talia. Only 2 out of the 16 inequalities/equations (see Appendix D) that were used in the problem-solving sessions were quadratic inequalities in factored form. Moreover, both of them, $x(6x + 8) < 0$ and $3x(500 – 2x) < 30(500 – 2x)$, do not involve repeated roots.

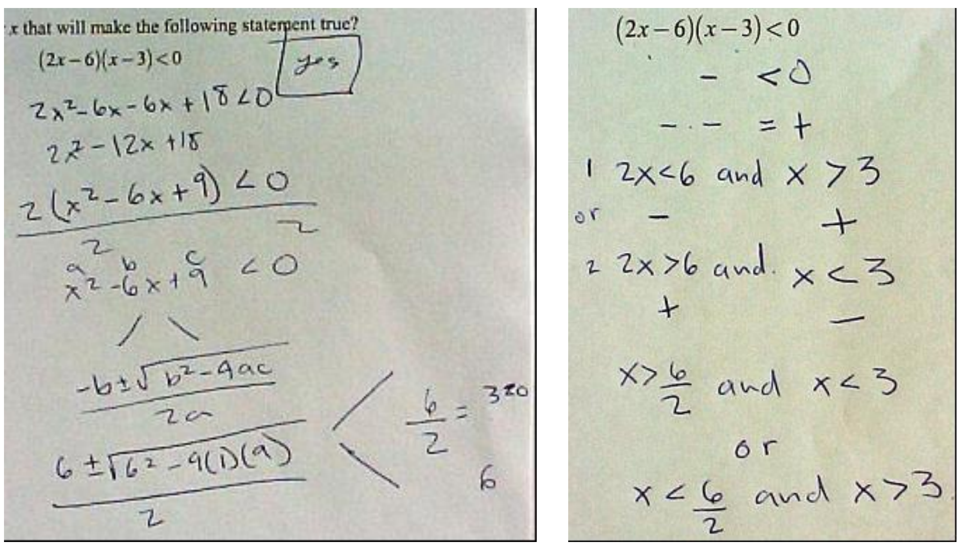

Figure 6.1: Pre- and post-interview comparison of Talia's initial work for Item S2

<u>Analytic anticipation in the post-interview</u>

Talia had a greater disposition to engage in analytic anticipation in the post-interview than in the pre-interview. For example, she did not engage in analytic anticipation in her first three approaches when she worked on Item Pre-T3: "Given that *m*

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[21] Procedure-oriented and goal-oriented can be viewed as features of the forward strategy and the backward strategy (Sweller, 1989) respectively. Both strategies are problem solving approaches, which are ways of thinking associated with the act of problem solving. The backward strategy is more desirable because it involves means-ends analysis, whereas the forward strategy does not.

is greater than n, can $m - 14$ ever be equal to $7 - n$?" She first approached the problem by plugging in 3 and 1 for m and n respectively prior to studying the characteristics of the two functions. She then considered manipulating symbols "I wanted to solve for one of the variables and I plug it back into the equation, but I remember this one problem (Item T1) that I did, when I did that I got 0 equals 0." She foresaw the use of a graphical method in her third approach: "I just thought about graphing both of them and finding where they intercept. And um, see what m and n is. But I think this is a different situation." Her foresights of plugging in numbers, of solving for one of the variables, and of graphing the two functions are considered explorative anticipation. However, in these cases, Talia was merely considering ideas and did not investigate further. In her fourth approach, Talia attended to the constraints of the problem: "Can m minus 14 equal 7 minus n. So this is a big number minus 14. Can that be equal to 7 minus a small number? … (she wrote "big # – 14 = 7 – small #") … Um, let me try making these two problems (functions) equal 28." Her analytic anticipation of equating both functions to 28 did not occur until after she had explored three different ways.

When attempting Item Post-T3, Talia engaged in analytic anticipation sooner. She began by comparing the two functions and predicted $7 - n$ was less than $m - 14$.

> Talia: Let's see, here you're subtracting 14, but this number (14) is always bigger than that number (7). So, this one, 7 minus n, is always less than that one ($m - 14$) … What if I look at what m is, hold on, m is equal to 14 plus 7 … (she wrote $m = 21 - n$) … I don't think that really helps me… OK, let me try plugging in numbers.

Having explored by plugging in 5 and 2 into m and n respectively and obtained -9 = 5, Talia continued to compare the two sides: "Here we are subtracting by 14. But here we are we are adding by 7. But it's a negative number there (-*n*)." She wrote, "$m > n$, $m - 14 = -n + 7$" and reiterated her goal: "I need to find 2 numbers that'll make this true, these two equal." She then equated each function to 20 and found that $m = 34$ and $n = -13$. Talia arrived at the same strategy in both interviews. The difference is that she engaged in analytic anticipation sooner in the post-interview than in the pre-interview.

In the post-interview, Talia's tendency to analyze the problem situation led her to a successful solution for the follow-up task: Given that $m = n$, can $m - 14$ equal $7 - n$?

> Talia: Right now, nothing comes to mind. … What if I put in 0? … (obtained $-14 = 7$) I don't know, but I am always subtracting 14 here, and I'm always adding 7 here. But this is a negative *n*. If these two numbers are the same, and I'm subtracting 14, but I'm adding 7 to a negative number. Hey, why don't I just solve for *a*? … (She solved $a - 14 = -a + 7$ and obtained $2a/2 = 21/2$) Why did I solve for *a*? Because, oh I, I set them to each other, and then, I look for a value for *a* that would make these two equal, is that what I did? Yeah, that's what I'm trying to do.

Her foresight of using the common symbol *a* is considered analytic anticipation because it emerged in the midst of her reasoning with the two constraints. Her pausing to check her goal, "why do I solve for *a*?", is an indication that she was analyzing her work.

In contrast, in the pre-interview, her analytic anticipation and explorative anticipation were in the sub-context of manipulating symbols or plugging in numbers. For example, she considered and rejected an idea that she had tried in Item Pre-T1: "solve for *m*, solve for *n*, set them equal to each other. Then you end up with the same thing, I think." She then explored by plugging in numbers: "Well, I know it can't be 0. Can't be 1, but what number could it be? Let me just make them (*m* and *n*) equal [to 50]… 36, -43, those aren't equal." She guessed, "I don't think there is something that will make these two equal … cause look how different these two ($m - 14$ and $7 - n$) are. This (36) is like bigger and this one (-43) is way smaller." Her reasoning was based on empirical evidence and perceptual differences. The proof scheme underlying her justification was the *empirical proof scheme* (Harel & Sowder, 1998).

To recapitulate, Talia demonstrated a substantial improvement from impulsive anticipation to analytic anticipation for a single-variable task. For a two-variable task, she demonstrated a greater tendency to engage in analytic anticipation in the post-interview than in the pre-interview. In addition, her analytic anticipation and explorative anticipation in the post-interview were generally more sophisticated because she was reasoning with symbols in a deductive manner, as compared to her earlier reasoning with numbers in an inductive or empirical manner.

Improvement in Ways of Thinking Associated with Predicting

Talia made more predictions in the post-interview than in the pre-interview, as depicted in Table 6.1. In terms of quality of prediction, she exhibited coordination-based prediction in the post-interview but not in the pre-interview. Moreover, she exhibited association-based prediction in the pre-interview but not in the post-interview. Her responses for items T2, T1, and S2 substantiate these observations.

<u>Making more predictions in the post-interview</u>

Talia made predictions in the post-interview, but not in the pre-interview, for Item T2: "p and q are odd integers between 20 and 50. For these values, is $5p - q > 2p + 15$ always true, sometimes true or never true?" Talia arrived at the same solution in both interviews. Her success in the pre-interview was a consequence of her engagement in explorative anticipation and analytic anticipation. She explored by plugging in (21, 23) and (23, 21) into $5p - q > 2p + 15$ and observing the results 82 > 57 and 94 > 61 respectively. She then compared the two sides: "5*p* has to be smaller than this *q* so that this number (94) can be less and that this number (61) can be bigger. … I'm trying to make this ($5p - q$) smaller. So, let me go [to] the extremes." She plugged in (21, 49) and found that it made the inequality false.

In the post-interview, Talia also engaged in analytic anticipation and explorative anticipation, but she made predictions and "checked" them from the start.

> Talia: [It] should sometimes be true. Um, this, because this number (5*p*) is bigger than this one (2*p*), wait, you are subtracting *q*, and I don't know this value, and this value (*q*) is probably between these two numbers (20 and 50). But still, this (5*p*) should be really big. This (2*p*) should be somewhat big but you are adding (15) to it. Um… I think this ($5p - q$) outweighs this one ($2p + 15$) though.

Based on this response, it was not clear whether Talia's initial prediction involved coordination. If it did, then it would be coded as coordination-based prediction; otherwise, comparison-based prediction. Her inference/prediction that $5p - q$ outweighs $2p + 15$ is definitely coordination-based because she coordinated 5*p*, 2*p*, *q*, and 15. Having plugged in (21, 23) and (23, 29) to verify her prediction, she concluded that it was always true. When she was asked if she would believe a student who said that he had a pair of numbers that would make the equation false[22], she responded "maybe, because I didn't try 49 over here, like 49 for *q*." She explored by plugging in (23, 49) and obtained $66 > 61$. She considered plugging in (21, 49) and predicted that it would still make it true: "It ($5p - q$) will still be bigger than this ($2p + 15$) because this (2*p*) is pretty small here, and you adding a really small quantity. You're adding something but it's not, I don't

[22] This strategy was employed whenever the interviewee had arrived at a conclusion and had the potential to lose momentum in thinking. The use of this strategy is an indication that the interviewee did not attempt to consider alternatives or to falsify her or his way of understanding of the problem situation.

think it's big enough to make a difference over here." Although she maintained[23] her way of understanding that $5p - q$ was larger, she reasoned with the symbols and demonstrated coordination-based prediction. Although she had not cultivated *considering falsity* as a way of thinking associated with problem-solving, she showed improvement in her ways of thinking associated with predicting. She predicted more often in the post-interview, and her predictions were coordination-based.

<u>Coordination-based prediction in the post-interview</u>

Talia exhibited coordination-based prediction in the post-interview, but comparison-based prediction in the pre-interview, for Item T1: "Given that $5a = b + 5$, which is larger: *a* or *b*?" Talia exhibited only one instance of prediction in the pre-interview. She predicted that, "if *a* and *b* were equal, then *a* would be larger because, I mean this ($5a$) value would be larger." This prediction is comparison-based prediction because she was comparing the two sides in terms of their arithmetic operations.

In the post-interview, Talia demonstrated one instance of comparison-based prediction and two instances of coordination-based prediction. She exhibited comparison-based prediction when she was asked if *a* and *b* could equal each other: "I don't think they can because, you're multiplying here and you are adding here." An instance of coordination-based prediction was observed when she predicted that "*b* will have to be larger, just because you need more adding than you do multiplying in order to get [$b + 5$] large." Her prediction incorporates change and compensation. A second instance occurred when she predicted that "they (*a* and *b*) probably could be equal to each other if

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[23] Her anticipating of plugging (21, 49) would make the inequality true is not coded as tenacious anticipation because she did re-evaluate her way of understanding of the problem situation.

they were big[ger] fractions." Her prediction was based on her plugging in 1/5 for both *a* and *b* and obtaining 1 = 26/5 for $5a = b + 5$, and her plugging in 4/5 for both *a* and *b* and obtaining 20/5 = 29/5 for $5a = b + 5$. The reasoning underlying her prediction involves an element of change and coordination.

> Talia: Because here ($5a$) you are multiplying and, so it's growing.
> Lim: M-hm.
> Talia: But it's [also] reducing (because of the denominator) at the same time, and this ($b + 5$) will always [be] growing. So may be this one ($5a$) could catch up to this one ($b + 5$).

Talia's prediction that a bigger fraction would probably work is characterized as coordination-based because she coordinated the multiplication of 5 by a fraction (the numerator increases the product, but the denominator decreases the product) with the adding of 5 to the fraction. However, she did not know how to find that fraction: "I'm thinking there could be, but I just don't know how to get it." These two instances of coordination-based prediction suggest Talia's reasoning in the post-interview was generally more "dynamic" than in the pre-interview. By dynamic, I mean it involves an element of change.

<u>Association-based prediction in the pre-interview</u>

Talia demonstrated association-based prediction in the pre-interview, but not in the post-interview, for Item S2: "Is there a value for *x* that will make the following statement true? $(2x - 6)(x - 3) < 0$". While working on this item, Talia demonstrated three instances of association-based prediction in the pre-interview. She predicted that there was a value for *x* that would make $(2x - 6)(x - 3) < 0$ true. Her prediction was based on her associating the inequality having a solution with her having a procedure for solving the inequality for *x*. Recall that in the pre-interview, she applied the quadratic formula and obtained 3 from $\frac{6\pm\sqrt{6^2 - 4(1)(9)}}{2}$. She predicted that 3 was not a solution because she saw that $3 < 0$ was false. Her prediction was based on her associating the result of $\frac{6\pm\sqrt{6^2 - 4(1)(9)}}{2}$ with the output of $x^2 - 6x + 9$. When she plugged in 3 for x into the inequality and obtained $0 < 0$, she predicted that 6 might a solution: "Maybe I'm supposed to multiply by 2." She doubled the resultant value of 3 because she thought she should compensate for the halving of $2(x^2 - 6x + 9) < 0$ to get $x^2 - 6x + 9 < 0$. Her prediction was again based on her associating the resultant value of $\frac{6\pm\sqrt{6^2 - 4(1)(9)}}{2}$ with the output of value $x^2 - 6x + 9$. She essentially conflated the root of a quadratic function with the output value of the function. In the post-interview, Talia did not exhibit association-based prediction because she attended to the meaning of the symbols.

To recapitulate, Talia predicted more often in the post-interview than in the pre-interview, and showed improvement from association-based prediction in the pre-interview to coordination-based prediction in the post-interview. This improvement is related to her improvement in her ways of understanding inequalities/equations.

Improvement in Ways of Understanding Inequalities/Equations

Table 6.1 highlights Talia's improvement from I/E-as-a-signal-for-a-procedure and I/E-as-a-static-comparison to I/E-as-a-comparison-of-functions. Her responses to items S2, T1 and T3 are used to explicate her improvement.

While attempting Item Pre-S2, Talia interpreted the inequality $(2x - 6)(x - 3) < 0$ as a signal for applying taught procedures. In Item Post-S2, she interpreted the inequality as a comparison of functions because her comparison was not based on specific input values for *x*, but rather, it was based on her coordinating the two linear factors.

While attempting Item Pre-T1, Talia interpreted the equation $5a = b + 5$ as a comparison between the two sides in terms of their arithmetic operations. Talia's interpretation was coded as I/E-as-a-static-comparison. While attempting Item Post-T1, her coordination-based predictions, as discussed previously, presupposed an I/E-as-a-comparison-of-functions interpretation because she allowed the values of $5a$ and $b + 5$ to vary.

While attempting Item Pre-T3, Talia exhibited the I/E-as-a-signal-for-a-procedure interpretation. For example, she equated the two functions as $m – 14 = 7 – n$ and commented, "That's what I'm going to be using to solve whatever it does allow me to solve for." She considered and rejected solving for one of the variables and plugging it back into the equation. She also considered "graphing both of them and finding where they intercept." Her tendency to be procedure-oriented is reflected in her comment: "I could probably do the same method (equating both functions to a particular number). But I'm wondering if there's some algebraic form to do this. There is, probably, but I don't know, or I forget it."

For Item Post-T3, Talia knew she could use the same method of equating both functions to a particular number to obtain another pair of values for *m* and *n*: "I just set it (both sides of $m – 14 = 7 – n$) to the same number and I solved for *m* and *n*, and see if they would meet the condition ($m > n$). But I'm sure there is a stopping point." Talia's prediction of a stopping point (i.e., critical point) presupposes her interpreting the equation as a dynamic comparison of the two functions. When the condition was changed to $m < n$, this way of understanding was probably what led her to immediately think of equating both sides to a negative number, -20. In the pre-interview, on the other hand, she equated both sides to 50 and then re-analyzed the results she had obtained from equating both sides to 28 before she could think of equating both sides to -20. Thus, she seemed to be interpreting the equation as a comparison of values in the pre-interview but as a comparison of functions in the post-interview.

To recapitulate, Talia showed improvement by advancing her way of understanding from I/E-as-a-signal-for-a-procedure and I/E-as-a-static-comparison to I/E-as-a-comparison-of-functions. This improvement is related to her improvement in her ways of thinking associated with foreseeing and her ways of thinking associated with predicting. Up to this point, I have presented her improvements in each area separately. In the next sub-section, I consolidate these improvements and discuss them using Cobb's (1985) notion of sub-context.

Improvement in the Sub-context in which Talia Operated

As previously stated in Chapter 2, according to Cobb (1985), one's anticipation is influenced by, (a) one's beliefs about mathematics and doing mathematics, (b) the sub-context in which one operates or the strategy one uses, and (c) one's conceptual structures or schemes that are currently elaborated. The second point, (b), means that the sub-context in which one is operating confines one's ways of thinking associated with anticipating. I extended this premise to include ways of thinking associated with the mental act of inferring, as well as ways of understanding inequalities/equations.

In the pre-interview, Talia's impulsive anticipation, association-based prediction, and the I/E-as-a-signal-for-a-procedure interpretation were related to her operating in the sub-context of manipulating symbols, especially when she was working with Item Pre-S1 and Pre-S2. When she was operating in this sub-context, she tended to be procedure-oriented and thus exhibited impulsive anticipation. Her failure to attend to the meaning of the symbols contributed to her association-based prediction.

In the post-interview, Talia's analytic anticipation, coordination-based prediction, and the I/E-as-a-comparison-of-functions interpretation were related to her operating in the sub-context of reasoning with symbols. When she was operating in this sub-context, she tended to be goal-oriented and thus exhibited analytic anticipation. Her reasoning with symbols allowed her to attend to the meaning of the symbols and coordinate change, thus allowing coordination-based predictions.

To highlight Talia's operating in the sub-context of reasoning with symbols, consider her response to Item Post-S2: "I don't know what this ($x > 3$, $x < 3$ and $x < 3$, $x > 3$) means. Let's see, here, I'm just going to plug in points, and see on which side of the 3 it is." When operating in the sub-context of reasoning with symbols, Talia attended to meaning and engaged in explorative anticipation. When she plugged $x = 5$ and $x = 2$ into $(2x – 6)(x – 3) < 0$, she was surprised to find that the output was positive in both cases: "So I think something's wrong with my critical point. Errr, maybe I can factor 2 right here." She simplified $(2x – 6)(x – 3)$ to $2(x – 3)(x – 3)$ and then rewrote it as $2(x – 3)^2$. She had difficulty interpreting it: "I don't know, not sure what this [$(x – 3)(x – 3)$] means? Like when they are both the same." She then thought of setting each factor to zero: "And they both give, x is equal to 3. … The solution for both of these is the same, which means, I don't know, which means, 3 and 3, which means that there is no solution?" Her prediction of "no solution" is considered coordination-based prediction because she could explain “whatever number will make this (x – 3) negative, will make this one (x – 3) negative. And whatever number that will make this positive, will make this positive.” Her reasoning with symbols and attending to their meanings facilitated her explorative and analytic anticipation.

In essence, Talia’s pre-to-post improvement can be summed up as a change in the sub-context in which she operates: from that of manipulating symbols to that of reasoning with symbols. Talia also operated in the sub-context of plugging in numbers during both interviews. Table 6.2 compares the sub-contexts in which Talia operated for each task.

Table 6.2: Pre-to-post Comparison in Terms of the Sub-contexts in which Talia Operated

	Pre-Interview				**Post-Interview**			
Interview item	**S2**	**T1**	**T2**	**T3**	**S2**	**T1**	**T2**	**T3**
Reasoning with symbols		1			1	1	1	1
Plugging in numbers			1	1		1	1	1
Manipulating symbols	1	1				1		

Table 6.3 summarizes the essential differences in Talia’s mathematical thinking and problem-solving between pre-interview and post-interview. To account for her improvements, data from the teaching intervention were analyzed. Recall that each teaching intervention consisted of five problem-solving sessions. The objective of these problem-solving sessions was to advance Talia’s ways of thinking associated with foreseeing/predicting and her ways of understanding inequalities/equations.

Table 6.3: A Summary of Pre-and-post Improvement for Talia

	Pre-interview	**Post-interview**
Ways of thinking associated with foreseeing	Impulsive anticipation	Analytic anticipation
Ways of thinking associated with predicting	Association-based prediction	Coordination-based prediction
Ways of understanding inequality/equation	I/E-as-a-signal-for-a-procedure	I/E-as-a-comparison-of-functions
Sub-contexts in which one operates	Manipulating symbols & Plugging in numbers	Reasoning with symbols

6.2 Talia's Trajectory from Manipulating-symbols to Reasoning-with-Symbols

Talia transition from the sub-context of manipulating symbols to the sub-context of reasoning with numbers and symbols began in the first problem-solving session. Her learning trajectory, depicted in Figure 6.2, consists of three transitions: (a) from manipulating symbols to working with specific numbers; (b) from working with specific numbers to reasoning with general numbers (e.g., large positive numbers, small positive numbers, and negative numbers); and (c) from reasoning with general numbers to reasoning with symbols. For each transition, the episode in which the transition occurred and probable factor(s) that could have contributed to the transition are discussed.

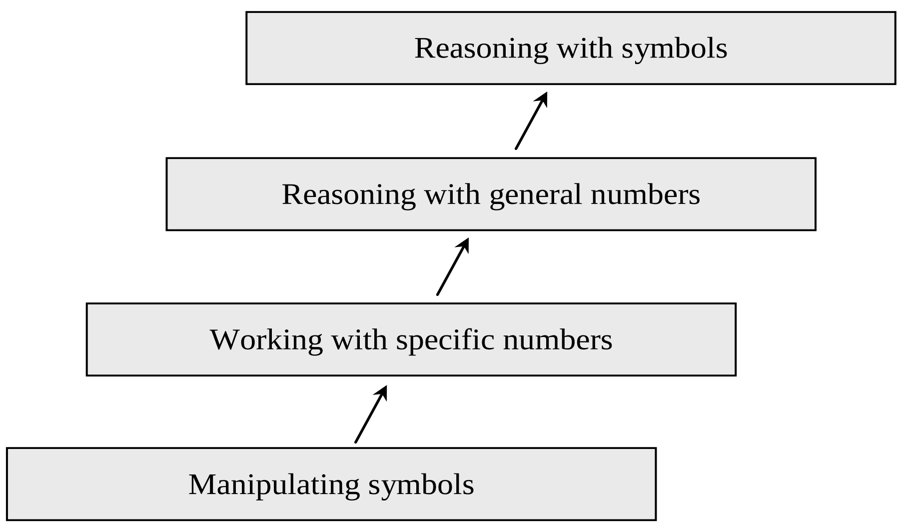

Figure 6.2: Talia's learning trajectory

Transition from Manipulating-symbols to Working-with-specific-numbers

The transition from the sub-context of manipulating symbols to the sub-context of working with specific numbers occurred when Talia worked on the first item, TE1-$_{T}$N1[24],

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[24] The code TE1-$_{T}$N1 stands for Teaching Episode 1 for Talia, New item, 1st item in the session. Likewise, TE5-$_{C}$R4 will stand for Teaching Episode 5 for Chela, Reuse-of-a-homework-task, 4th item in the session.

in the first problem-solving session. She was presented with the inequality $\frac{x-5}{x-10} < 0$ with no accompanying instruction. She interpreted the inequality as a signal to solve for *x*. Her written work is shown in Figure 6.3.

Lim: Alright, this is the first problem.
Talia: OK. So I just solve it? Alright, arrr, so I'm trying to solve for *x*. So I'm just going to multiply both sides by *x* minus 10, *x* minus 10, and it's *x* minus 5 is less than 0. And then you just add 5 to both sides. *x* is less than 5. Um, I think that's my answer.
Lim: What does this answer ($x < 5$) mean?
Talia: Um, that, this equation is true for any value of *x* that are less than 5, so, let me just try that out. So, 4 minus 5 over 4 minus 10.

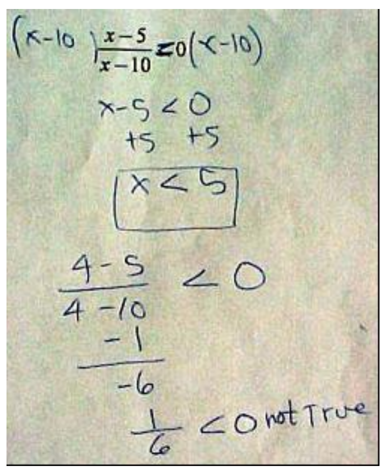

Figure 6.3: Talia's written work for Item TE1-$_T$N1

Having found that $x = 4$ did not make the inequality true, Talia continued to think of alternative means for manipulating the inequality: "How am I supposed to solve this? Um, maybe I can factor something out." It is likely that she was interpreting solving for *x* to mean isolating *x*. When asked, "What does solve for *x* mean?" she responded "To find the values for this problem where the statement is true." She then foresaw plugging in numbers.

Talia: So, umm, I'm just going to try some random values for this, 2. 2 minus 5 is -3. 2 minus 10 [is] -8. Umm, it has to be a number that is positive on the top and negative on the bottom, so I can get a negative number, and then the statement will be true. So, something that will give me positive is 6 minus 5, and 6 minus 10. This is negative, this is positive 1, over negative, um, 10, 3 4 5 6 (finger counting), 4 and that's less than 0. So one-fourth is a value that makes this statement true. … I'm sorry, 6.

Within the context of working with specific numbers, Talia could reason in a goal-oriented manner and foresaw plugging in 6 to make the numerator positive and the denominator negative. She even extended her reasoning to obtain all the values that would make the inequality true: "So *x* can be anything that is, um, bigger than 5, but less than 10. So 6 7, 6 7 8, 9."

Discussion. The change in sub-context from manipulating symbols to plugging in numbers was probably initiated by questions such as "What does this answer mean?" and "What does solve for *x* mean?" In items Pre-S1 and Pre-S2, Talia's lack of attendance to the meaning of the symbols had caused her to remain in the sub-context of manipulating symbols. Therefore, I infer that the transition from the sub-context of manipulating symbols to the sub-context of working with specific numbers is a consequence of her attending to meanings. This observation highlights the importance of getting students to recognize that the referents of literal symbols in algebra are numbers in context-free situations, or quantities in situations where the function or inequality/equation models certain physical phenomenon.

Transition from Working-with-specific-numbers to Reasoning-with-general-numbers

The transition from working with specific numbers to reasoning with general numbers occurred in Talia's initial response to the second item (TE1-$_{T}$N2): "Is $x(6x + 8) < 0$ always true, sometimes true, or never true?"

> Talia: Is *x* [times] quantity of 6*x* plus 8 less than 0 always true, sometimes true, or never true? Mmm, I'm thinking if I make *x* into a negative number so that, um, so that the whole function will be negative. So if there is an answer, it will probably have to be negative because if I make *x* positive, it's going to be greater than 0 all the time. Right? … OK. Um, so let me just try a negative number, -1.

Talia's initial way of thinking in this task was substantially different from that in the first task. While she impulsively tried to isolate *x* in the first task, she approached this task in a goal-oriented manner. Her prediction that only a negative number would have a chance of making the inequality true is considered coordination-based prediction because she knew that a positive value of *x* would make $6x + 8$ positive and the product $x(6x + 8)$ positive. Her foresight of plugging in -1 is considered analytic anticipation because it was aimed at making the whole function negative.

<u>Discussion</u>. Task TE1-$_{T}$N2 allowed Talia to repeat the goal-oriented way of thinking that had emerged when she was operating in the sub-context of working with specific numbers in task TE1-$_{T}$N1. While working on task TE1-$_{T}$N2, Talia began to reason with general numbers: *x* being positive would make the inequality "greater than 0 all the time." The transition from working with specific numbers to working with general numbers might be due to the inequality having *x* as a factor. An implication for teaching is that instructional tasks should be designed to allow students to apply, and then extend,

their ways of understanding. The quadratic inequality $x(6x + 8) < 0$ is considered a good follow-up to the rational inequality $\frac{x-5}{x-10} < 0$ because the two functions are structurally different, yet they both foster the same way of thinking, namely the backward strategy (a goal-oriented way of thinking) of making one factor positive and one factor negative. Hence, the use of the $x(6x + 8) < 0$ is in keeping with the Repeated-reasoning Principle (Harel, 2001), which stipulates that students should be provided with opportunities to repeat their reasoning in a variety of situations.

Transition from Reasoning-with-general-numbers to Reasoning-with-symbols

The transition from reasoning with numbers to reasoning with symbols began with Item TE1-$_T$N2 and continued through the entire teaching intervention. The following excerpt is a continuation of the previous excerpt.

> Lim: Are you able to find another value that'll make it true besides -1?
>
> Talia: Um, yeah. This number, well, let me see, this number (6*x*) has to be less than 8 so that, um, so that I can get a positive value here (6*x* + 8).
>
> Lim: M-hm.
>
> Talia: But the number (*x*) also has to be a negative number so that … when I get positive here (6*x* + 8), I'll have a negative here (*x*), and I'll get a negative answer, that'll be less than 0.
>
> Lim: M-hm. M-hm.
>
> Talia: So, well, if I put 2 (she interpreted -2 as 2 with a negative sign), it's going to be 12, and that's too big. That's bigger than 8. So, I think it has to be less than 2 (i.e, greater than -2), it can be -1.5. … May be it can be between 0 and -1.

This was the first instance where Talia was seen to reason with the symbols in a goal-oriented manner. She established the sub-goal of making $6|x|$ less than 8 and making x negative so that she could obtain $(-)(+) < 0$. Thus, her foresight of plugging -2 is considered analytic anticipation. Her prediction that numbers in the interval [-1, 0] would work is considered coordination-based because she foresaw that multiplying 6 by a proper fraction would result in a value less than 8: “It reduces this number because I’m multiplying [6] by 1 over something. So, it, multiplying by 1 over something, it’s the same thing as dividing 6 by something, and that will give me a number smaller than 8.” When Talia was operating in the sub-context of reasoning with symbols, she exhibited both analytic anticipation and coordination-based prediction.

Discussion. The transition from reasoning with general numbers to reasoning with symbols inevitably requires students to engage in analytic anticipation and explorative anticipation. In addition, certain ways of understanding may be necessary. I conjecture that Talia’s foresight of making $6|x|$ to be less than 8 would not have occurred without the way of understanding of making one factor positive and one factor negative. This way of understanding would not have emerged without her plugging specific numbers into $\frac{x-5}{x-10} < 0$ and reasoning about its structure using numbers. An implication of this conjecture is that mathematics educators should use reasoning with numbers as a sub-context to facilitate the emergence of critical ways of understanding that can foster reasoning with symbols.

Talia was unable to represent her goal of making $6x + 8$ positive as $6x + 8 > 0$, from which she could easily obtain $x > -8/6$. As a result, she had to proceed in the sub-context of plugging in specific numbers, such as -2, and reasoning with general numbers, such as a number in the interval [-1, 0]. This implies that mathematics educators should use reasoning with numbers as a platform for students to explore their ways of understanding related to symbolic structure. I conjecture that Talia’s undesirable ways of thinking—such as impulsive anticipation, association-based prediction, and non-referential symbolic way of thinking—probably resulted from her working with algebraic symbols and structures without the support of numbers. A lack of numerical support for algebraic reasoning is a probable cause for students' perceived disconnection between the world of algebra and the world of arithmetic, as observed by Lee and Wheeler (1989).

Talia's way of understanding negative numbers as positive numbers with a negative sign occurred on many occasions. This way of understanding caused some confusion for Talia in her attempt to symbolize the constraint of $6|x|$ being less than 8. She experienced cognitive conflict when her representation $6x < 8$, which she thought should be correct, did not agree with the table of values she had created. I only detected this way of understanding during the analysis phase, so I did not have a chance to help her address this way of understanding in the teaching intervention. This example highlights how important it is for teachers to uncover students' ways of understanding in order to effectively help students advance their ways of understanding. Chela also demonstrated this way of understanding when she was reasoning with $6x + 15 < 0$: "What number will make $6x$ greater than 15. … I mean like if I multiply, what number will come out as a negative number, to be greater than 15?"

In this section, Talia's trajectory from manipulating symbols to working with specific numbers, to reasoning with general numbers, and to an initial phase of reasoning with symbols was discussed. A few implications for instruction were suggested: (a) get students to attend to meaning when they solve problems; (b) design instructional tasks that allow students to repeat their reasoning and to apply/explore their ways of thinking and ways of understanding; (c) use reasoning with numbers as a sub-context to facilitate the emergence of certain critical ways of understanding that could foster reasoning with symbols and structures; and (d) be sensitive to students' ways of understanding.

6.3 Accounting for Talia's Improvement

While the previous section outlines Talia's improvement in terms of transitions from one sub-context to another, this section presents factors that could account for Talia's improvement. Three factors were identified: (a) attending to meaning and the referents for symbols, (b) opportunity to explore, and (c) opportunity to predict. Each of these factors is discussed in detail in this section.

Attending to Meaning and the Referents for Symbols

The importance of attending to meaning is highlighted in the discussion on Talia's transition from manipulating symbols to working with specific numbers. As mentioned in that discussion, once Talia had established numbers as referents for literal symbols, she could engage in coordination-based prediction and analytic anticipation.

However, Talia was still procedure-oriented when she worked on a list of homework items after the first session. She incorrectly applied the newly learned strategy. For example, she solved $\frac{x+7}{x+11} > 0$ by finding the critical values using $x + 7 = 0$ and $x + 11 = 0$, and converting $x = -7$ and $x = -11$ to $x > -7$ and $x > -11$. Having found that -8 did not make the inequality true, she discarded $x > -11$, and circled $x > -7$ as her answer.

In the second problem-solving session, Talia's response to Item TE2-$_{T}$R1 ("Is there a value of x that makes $\frac{x+7}{x+11} > 0$ true?") was substantially different from her written response to the homework item ("Find the solution set[25] for $\frac{x+7}{x+11} > 0$.").

Talia: Is there a value of *x*? Um, it should be a big positive number, or a small negative number.
Lim: Give me an example of each.
Talia: Errr, it can be -2 because this top number (7) over-power the -2, or it could be any positive number because positive plus positive is positive. Positive over positive is bigger than 0.

By "a small negative number" and "no bigger than -7," Talia meant $-7 < x < 0$. As mentioned previously, Talia conceived a negative number as a positive number with a negative sign. Although she overlooked the other possibility (that is, negative over negative is positive), her prediction of *x* being a big positive number or a small negative number is considered coordination-based because she was reasoning with an interval of numbers. In addition, she made an initial prediction on her own. Her foresight of plugging in -2 is considered analytic anticipation because it was directed at making both numerator and denominator positive.

When asked if she was "able to find the 'biggest' negative number that will make it true," Talia first thought, "-7 … it's because -7 plus positive 7 is 0. Or actually, also -11." She then reasoned with specific numbers like -6, -10, and -10.9999. Eventually, the negative-over-negative-is-positive way of understanding emerged.

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[25] The notion of solution set emerged in the first problem-solving session and the meaning was discussed. The homework was to an attempt to have her apply certain ideas such as finding critical value and checking the intervals that were discussed in the session.

Talia: This number (-10.9999) is still smaller than -11 by very very little, um, so the bottom will be positive, it (-10,9999) is more powerful than a positive 7, so the top will be a negative. … so it won't be true. So you have to make it the same sign on the both sides.
Lim: M-hm. M-hm.
Talia: Umm… so no, -10 isn't the one. May be negative… anything more negative than -7, it's going to… make the top negative and the outcome [will still be] negative.
Lim: M-hm.
Talia: Unless we have like a negative over a negative.
Lim: So if you want to make a negative and a negative, is that possible?
Talia: Yeah, you could put like -20.

Discussion. The difference in Talia's responses in her homework and in the subsequent problem-solving session might be due to the differences in setting (at home versus in the presence of a researcher) and in the nature of the two tasks. These differences suggest that the instantiation of a particular way of thinking depends on the circumstantial conditions and task characteristics. An implication for instruction is that it is important to foster a learning environment that is conducive for nurturing desirable ways of thinking. Another implication is that it is advantageous to use of non-directive tasks, instead of directive tasks. A directive task, such as "find the solution set" and "simplify this expression," tends to elicit impulsive anticipation, whereas a non-directive task tends to encourage explorative anticipation and making predictions.

Talia's response to the homework item also suggests that her ways of thinking are robust. In the homework example, her way of thinking is procedure-oriented, which usually manifests itself as impulsive anticipation. I conjecture that the experience that Talia gained from the teaching intervention might have little long-term impact if her learning environment fosters impulsive anticipation. An instructional implication is that mathematics teachers should foster a learning environment that encourages students to explore and to analyze, instead of to remember and to apply. Desirable exploration and analysis would require students to reason in a symbolic referential manner, that is, to attend to meaning.

When Talia did not attend to meaning she tended to be procedure-oriented; in contrast, when she attended to meaning she could engage in coordination-based prediction and analytic anticipation. Repeated experience of using numbers to uncover symbolic structure(s) can help students cultivate a disposition to make a prediction, and to engage in coordination-based prediction and analytic anticipation. After all, a way of thinking cannot be developed in a single experience.

The question "are you able to find the 'biggest' negative number that will make it ($\frac{x+7}{x+11} > 0$) true?" presented Talia with a need to find a critical value. This is an instantiation of the Necessity Principle (Harel, 1998) which stipulates that for students to learn a particular concept they must perceive a need for it. Talia's experience of needing to find a critical value might have contributed to her confidence in resolving cognitive conflicts involving critical values. Recall previous discussion regarding Talia's response to Item Post-S2 (on sub-section *Improvement in terms of Sub-context in which Talia Operated*) Talia thought, "something's wrong with my critical point", when she found that both $x = 2$ and $x = 5$ made the inequality false. She resolved her conflict by exploring another approach: factoring out a 2 and obtaining $2(x - 3)(x - 3) > 0$. When she was not sure what $(x - 3)(x - 3)$ meant, she set each factor to zero and realized that there was only one critical value.

The sub-context of plugging in numbers can be a platform for students to develop the I/E-as-a-comparison-of-functions way of understanding. Talia's search for a critical value by plugging in numbers may have set the stage for developing this way of understanding. This is because she was constantly considering alternative numbers (e.g., -7, -6, -10 and -10.9999) when she reasoned with a specific number. Allowing a literal symbol to assume different values could foster her interpreting an algebraic expression as a function, rather than a non-varying entity.

With numbers as referents for her symbols, Talia could explore the symbolic structure and encounter the negative-over-negative-is-positive way of understanding. She used a similar way of understanding—making both linear factors positive or both negative—in her work on Item Post-S2, which involved $(2x - 6)(x - 3) < 0$. According to one direction of the Duality Principle, which stipulates that "how students come to understand mathematical content influences their ways of thinking" (Harel, 2001, p. 207), this way of understanding could help Talia foster analytic anticipation and the backward strategy, which is a goal-oriented way of thinking associated with the act of problem solving.

Opportunity to Explore

Despite being a Calculus student, Talia encountered difficulties when she operated in the sub-context of reasoning with symbols because of deficiencies in her ways of understanding certain algebraic concepts. Nevertheless, within the DNR-based instructional environment I implemented, these deficiencies were opportunities for learning. The opportunities for exploration in the problem-solving sessions enabled these deficiencies to surface and be addressed.

Talia's solution process for Item TE2-$_T$N4 provides a useful example for discussing how an opportunity for Talia to explore her ways of understanding could contribute to learning. TE2-$_T$N4 asks, "Is the following statement always true, sometimes true, or never true? $5 + (8x – 20) < 10 + (8x – 20)$". Talia simplified the inequality to $8x – 15 < 8x – 10$ and compared the two sides by reasoning with general numbers.

> Talia: Let me think in term of negatives and positives first. Um, sometimes true, this (x) can be a positive number, big, this ($8x$) becomes a big number, and this ($8x – 15$) is big outcome. This ($8x$) is a big number, this ($8x – 10$) becomes big number, and I think it's sometimes true. That's my prediction.

Talia predicted "sometimes true" because she thought that the inequality would be true for positive numbers and false for negative numbers.[26] Nevertheless, her prediction is considered coordination-based because she was not thinking with specific numbers and was coordinating the input x with $8x$, $8x – 15$ and $8x – 10$. The predicting-prior-to-performing way of thinking was absent when she worked on a similar item in the pre-interview, Item Pre-S1.

Talia then explored the idea of finding a critical point: "I wonder if I can move this thing ($8x – 10$) to this side and set it equal to 0 to find the critical points." She foresaw from $(8x – 15) – (8x – 10) < 0$ that the $8x$'s would cancel, but added, "that's probably not the best thing [to do]." When she was asked to proceed, she obtained $-5 < 0$ and became perplexed.

> Talia: -5 less than 0, that's true. But then the x's are gone. So this (-5) isn't, is this really a critical point? I'm not sure if it is. … This

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[26] Treating a negative number as a positive number with a negative sign, she appended the negative sign to the result she obtained from operating on its positive counterpart; so she could be visualizing the inequality as something like $-(8|x| – 15) < -(8|x| – 10)$.

> inequality, I know this statement ($-5 < 0$) is true, but I'm looking for the critical point of *x*.

Talia eventually saw the connection between the equation $8x - 15 = 8x - 10$ not having a solution, and the inequality not having a critical point. This episode allowed Talia's deficient way of understanding of critical points to surface and be addressed.

When Talia was asked whether she was "able to just study this (original) inequality without doing any work … [and answer] is there a value of *x* that will make the inequality true." Talia responded, "No, because I know that the *x*'s are going to cancel." As with Item Pre-T1, Talia did not realize the significance of the cancellation of the *x*'s. Instead, she associated the disappearance of *x* with the inequality not having a solution. When she plugged in 2 for *x*, she noticed that "these two (8*x*'s) will be the same numbers." She admitted that she did not realize this solution earlier.

When she foresaw plugging in -10 for *x*, I wrote out the arithmetic expressions and drew two pairs of parentheses, in which she could write in the result of $8(-10) - 20$ (see Figure 4.5). Upon writing $5 + (-100) < 10 + (-100)$, Talia noticed the structure and wrote $5 + X < 10 + X$.

> Talia: OK, is this (*X*) always going to be the same number (*X*), plus something, since this is always plus 10 and this is always plus 5? This number is always going to be bigger than this number regardless of what *x* is. It's still the same thing. Yeah. So, the answer is always true.

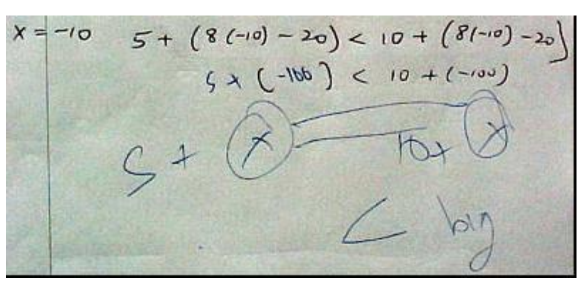

Figure 6.4: Talia's observation of the structure

Discussion. Task TE2-$_T$N4 is similar to Item Pre-S1, which involved a more elaborate inequality $(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$. While Talia manipulated symbols without attending to meaning in Item Pre-S1, in this new task she analyzed the inequality by reasoning with general numbers and made a prediction. These experiences allowed her to engage in analytic anticipation and coordination-based prediction.

Task TE2-$_T$N4 allowed Talia to explore her understanding of critical points, to exhibit association-based prediction, and to learn that the absence of a critical point does not imply the absence of solution for an inequality. Talia's plugging in -10 allowed her to "see" the symbolic structure of the inequality. This experience might have promoted the plugging-in-numbers-to-notice-patterns-or-structure way of thinking.

Talia's way of understanding the structure of the inequality, reconceived as $5 + X < 10 + X$, was facilitated by my writing out the expressions and drawing those parentheses. One may question the appropriateness and timing of the my involvement in Talia's work. As the instructor, I felt that Talia was ready to abstract the structure of the inequality from her work with numbers. This example highlights the importance of strategic intervention on the part of a teacher, an idea that is based on Vygotsky's notion of zone of proximal development.

Opportunity to Predict

Talia's improvement in making more predictions, especially coordination-based ones, during the post-interview was related to her opportunities to predict in the problem-solving sessions. To examine how these opportunities might have contributed to her improvement, I discuss Talia's work on a sequence of three tasks.

Item TE4-$_T$N3a: “Plugging x = 127 into $4x – 20 > 3x – 20$, we get 361 for the right hand side. What is the value on the left hand side?”

Talia foresaw plugging in 127 into $4x – 20$. When asked to predict, she responded, “I’m trying to think how much more is this ($4x – 20$) than this ($3x – 20$) but I’m not sure how.” She noticed that $4x$ “is 1 times more” but had to struggle with its meaning, because the difference between $4x$ and $3x$ involves both multiplicative and additive comparisons.

Lim: 1 times more. So, what does that mean?
Talia: So, um, I don’t know, multiply it by 1? Because $4x$ is 2 times more than 2, so you multiply it by 2, and $4x$ is 1 times more than 3, so you multiply it by 1?
Lim: $4x$ is 1 times, what do you mean by 1 times more?
Talia: Well, 1 more, also 1 times more? 1 times 3 (wrote and then cancelled “1 × 3”). No, never mind. It’s 1 more.
Lim: 1 more?
Talia: M-hm.
Lim: What do you mean by 1 more?
Talia: 1 more x. So I guess you add this (127) to this (361)? Is that it? Because this (-20) is the same on this both sides. So you just add 127. Um, yeah.

Discussion. Talia did not make a prediction on her own. The explicit request to get Talia to predict was necessary since, according to Vygotsky’s (1978) notion of *zone of proximal development*, students are unlikely to engage in certain ways of thinking without the influence of a teacher or a more advanced peer.

Talia’s lack of predicting could have been due to her deficient way of understanding $4x$. Failing to interpret $4x$ as $x + x + x + x$, she could not spontaneously notice that $4x – 20$ is one x more than $3x – 20$. Instead, she was probably interpreting $4x$ as 4 times x. The opportunity to predict and to explore enabled Talia to struggle with her way of understanding $4x$, and to appreciate the advantage of having alternative ways of interpreting $4x$. According to Harel and Sowder (2005), beliefs such as “a concept can have multiple interpretations" and "it is advantageous to possess multiple interpretations of a concept" (p. 32) are desirable ways of thinking that should be developed in elementary and secondary school mathematics.

Item TE4-$_{T}$N3b: "Plugging $x = 8.01$ into $4x - 20 > 3x - 20$, we get $12.04 > 4.03$. What will we get if plug in $x = 16.02$."

For this item, Talia made a prediction: "So why don't we multiply these by 2." Her prediction, however, is association-based because she associated doubling the input value of *x* with doubling the output value of each function. She then doubted (i.e., analytic-anticipated) her prediction and considered whether the doubling should be applied to the *x*-terms or to the entire function. She chose the latter because, "this (12.04) should be 2 times more, and this (4.03) should be two times more as well, because I'm doing the same thing to both sides." Talia's confidence increased after she associated the justification for doing the same to both sides with the justification for doing the same to the input and to the outputs. She had to work with specific numbers to see that the constant term, -20, was not doubled.

Discussion. This task gave Talia an opportunity to predict, to make association-based predictions, and to plug in numbers to uncover the mathematics underlying the result. She had the opportunity to engage in analytic anticipation and to apply the plugging-numbers-to-see-the-underlying-structure way of thinking.

Item TE4-$_{T}$N3c: "Plugging $x = 9.11$ into $4x - 20 > 3x - 20$, we get $16.44 > 7.33$. What will we get if we plug in $x = 10.11$?"

Noticing that 10.11 was 1 greater than 9.11, Talia predicted, "you could add 1 more, but you have to add only to the *x*'s. … But how do you disregard the 20. … I think I just add 1 (wrote $17.44 > 8.33$)." Talia's focus on the constant term of -20 was influenced by her learning from the preceeding task. The phenomenon of using recently learned ideas, which I call *Recency Effect*, is discussed in the last section of this chapter. Talia did not notice that $4x + 1$ was different from $4(x + 1)$ until after she had written $4x + 1 – 20$ and $4(9.11 + 1) – 20$, and compared them. In addition, her trying to disregard the -20 conflated the issue because she was not aware that $(4x + 1) – 20$ and $(4x – 20) + 1$ were equal. To resolve her confusion, she plugged in 9.11 for x, and then 10.11 for x. Comparing the results (see Figure 4.6), she noticed the difference of 3 and 4: “So I could’ve just added 3 and 4.” She associated them to $3x$ and $4x$ but was not able to explain the association in terms of the distributive property.

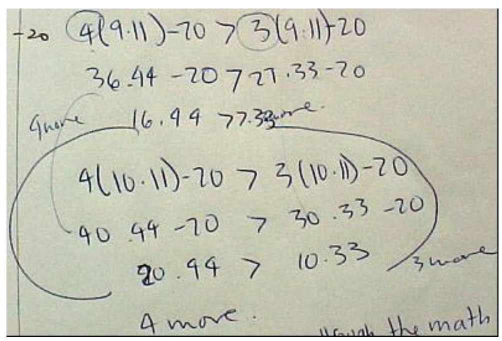

Figure 6.5: Talia’s observing “4 more” and “3 more” from her numerical work

To test whether Talia had internalized the symbolic structure, I asked her, “what if x is 19.11?” She appeared to have internalized the distributive property: “19.11 is 10 more. So, 4 times 10, you’ll have to add 40. So it’s, um, 4 times 9.11 plus 40, minus 20? … So, essentially it’s just adding adding 40 to whatever your output of 9.11 is.”

Discussion. This task allowed Talia to analyze the inequality, to predict, and to practice plugging-numbers-to-see-the-underlying-structure way of thinking. Using those numbers, Talia understood why her initial prediction of adding 1 was incorrect and she was able to see the relevance of the distributive property in this problem.

This task also allowed Talia's deficient ways of understanding, which were related to the associative property and the distributive property, to surface. A single episode such as this might not have helped Talia to address those issues effectively. However, repeated reasoning in similar situations would help her to interiorize associative and distributive properties, as well as to develop checking-to-ensure-preservation-of-values (arithmetic invariance) way of thinking, which is foundational to developing changing-the-form-without-changing-the-value (algebraic invariance) way of thinking.

<u>Evidence of Talia's using numbers as a means to reason with structure</u>

Talia's improvement in terms of using numbers to get a feel of an algebraic expression is evident in her response to Item Post-S5: "Is the following statement always true, sometimes true, or never true? $(x + 1) + (x + 2) + (x + 3) + \ldots + (x + 99) + (x + 100) < 100x$". Talia predicted sometimes true, and plugged in 2 to show that that the inequality could not be always true. Exploring with $x = 100$, she noticed the pattern in $101 + 102 + \ldots + 199 + 200$ and thought of the pairing strategy, although incorrectly: "You add the first number and you add this [last] one, and then you would divide it by the number of total terms." She applied the idea that emerged from the arithmetic to the algebraic inequality and obtained $\frac{(x+1)+(x+100)}{100}$. She then suspected that it should be divided by 2 because, "you want to get the value of one term." Next, she tried to recall the formula for arithmetic series: "It's like N over 2 times first term and then last term. I don't remember." So Talia went back to reasoning with $x = 100$, but was unable to decide which side would be bigger. She then thought of factoring, but realized, "I can't factor an *x* out of all them." She then returned to applying her summation idea

$\frac{n}{2}(\text{1st term} + \text{last term}) < 100x$, but added, "I don't think this is the right way to do it."

After exploring by plugging in -100 for *x*, Talia then thought of ignoring the *x*'s and focused on the numbers: "It's like adding 1 to 100. Let me find the sum from 1 to a 100. This is 101, and then times how many terms you have, divided by 2. That's 50, times 101." Seeing the product as 5050, Talia questioned, "How's that going to help me? I'm going to see if it's less than 100*x*." She then noticed that there were 100 *x*'s. When she obtained $100x + 5050 < 100x$, she guessed that the left function was greater than 100*x*. She spontaneously thought of negatives: "But then, if it's really negative, it's also smaller. So I think it's sometimes true." In this case, Talia seemed to be interpreting the inequality as $-(100p + 5050) < -(100p)$ where $-p$ represents a negative number. This inference was drawn because Talia treated negative numbers as positive numbers, worked with these positive numbers, and then negated the result. When she tested her prediction with $x = -500$, she changed her answer: "but you are adding, never mind. Since you are adding over, this number is always going to be bigger than this one. So this is never true."

<u>Discussion</u>. This item was not discussed earlier in the pre-post comparison because Talia did not work on this item in the pre-interview. This episode highlights how Talia capitalized on numbers to help her to reason with symbols and structure. In other words, she demonstrated the using-numbers-as-a-means-to-reason-with-structure strategy, which is a way of thinking associated with problem-solving. With numbers as referents for her symbols, she could rectify her mistake of dividing the sum of the first and the last terms by the number of terms. Her attempt to recall the formula for arithmetic

sequences emerged out of a need to sum those numbers instead of being a conditioned response. Her abandoning $\frac{n}{2}(\text{1st term} + \text{last term}) < 100x$ was probably due to her difficulty in foreseeing its usefulness when the terms have x's in them. When she ignored the x's, she could use it successfully for summing 1 to 100. This example highlights Talia's ability to reason with structure using numbers and her continued difficulty in reasoning with structure at a symbolic level.

Talia's improvement in her tendency to predict was demonstrated in this episode. For example, she immediately predicted "sometimes true" at the beginning and predicted "sometimes true" at the end, after she considered the possibility of negative numbers making the inequality true.

Talia's failure to spontaneously detect that there were 100 x's on the left side was an indication that she had only internalized and not interiorized the way of understanding $4x$ as $x + x + x + x$. If she had interiorized this way of understanding, she would have arrived at her conclusion much sooner. This incident highlights the importance of Harel's (2001) Repeated-reasoning Principle; that is, students need to reason repeatedly in order to interiorize a way of understanding.

6.4 Difficulties Talia Faced

The previous sections of this chapter have discussed Talia's progress and improvement. However, there were many times during the teaching intervention in which Talia encountered difficulties. This next section examines some of those difficulties so as to provide a more complete picture of Talia's learning.

Talia's Difficulties with Solution Set and Invariance of an Inequality

Talia demonstrated some improvement in her ways of understanding inequalities/equations. Because she was in the sub-context of manipulating symbols, Talia initially interpreted an inequality as a task to isolate x (items Pre-A1 and TE1-$_T$N1), a task to apply a certain procedure (Item Pre-A2), or an equation with an unequal sign. By the end of the teaching intervention, she began to view an inequality as a comparison between two sides and as a proposition that could be true or false depending on the input values. In her reflection on what she had learned over the five problem-solving sessions, she wrote, "I looked at inequalities as if they were a comparison of two functions. … I also looked at them as different propositions, and I tried to find ways to make them true or false by thinking in terms of positive numbers and negative numbers."

Talia demonstrated difficulties with the notions of solution set and invariance. Talia began with a weak understanding of solution of an inequality. For example, when she solved $1.2x + 3456 < 7 + 8.9x$ (Item Pre-A3) and obtained $447.9 < x$, she predicted that 447.9 would make the inequality true. Similarly, when she obtained $x > -4/3$ from $6x + 8 > 0$ (Item TE1-$_T$N2), she did not expect the value of $6(-4/3) + 8$ to be zero.

In her second session, Talia conflated the invariance aspects of an inequality with variable aspects (e.g. the truth value depends on input value). For Item TE2-$_T$N3, she did not see the invariance that allows simplifying $5 + (8x – 20) < 10 + (8x – 20)$ to $5 < 10$. As such, she predicted that $5 + (8x – 20) < 10 + (8x – 20)$ was sometimes true. She predicted true because she saw $5 < 10$ as "one situation that is true," and she predicted false because she associated the falsity of the inequality with the disappearance of x and the inequality not having any critical point. She did not know that critical points of an inequality are invariant. She also did not seem to know that $5 < 10$ was equivalent to the original inequality. Instead of seeing them as invariant properties, she saw them as circumstantial results that could vary, similar to how she saw the truth-value of an inequality as varying according to the input value of *x*.

By her third session, Talia seemed to have some understanding of solution set, but she still had not understood algebraic invariance. This was revealed in her work for Item TE3-$_{T}$N1: "Consider these two inequalities: $x < 1$ and $8x + 3 > 8 + 3x$. Is there a value for *x* that will make one of them true but will make the other false?" When Talia solved $8x + 3 > 8 + 3x$ and obtained $x < 1$, she knew that the two inequalities "are the same, [have] the same, um, solution set." She added, "whatever value I put in here ($8x + 3 > 8 + 3x$) will make this true as well. Right? So, to test that out let me just try …" After plugging in 0, -10 and 10, she commented, "I don't think there is a value." Later on, she was asked to considered $x < 1$ and $5x < 5$. Talia entertained the possibility of having a value for *x* that would make one of them true and the other false: "0? No. 1? 5 is less than 5? 1 is less than 1. Maybe 1, cause 5 isn't less than 5. No. One isn't less than 1. False again." While Talia knew that $5x < 5$ could be simplified to $x < 1$, she could not explain their equivalence in terms of preservation of a solution set.

Despite efforts to help Talia understand the notion of preserving solution sets while simplifying an inequality, she did not internalize the ways of understanding associated with equivalent inequalities. Talia repeatedly divided both sides of an inequality by a function without ensuring that the function be positive. For example, she simplified $5(8x – 20) < 10(8x – 20)$ to $5 < 10$ (Item TE3-$_{T}$N4), and $(x + 1)(2k – 7) = 3(2k – 7)$ to $x + 1 = 3$ (Item Post-T4). These results prompt the question:

What made it difficult for Talia to develop the way of understanding related to preservation of solution sets? Is it because she lacked the algebraic invariance way of thinking? To answer these questions, further research is needed to study students' development of the preserving-solution-set way of understanding, their development of algebraic-invariance way of thinking, and the interplay between the two.

Talia's Difficulties with the Considering-Falsity Way of Thinking

Talia exhibited the considering-negative-numbers way of understanding in the third session when she worked with $5(8x – 20) < 10(8x – 20)$. However, she was not able to invoke it in the next session to solve Item TE4-$_{\mathrm{T}}$N1: "Given that $x > 10$, is $3x(500 – 2x) < 10(500 – 2x)$ always true, sometimes true, or never true." Even when she was asked, "If your classmate says that he found a value of x that makes the inequality true, will you believe him?" she still maintained her way of understanding and reasoned, "I don't see how I could because 3 times 10 is already 30. And 30 is not less than 30, right? And then 3 times 11 is 33… plus you are multiplying this. So I don't think there is, unless it was less than 10." Talia's lack of consideration for the other case (i.e., not considering-falsity) might be a consequence of her way of understanding that $x > 0$ implies $f(x) > 0$, which probably caused her to ignore $500 – 2x$ and instead focus on $3x$ and 30. Once she was reminded that she had said in the previous session "remember, think of negatives", she thought of the other case.

Talia: Oh, yeah. I can get a negative out of this. If $2x$ is bigger than 500, I could get a negative number, and then this will be really negative, so this number will be more than this one. And it will be sometimes true. So 2 times what is going to give him bigger than 500? Um, let's try 2 times 400.

Talia's way of understanding of making $2x$ larger than 500 could not emerge without the considering-falsity way of thinking. This example provides an instantiation of the second direction of the Duality Principle: "ways of understanding [students] produce are determined by the ways of thinking they possess" (Harel, in press c). In Tzur's (2003) terms, Talia is said to be in the *participatory stage* of her development of the considering-negatives way of understanding, and not yet in the *anticipatory stage*, because she was not able to anticipate the usefulness of considering negatives beyond the context in which she learned, and cuing was needed to evoke this way of understanding. Further research is needed to study student's transition from the participatory stage to the anticipatory stage in the development of considering-falsity as a way of thinking associated with problem-solving.

Talia's Weakness in Dealing with Numbers

Talia used finger-counting to compute additions and subtractions such as 6 – 10, -8 + 12, -8 + 6, and -50 + 6. She had to use the procedure for multiplying fractions to compute 1.5 × 6 and to find the value for one-fourth of 6. In Item Post-S2, she misinterpreted 6/2 as 6½. Furthermore, Talia had difficulty noticing the multiplicative relationship between $8x + 20$ to $6x + 15$, even after she had generated a table of values with number-pairs such as (15, 20), (33, 44), (9, 12), (3, 4), and (-3, -4). Chela, an Algebra 2 student, on the other hand, was able to detect the multiplicative relationship and ratio much sooner: "There might be a number when you multiply [by 30], you'll get 40. … so this ($30x = 40$) will be the equation to find it."

As mentioned earlier in a few places, Talia's ways of understanding a negative number as a positive number with a negative sign was an obstacle to her progress in reasoning with symbols. For example, while attempting Item TE1-$_T$N2, Talia had difficulty constructing $6x > -8$ because she was reasoning with $6|x|$ being less than 8 but was representing the constraint as $6x < 8$. Her insensitivity to the order of operations confounded this problem. For Item TE2-$_T$N4, she seemed to append the negative sign after the operations were performed with their corresponding positive number. For example, when x was negative, she would treat the inequality $8x – 15 < 8x – 10$ as $-(8p – 15) < -(8p – 10)$, where $x = -p$.

Talia was also insensitive to properties such as the associative property and the distributive property. As discussed previously, while working on Item TE4-$_T$N3c, she conflated $4(x + 1)$ with $4x + 1$ and she was not aware that $(4x + 1) – 20$ was equivalent to $(4x – 20) + 1$.

In conclusion, Talia's foundation in arithmetic and algebra was rather weak, even though she was a Calculus student. The problem-solving sessions offered her an opportunity to explore and improve her ways of understanding. Once she began to attend to meanings, she could reason in a symbolic referential manner and thereby exhibit more desirable ways of thinking associated with foreseeing and predicting, as well as sophisticated ways of understanding inequalities/equations.

6.5 Three Other Learners

Among the four learners that participated in the intervention portion of this study, Talia showed the greatest improvement, which is why I provided a detailed explanation of Talia's learning process in the preceding sections. In this section, the three other learners' (Chela, Ali, and Vito) improvements as a result of the intervention are compared. Chela is compared with Talia, and Ali with Vito. These comparisons were chosen because the teaching interventions for Vito and Ali were conducted in the same period of time, while those for Chela and Talia were conducted at a later period. Because of the experience gained from working with Vito and Ali, the teaching interventions for Talia and Chela were more effective. As a result, I chose to analyze the problem-solving sessions for Talia and Chela only, but not for Vito and Ali. The discussions that follow are mainly based on the data analysis of the four learners' pre- and post-interviews.

The Case of Chela

Chela was enrolled in Algebra II and averaged a B. Algebra II is two mathematics class levels behind Calculus, the course in which Talia was enrolled. Compared to Talia, Chela exhibited less non-referential symbolic reasoning and more coordination-based prediction during the pre-interview. This is because Chela operated mainly in the sub-context of plugging in numbers and appeared to be in control of the situation.

Chela improved from operating in the sub-context of plugging in numbers (in all items except Pre-T4) and manipulating symbols (Pre-S5) in the pre-interview to operating in the sub-context of reasoning with symbols (Post-S2, S5, T2, T4) and plugging in numbers (Post-T1 and T3) in the post-interview. For example, consider her response to Item S5: "Is the following statement always true, sometimes true, or never true? $(x + 1) + (x + 2) + (x + 3) + \ldots + (x + 99) + (x + 100) < 100x$". In the pre-interview, Chela was tenacious in using the formula $a_n = d(n - 1) + a_1$ (this episode is discussed in the section titled *Two Interesting Phenomenon* at the end of this chapter). In the post-interview, Chela temporarily ignored the x's, found the sum of 1 to 100 to be 5050 using the pairing strategy, noticed "you are adding it (x) 100 times", and concluded "never true" because "basically, you are adding what you get here ($100x$) to this (5050)."

In terms of ways of thinking associated with foreseeing, Chela exhibited analytic anticipations in both interviews, but those in the post-interview were more sophisticated because they were in the sub-context of reasoning with symbols. While attempting Item Post-S2, Chela spontaneously recognized the factored-form structure and foresaw that, "this [factor] has to come out negative and this [factor] has to come out positive, or the other way round" (interiorized anticipation). She also reasoned with the function ("maybe I can try to look at the equation and try to make some reasoning out of it") and noticed that $2x – 6$ is twice of $x – 3$ ("it's a double of it, because if you multiply this by 2, it'll be $2x$ minus 6."). Her conclusion, "if this ($x – 3$) is a negative number, then this ($2x – 6$) will be a negative number" is considered deduction-based. In the pre-interview, Chela's way of understanding ("I think one side (factor) has to be negative, one side has to be positive, so it will be, stay negative") arose from plugging in numbers, and her inference that both factors had the same sign was based on empirical evidence.

In terms of ways of thinking associated with predicting and ways of understanding inequalities/equations, Chela's improvements were also related to her operating in the sub-context of reasoning with symbols. By reasoning with symbols, she could coordinate the quantities $2q$, $5p – q$, and $2p + 5$ in Item Post-T2: "p and q are odd integers between 20 and 50. For these values, is $5p – q > 2p + 15$ always true, sometimes true or never true?"

Chela: It ($5p – q$) will still be bigger than this ($2p + 15$) because this ($2p$) is pretty small here, and you adding a really small

> quantity. You're adding something but it's not, I don't think it's big enough to make a difference over here.

In the pre-interview, she reasoned by plugging in numbers such as (23, 21), (49, 41), (33, 27), and (27, 33). Having found that none of those cases made the inequality false, she responded "I'm leaning more on always true but only between the numbers 20 and 50." Her guess of "always true" lacked the coordination that she had in the post-interview.

Chela's improvement in her reasoning with symbols in the post-interview was a consequence of her opportunities during the teaching intervention to practice reasoning with symbols by attending to structure and exploring functions. In the learning process, Chela encountered certain difficulties while working on inequalities that involved a product or a quotient of two linear functions. The difficulties included: (a) not considering falsity (e.g., she thought that "if this outcome ($4x - 20$) is a positive, this outcome ($3x - 20$) will be a positive … it'll be greater than 0. And if this outcome is a negative, and this is a negative, it'll be a positive, which will make it greater than 0."); (b) not autonomously applying the standard procedure for solving linear inequalities (e.g., she tried solving $3x - 20 < 0$ by plugging in numbers to find the "breaking point"); and (c) not attending to the meaning of symbols (e.g., she solved $\frac{2x-4}{x-12} \geq 0$ by working on each linear function individually and combining the results as $\frac{x \geq 2}{x \geq 12} = x \geq \frac{1}{6}$).

The teaching interventions for Chela and Talia were quite similar, but Talia demonstrated greater improvement. This is likely because Talia began with a non-referential symbolic way of thinking and exhibited more instances of non-referential symbolic reasoning in the pre-interview than Chela. Talia made more progress through the teaching intervention because once she attended to meanings, she could engage in analytic anticipation and explorative anticipation in the sub-context of reasoning with symbols. Being a Calculus student, Talia had more experiences dealing with algebra, and that presumably contributed to her progress. Chela, on the other hand, having only Algebra 2 experiences, needed more time to move from her familiar territory of working with numbers to the the higher level of reasoning with symbols.

The Case of Vito

Vito, a Pre-calculus student with an A average at the time of this study, was one mathematics course behind Talia. Compared to the other interviewees, Vito had a greater tendency to engage in tenacious anticipation. He demonstrated tenacious anticipation while solving items Pre-T1, Pre-S5, and Post-S5.

Vito seemed to be rather systematic in his approach to solving problems. For example, he tended to use 2 and -1 as the first two numbers for trial-and-error substitutions (e.g., items Pre-S2 and Pre-S5).

> Lim: M-hmm, m-hmm. So did you choose 2, arrr, why did you choose 2?
> Vito: Oh, because this is the first one. If I plug in 1, well, then, it's easier but then 1, 1 will not always work. So I always pick 2.

Based on my working with Vito, I inferred that he viewed mathematics as a collection of rules and procedures. Vito liked to use rules and "short-cuts". He enjoyed working on the set of problems that involved linear inequalities of the form $Ax + B < Cx + D$ after having abstracted certain rules for determining whether such an equality is always true, sometimes true, or never true. Throughout the teaching intervention, Vito was dedicated to learning and appeared to make progress. The intervention was tailored to respond to his way of learning and, as a result, unintentionally focused more on developing ways of understanding (e.g. solution set, equivalent inequalities, preservation of solution set) than on ways of thinking.

Vito's improvement from the pre-interview to the post-interview was marginal. One improvement was that he exhibited coordination-based prediction while working on Item Post-T1, but not on Pre-T1. For Post-T1, he initially predicted b was larger because he foresaw that it had to compensate for the greater effect of multiplication to satisfy the constraint $5a = b + 5$. He illustrated by plugging in 2 for a, and found b to be 5. While attempting Pre-T1, he initially predicted, as discussed in Chapter 4, that a was larger because he conflated a with $5a$, and b with $b + 5$.

Another of Vito's improvements concerns his problem-solving approach for Item T2: "p and q are odd integers between 20 and 50. For these values, is $5p – q > 2p + 15$ always true, sometimes true or never true?" He considered exhaustive, mutually exclusive cases ($p > q$, $q > p$, and $p = q$) in the pre-interview, but considered extremes cases ($p = q = 49$; $p = 49$, $q = 21$; $p = 21$, $q = 49$) in the post-interview.

Surprisingly, Vito performed worse on Item Post-S2 than on Item Pre-S2. In the pre-interview, he approached the problem by plugging in 2, and then -1, for x into $(2x – 6)(x – 3) < 0$. With the aid of the numerical results, he could attend to the structure of the inequality, although he did not seem to treat $2x – 6$ as a function.

> Vito: So you'll never have like, a negative and a positive … you have 2 (the coefficient in $2x – 6$), and it was multiplying this. So it's double, it will double it, -6 (the constant term in $2x – 6$), is the same thing. So it's just double it, times 2.

In the post-interview, he approached Item Post-S2 by expanding the factors in $(2x-6)(x-3) < 0$. He obtained $2x^2 - 6x - 6x + 18 < 0$, incorrectly simplified it to $x^2 < -9$, and concluded, “there is no x values” because “you can’t square a number and get a negative.” But when asked if he could tell by looking at $(2x-6)(x-3) < 0$ without doing any work, he replied “no” without any intention to study the inequality.

The Case of Ali

Ali, a Pre-calculus student averaging a B at the time of his participation in the project, had the same mathematics teacher as Vito. Prior to the teaching interventions, their teacher commented that, “if any student should get an A, it should be Vito”, and “Ali turns in his work but he doesn’t seem to understand the math” (these paraphrased comments were recorded later). The teacher stated that Ali did not deserve a B in the course, and pointed out that he received help from tutors on his homework.[27]

Interestingly, Ali demonstrated greater improvement from his pre- to post-interview than Vito[28], especially in terms of reasoning with symbols. In both interviews, Ali approached Item T2 by plugging (21, 21) into $5p - q > 2p + 15$. In the pre-interview, he proceeded by considering cases such as (-21, -21), (49, 49) and (21, 49). In the post-interview, he reasoned with the structure of the inequality $5p - q > 2p + 15$, and his

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[27] Based on my work with Vito and Ali, I would consider Vito to be more studious and Ali to be more pragmatic. Because of his part-time job, Ali came in mentally exhausted for the first two problem-solving sessions. At the beginning of the third session, he was explained the importance of being alert in order for his participation to count and he was given the option to discontinue his participation. From then on, his participation improved.

[28] It was Vito who failed his Pre-calculus course by the end of his school year. He later told me that he failed because he did extremely badly in the final examination, which was weighted heavily by his teacher. I speculate that Vito had not interiorized, but only internalized the procedural aspects of, the mathematical concepts that were taught to him.

consideration of (21, 49) was aimed at making “the right side a big number [so that] I can make this false.”

While working on Item S5, which involves $(x + 1) + (x + 2) + (x + 3) + \ldots + (x + 99) + (x + 100) < 100x$, Ali, during the post-interview, noticed the structure from his trial-and-error substitution: “See all these numbers, we are going to get [100]x plus these, you know, the 1 through 100. … OK, then, you know, it’s never true.” In the pre-interview, he could not relate his numerical results to the structure, and commented, “There is an equation (formula) to this that I learned last year in algebra, sequences. But right now, I can’t remember what the equation is. If I knew, I think I could have found out real quick.”

As for Vito, his responses to Pre-S5 and Post-S5 were rather similar. Vito explored by plugging in numbers. He was careless with his interpretation of symbols; he conflated the input value of 2 with the common difference of 2 in the pre-interview, and he interpreted $x + 99$ as $99x$ in the post-interview. In both interviews, Vito did not attend to the significance of the coefficient of $100x$, and was tenacious in his way of understanding that the sum of the last three terms on the left side was greater than the value on the right side.

Vito was more likely to be tenacious while Ali was more likely to consider alternatives (e.g. negative values, large numbers, and cases to falsify). For example, consider their responses to Item Pre-S3: “Is there a value for x that will make the following statement true? $1.2x + 3456 < 7 + 8.9x$”. Both Vito and Ali initially plugged in 10 for x and obtained $3468 < 96$. Vito was extremely confident that the left side would never be smaller than the right side because he focused on the constant terms: “You are always going to add 3456, and this is higher than this one, than 7". He considered plugging in 100 for x after being asked whether the coefficients of 1.2 and 8.9 would make a difference. Upon obtaining $3576 > 897$, Vito maintained that "there is no value for x because you are always going to add 3000 more than, compared to 7."

Ali, on the other hand, pondered the results he obtained: "That's weird because when I was doing this right now, I was thinking that, I don't know, wait, I was thinking that, you know, maybe… OK, now I'm thinking that if I, just try negative, -10." Upon noticing the results for $x = -10$, Ali then considered plugging in a large number: "3444 isn't less than -82. So, what if I try a bigger number like 3000."

On two occasions, Ali foresaw changing the problem situation to elaborate his way of understanding. For Item Post-S2, Ali had difficulty applying his observation that $2x$ and 6 were twice x and 3, respectively, to explain why there would be no values of x that would make his $(2x - 6)(x - 3) < 0$ true. Instead, he considered changing the inequality to $(2x - 6)(x - 1) < 0$: "If I were to have x minus 1, I could probably get, I can make this, err, statement true." For Item Pre-T1, as discussed in Chapter 4, Ali considered changing $5a = b + 5$ to $5a = b + 20$ to illustrate how a could be bigger than b.

Revisiting The Case of Talia

Talia's case reflects the reality of students who perform relatively well in their mathematics courses yet still reason in a non-referential symbolic way. This phenomenon raises the issue of whether current high school algebra curricula are effective. This issue is discussed in the conclusion chapter on the section on *Implications for Instruction on Middle/High-School Algebra*.

On the other hand, Talia's case constitutes an existence "proof" that students ways of thinking associated with foreseeing/predicting can be advanced when appropriate learning conditions are in place. Talia showed substantial improvement in just five one-on-one problem-solving sessions. Her success was due to a combination of factors including the effectiveness of the teaching intervention, her motivation to improve her algebra, and her effort in reviewing her work and doing the homework after each of the first four problem-solving sessions.

The caveat to this proposition is that Talia's improvement is likely to dissipate if she does not continue to reason with the desirable ways of thinking that she began to develop during the problem-solving sessions. Nevertheless, the important point here is that students' ways of thinking associated with foreseeing/predicting can be advanced under appropriate conditions, such as a learning environment that promotes attending to meaning, builds on their existing ways of understanding and ways of thinking, and provides opportunities for students to explore and to predict.

The learners in this study were asked to write about their participation during the teaching intervention; one writing task was given at the end of the teaching intervention and a second writing task was given approximately a year later. Below are excerpts of Talia's comments a year after the intervention (see Appendix F).

> I started thinking in terms of signs (positive or negative) whenever I was asked to solve for an inequality. Though in pre calculus and calculus I hardly dealt with inequalities, I was able to utilize the skills I learned in the SAT's. I was able to do the inequalities in the SAT's very easily because instead of thinking in terms of numbers I thought in terms of signs and greater vs. smaller quantities.
>
> Through the "Students' Reasoning in Algebra" project, I learned that math is reasoning and using logic. One of the most important lessons I

> learned though was to try different approaches to solve problems. I also learned that it is important to review the material that you learn so that it sticks to you. The project did help me and by the end of the whole project I think I was able to reason better. I found out that my thought process was slow and that this could be a problem for me in the future but I also found that I was able to get through the training alright. In conclusion, I learned how to ask questions when solving a problem and how to guide myself through a problem by using reasoning skills.

In this section, the improvements of the four learners were compared. Talia and Chela showed greater progress than Vito and Ali. This is partly due to the effectiveness of the teaching interventions for Talia and Chela, which were held after I gained experience from working with Vito and Ali. In addition, the characteristics of each individual learner greatly influenced the amount of progress. For example, once Talia addressed her non-referential symbolic way of thinking by attending to the meanings of symbols, she could reason with symbols meaningfully.

6.6 Two Interesting Phenomena

By studying students' problem-solving behaviors in terms of the mental acts of foreseeing and predicting, I encountered two phenomena that emerged from the data analysis. They are discussed here because they are important for the field of mathematics education to recognize. The two phenomena are discussed at the end of this chapter because they were not part of my research questions.

The Recency Effect

Students' disposition of using recently learned ideas to solve a problem was observed during the teaching experiment. For example, Talia was compelled to use the critical value idea she had learned in solving $x(6x + 8) < 0$ (Item TE1-$_{T}$N2) to solve

$2y + (4y - 9) \le 0$ (Item TE2-$_T$R2). In her homework she equated $2y = 0$ and $4y - 9 = 0$ to obtain the critical values $y = 0$ and $y = 9/4$. She concluded $y \le 9/4$ and wrote, "Since you are adding, you include negative #'s in your solution set." I use the term *Recency Effect* to refer to the phenomenon in which a person applies a recently learned idea to a problem situation without checking its validity.

A similar phenomenon was observed in Chela's solution to one of the homework items: "What is the solution set for $(2x - 4) + (x - 12) \ge 0$?" Chela solved $2x - 4 \ge 0$ and $x - 12 \ge 0$, obtained and represented $x \ge 2$ and $x \ge 0$ on a number line, and boxed $x \ge 2$ as the solution. However, when the same task was re-posed in a problem-solving session (as Item TE5-$_C$R3a) after she had worked on Item TE5-$_C$N2, Chela spontaneously thought of combining like terms. This is likely because the idea of combining like terms emerged from her work on the previous item (Item TE5-$_C$N2): "Is the following statement always true, sometimes true, or never true? $(x + 2 + 3x) + 4 + (5x + 6 + 7x) < 2(8x + 9)$" This example illustrates how students' reasoning tends to depend on the recently encountered ways of understanding.

Another example of the Recency Effect was Vito's spontaneous use of a rule, which he had abstracted from working on a set of problems, for determining whether a linear inequality of the form $Ax + B < Cx + D$ is always true, sometimes true, or never true. Vito applied the rule to both the inequalities in Item TE2-$_V$N2: "Consider these two inequalities: $5x + 10 > x + 5000$ and $4x > 4990$. Is there a value for x that will make one of them true but will make the other false?"

> Vito: Mm, I think so. If x, A (i.e., the coefficient 5), this [first inequality] is, um, this is sometimes true, and this one ($4x > 4990$) is, um, sometim-, no, I don't know what it is,

> because it doesn't have a *C* … Because it doesn't have a *C*, so *A* doesn't equal *C*, but not only that, because this (4990) never changes, this will always stay the same.

The recency in learning the rule caused Vito to spontaneously apply it to each inequality without attending to what the question was asking. Vito was not tenacious in this case because he proceeded by plugging in numbers to find a value that would make one inequality true while the other false. Nevertheless, Vito's response is an instantiation of the Recency Effect.

The Recency Effect is not necessarily undesirable. Chela's ingeniously adapted her newly learned idea of making one part positive and one part negative to find a value of x that would make the inequality $(7x - 31 + 5x) - 43 < (7x - 31 + 5x) - 18$ false (Item TE4-$_C$N2). She first observed that both sides of the inequality have the common function, and predicted that, "it's going to be true … 'cause it's going to be minusing, -43, and this one is minusing -18." Chela then proceeded to "use the goal-oriented thing where … I want this one (left-side) to be positive and this one (right-side) negative … to make it false." She then found that x has to be "greater than 6.11 but less than 4.0833. … So that means it's always true." Unfortunately Chela's reasoning was flawed here because she had changed the constraint from $(7x - 31 + 5x) - 43 > (7x - 31 + 5x) - 18$ to $(7x - 31 + 5x) - 43 > 0 > (7x - 31 + 5x) - 18$. Nevertheless, the point is that applying a newly learned idea in novel situations involves repeated-reasoning and is a chance to learn from one's mistakes.

The Recency Effect may be a viable construct to account for certain unexpected results. For example, in Vaiyavutjamai and Clements's (2006) study on effects of traditional instruction on student performance of linear equations and inequalities, 9th

grade students in Thailand scored worse in the post-teaching stage (after 13 lessons on linear equations and inequalities) than both pre-teaching stage and retention stage (6 months later) on solving the linear equations, but the reverse pattern was found for inequalities. Their results are summarized in Table 6.4. The authors did not offer any explanations to account for this interesting result: 23% for the equations portion of the assessment at the post-teaching stage was lower than (a) 29% at pre-teaching stage, (b) 29% at the retention stage, and (c) 35% of the inequalities portion at the post-teaching state. The Recency Effect of students' learning about linear inequalities just before they took the post-teaching assessment is a plausible explanation for the unexpectedly low percentage of 23%, as well as for 35% (for linear inequalities at post-teaching stage) being the highest among the six percentages.

Table 6.4: Average Percentage of 231 Students Giving a Correct Solution to Five Inequalities and Corresponding Equations

	Linear Inequalities ($3x \leq 6$; $x/2 > 4$; $x - 3 \geq 7$; $1 - x \leq 0$; $3 - 4x \leq 6x - 7$)	Corresponding Equations ($3x = 6$; $x/2 = 4$; $x - 3 = 7$; $1 - x = 0$; $3 - 4x = 6x - 7$)
Pre-teaching stage	11%	29%
Post-teaching stage	35%	23%
Retention stage	24%	29%

Note. Adapted from "Effects of Classroom Instruction on Student Performance on, and Understanding of, Linear Equations and Linear Inequalities," by P. Vaiyavutjamai and M. A. Clements, 2006, *Mathematical Thinking and Learning, 8*(2), p. 132.

The Presence Effect

Another observation that emerged during the analysis of the teaching intervention was students' tendency to fixate on a particular way of understanding. This phenomenon led to the identification of tenacious anticipation as a way of thinking associated with foreseeing.

For instance, while working on Item TE2-$_T$R2, Talia did not think of combining like terms when she had the critical value idea for dealing with $2y + (4y - 9) \leq 0$. I asked her to "imagine a year ago, before you learned this critical value idea. … What would you do when you are asked to solve for y?" Talia responded, "a year ago I would have done this [the same thing]" and wrote $2y = 0$ and $4y - 9 = 0$. I had to offer this inequality $2n + 4n - 9 \leq 0$ in order to get Talia to think of combining like terms. The removal of the parentheses[29] in this example was necessary in order for Talia to breakaway from the critical value approach. I use the term *Presence Effect* to refer to the phenomenon in which a particular way of understanding prevents a person from considering alternative ideas.

In Item Pre-A5, the presence of the formula $a_n = d(n - 1) + a_1$ prevented Chela from noticing that $3 + 4 + \ldots + 102$ is greater than 200. She used the same formula three times, for different values of x, to determine the value of n. When she used it for the first time for $x = 2$, she found $n = 51$, but she did not attend to the meaning then. When she used it for the second time for $x = 5$, she found $n = 100$ and related it to there being 100 terms. However, when she worked with $x = 100$, she used it to obtain $n = 100$, even though she already knew there were 100 terms. As suggested in these examples, the Presence Effect seems to prevent a person from "thinking outside the box."

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[29] The presence of the parentheses was not the only factor that resulted in Talia's not combining like terms because she combined like terms when she was working with $(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$ in Item Pre-A1.

The Relevance of the Recency Effect and the Presence Effect to Mathematics Education

The Recency Effect is important to mathematics educators because an awareness of it may cause educators to think differently about their choice and sequencing of instructional activities. For example, a teacher may create a learning situation to perturb students by capitalizing on his knowledge of students' tendency to apply certain recently learned ideas. This strategy was used in the teaching intervention in this study. For example, the inequality $2y + (4y - 9) \leq 0$ was presented after Talia had learned the critical idea approach.

An awareness of the Presence Effect allows teachers to be more sensitive to students' ways of understanding. Recognizing students who are tenacious in their ways of understanding may influence some teachers to deal with those students differently. Vito's lack of success in advancing his ways of thinking during the intervention was in part due to my lack of awareness of Vito's tenacity. As such, the teaching actions during the intervention were not aimed at helping Vito address this undesirable way of thinking.

The Recency Effect and the Presence Effect also suggest that students have a tendency to minimize their effort in thinking. If this is the case, then the Recency Effect and the Presence Effect may be thought of as instantiations of a more general phenomenon, which I call the *phenomenon of applying minimal cognitive effort*.[30] While working on mathematics problems, some students prefer to "do" rather than to "think". My conjecture is that students tend to feel more productive when they are "doing"

[30] I speculate that there might be a theory that could account for the phenomenon of applying minimal cognitive effort because a person generally wants to get the most by doing the least. I speculate that a person's non-volitive actions are naturally governed by the *principle of minimal effort*.

something than when they are "thinking" about the problem situation. A possible area for future research is the relation between students' ways of thinking associated with anticipating and their tendency to engage in minimal cognitive effort.

6.7 Recapitulating the Main Points

In this chapter, the case of Talia confirms that students' ways of thinking associated with foreseeing/predicting can be improved via a teaching intervention that is guided by DNR-based instruction. Talia's improvement can be summarized in terms of the change in the sub-context in which she operated; she moved from the sub-context of manipulating symbols in the pre-interview to the sub-context of reasoning with symbols in the post-interview. When operating in the sub-context of manipulating symbols, Talia tended to interpret an inequality/equation as a signal to do something and thus exhibited impulsive anticipation. She also tended to manipulate symbols without attending to meanings and thus made association-based predictions. In contrast, when operating in the sub-context of reasoning with symbols, Talia tended to interpret inequalities/equations in a meaningful manner, which allowed her to be goal-oriented and thus exhibit analytic anticipation. She also tended to study the inequality and thus made coordination-based predictions.

Data from the teaching intervention reveals that Talia's transition from the sub-context of manipulating symbols to the sub-context of reasoning with symbols involved her working with specific numbers and reasoning with general numbers. Three main factors that could account for Talia's improvement were identified: (a) attending to the meaning and referents of symbols, (b) exploring her ways of understanding, and

(c) making predictions. Other factors include using numbers as a platform to investigate symbolic structures and using non-directive tasks to promote exploration and prediction. The difficulties Talia encountered during the teaching intervention include difficulty developing: (a) ways of understanding related to solution sets and equivalent inequalities, (b) the considering-falsity way of thinking, and (c) ways of understanding related to arithmetic such as negative numbers and distributive property.

In this study, the four learners' ways of thinking were rather distinct. In the pre-interview, Talia had a tendency to engage in impulsive anticipation and association-based prediction while she was operating in the context of manipulating symbols. Chela had a tendency to engage in explorative anticipation while operating in the sub-context of plugging in numbers. Among the four learners, Vito had the greatest tendency to be tenacious in his ways of understanding, while Ali had the greatest tendency to consider alternatives.

Finally, two interesting phenomena emerged during this study. The Recency Effect refers to the phenomenon in which a student applies a recently learned idea to a problem situation without checking its validity. The Presence Effect refers to the phenomenon in which the presence of a particular way of understanding prevents a student from considering alternative ideas.

CHAPTER 7: CONCLUSION

This chapter is organized into four sections: (a) a summary of the major results, (b) contribution to the field of mathematics education, (c) implications for instruction, (d) limitations of this research, and (e) directions for future research.

7.1 A Summary of the Major Results

In Chapter 1, a discussion of the pilot study demonstrates that students exhibit different problem-solving behaviors. Some problem-solving behaviors are more desirable than others. This dissertation study examined students' problem-solving behaviors by focusing on the mental acts of foreseeing/predicting. In Chapter 4, a comparison between two students' responses (Talia and Pham) to Item Pre-S1 illustrates the usefulness of examining students' ways of thinking associated with foreseeing/predicting for differentiating between desirable and undesirable problem-solving behaviors. Identifying ways of thinking associated with foreseeing provides educators with the vocabulary necessary to communicate the way a student approaches a problem. For example, they can describe whether the student hastily applies a procedure (impulsive anticipation), is tenacious in her or his way of understanding (tenacious anticipation), explores different ideas (explorative anticipation), analyzes the problem situation and identifies a goal (analytic anticipation), or spontaneously applies well-established ways of understanding (interiorized anticipation). Ways of thinking associated with predicting allow educators to communicate about the bases for student predictions: for example, whether the prediction is based on an association, a comparison, or some coordination.

The categories of ways of thinking associated with foreseeing, ways of thinking associated with predicting, and ways of understanding inequalities/equations that emerged from the data are listed in Table 4.1 and Table 5.1. Some of these categories are more desirable or sophisticated than others, as presented in Table 7.1.

Table 7.1: A Summary of Ways of Thinking and Ways of Understanding in Terms of Desirability/Sophistication

	More Desirable Categories	**Neutral Categories**	**Less Desirable Categories**
Ways of thinking associated with foreseeing	Analytic anticipation Interiorized anticipation	Explorative anticipation	Tenacious anticipation Impulsive anticipation
Ways of thinking associated with predicting	Coordination-based prediction	Comparison-based prediction	Association-based prediction
Ways of understanding inequalities/ equations	I/E-as-a-comparison-of-functions I/E-as-a-constraint I/E-as-a-proposition		I/E-as-a-static-comparison I/E-as-a-signal-for-a-procedure

Relationship between Ways of Thinking Associated with Foreseeing/Predicting and Problem-solving

One of the objectives of this study was to explore the feasibility and usefulness of using characteristics of students' foresight/prediction as a means to communicate the quality of students' problem solving. Major findings concerning the relationship between ways of thinking associated with foreseeing/predicting and problem solving were discussed in detail in Chapter 4, and are briefly summarized below.

1. Association-based prediction is related to the non-referential symbolic way of thinking, in which one reasons without attending to the referent or the meaning of symbols.
2. Impulsive anticipation is related to forward-strategy, which is a problem solving approach that does not involve means-ends analysis, whereas analytic anticipation is related to backward-strategy, which involves means-ends analysis or goal-oriented reasoning.
3. Tenacious anticipation is related to inflexible reasoning, in which one tends to engage in mathematics without an element of doubt or consideration for alternatives.
4. Coordination-based prediction is related to reasoning with change, a way of thinking essential for developing the *process conception* of function.
5. Analytic anticipation facilitates problem solving because it guides one's exploration. However, analytic anticipation does not always lead to success, especially when critical ways of understanding do not emerge from exploration and analysis. In that case, input from the teacher or classmates may be necessary.
6. Interiorized anticipation provides efficiency because one can capitalize on one's existing ways of understanding instead of engaging in analytical or exploratory actions.

Relationship between Ways of Thinking Associated with Foreseeing/Predicting and Ways of Understanding Inequalities/Equations

The way a student interprets a problem situation influences the actions he (or she) foresees or the results he predicts. Conversely, what he foresees/predicts may subsequently modify his understanding of the problem situation. Four specific relations between interviewees' ways of thinking associated with foreseeing/predicting and their ways of understanding inequalities/equations (I/E) were discussed in Chapter 5. These are summarized below.

1. The I/E-as-a-signal-for-a-procedure interpretation tends to result in impulsive anticipation and association-based prediction. When a student interprets an inequality or equation as a signal to do something, the student is more likely to rush into actions than to analyze the problem situation and is also more likely to make associations without attending to the meanings or the referents of the symbols.
2. The I/E-as-a-constraint interpretation is related to analytic anticipation. When a student interprets an inequality/equation as a constraint, the student tends to analyze the inequality/equation in terms of what would make it true or false.
3. The I/E-as-a-comparison-of-functions interpretation is usually related to coordination-based prediction. When a student interprets an inequality/equation as a comparison of functions, the student tends to coordinate the input value with the output values of the functions.
4. Interiorized anticipation requires one to have interiorized the relevance of the anticipated action to the situation at hand. For problems involving inequalities/equations, interiorized anticipation usually involves sophisticated ways of understanding inequalities/equations such as I/E-as-a-comparison-of-functions and I/E-as-a-constraint.

Change in Learners' Ways of Thinking and Ways of Understanding

To understand change in a learner's ways of thinking and understanding, five 60-minute one-on-one problem-solving sessions were conducted with four high school students. Among the four learners who participated in the teaching interventions, Talia (Calculus) showed the greatest improvement, followed by Chela (Algebra 2), Ali (Pre-calculus), and Vito (Pre-calculus). The teaching interventions for Talia and Chela were more effective because they built on the instructor's experiences working with Ali and Vito. Talia demonstrated more improvement than Chela, probably because Talia's baseline was lower as a result of her non-referential symbolic way of thinking. Chela, on the other hand, lacked the prior experience Talia had in reasoning with symbols because Chela was two mathematics courses behind Talia.

The case of Talia demonstrates that students' ways of thinking associated with foreseeing/predicting can be advanced through an instructional intervention that is guided by the pedagogical principles of Harel's (2001) DNR-based instruction. Talia's improvements are summarized below.

1. In terms of ways of thinking associated with foreseeing, Talia showed improvement by moving from impulsive anticipation to analytic anticipation. Her analytic anticipation and explorative anticipation were generally more sophisticated in the post-interview because she was reasoning with symbols in a deductive manner. During the pre-interview, she reasoned with numbers in an inductive manner and manipulated symbols in a non-referential symbolic manner. In addition, Talia demonstrated the using-numbers-as-a-means-to-reason-with-structure way of thinking in the post-interview but not in the pre-interview.
2. In terms of ways of thinking associated with predicting, Talia showed improvement by moving from association-based prediction to coordination-based prediction. In addition, she predicted more often in the post-interview than in the pre-interview.

3. In terms of ways of understanding inequalities/equations, Talia showed improvement by moving from I/E-as-a-signal-for-a-procedure and I/E-as-a-static-comparison to I/E-as-a-comparison-of-functions.

The above improvements are related to Talia's progress from operating in the sub-context of manipulating symbols in the pre-interview to operating in the sub-context of reasoning with symbols in the post-interview. While operating in the sub-context of manipulating symbols, Talia tended to (a) interpret an inequality/equation as a signal to do something and thus exhibited impulsive anticipation, and (b) manipulate symbols without attending to meanings and thus made association-based predictions. In contrast, when operating in the sub-context of reasoning with symbols, she tended to (a) interpret inequality/equation in a meaningful manner that would allow her to be goal-oriented and thus exhibited analytic anticipation, and (b) study the inequality and coordinate among various quantities and thus made coordination-based predictions.

Transition from Manipulating Symbols Non-referentially to Reasoning with Symbols Structurally

Talia's trajectory from manipulating symbols non-referentially to reasoning with symbols involved three transitions: (a) from manipulating symbols to working with specific numbers, (b) from working with specific numbers to reasoning with general numbers, and (c) from reasoning with general numbers to reasoning with symbols. Each of these transitions is described below.

1. A critical way of understanding underlying the transition from manipulating symbols to working with specific numbers is the realization that the referents of the literal symbols in an inequality/equation are numbers. Posing questions such as, "What does

this mean?" could help students develop this way of understanding and foster the referential symbolic way of thinking.

2. The transition from working with specific numbers to reasoning with general numbers (e.g. negative numbers or an interval of values) depends on task characteristics. For example, the factor x in the inequality $x(6x + 8) < 0$ led Talia to reason with positive numbers and with negative numbers.
3. The transition from reasoning with general numbers to reasoning with symbols may involve certain ways of understanding and ways of thinking. In Talia's case, the critical way of understanding was making-one-function-positive-and-one-function-negative-to-make-the-product/quotient-negative. Reason with symbols and structure in unfamiliar situations inevitably involves analytic anticipation and explorative anticipation.

Factors that Could Improve One's Ways of Thinking Associated with Foreseeing/Predicting

Factors that could account for Talia's improvement from pre-interview to post-interview were identified. Listed below are five factors that can generally help students advance their ways of thinking associated with foreseeing/predicting and their ways of understanding.

1. *Attending to meaning and referents for symbols*. To counter the non-referential symbolic way of thinking and association-based predictions, students have to attend to meanings or the referents of the symbols. In non-contextualized problem situations, the referents for literal symbols are usually numbers.

2. *Numbers as a platform for investigating algebraic expressions*. Once students view numbers as referents for symbols, they could explore the symbolic structure(s) in an inequality/equation and encounter certain ways of understanding that are critical for reasoning with symbols. Such experiences foster analytic anticipation and explorative anticipation.
3. *Opportunity for students to explore*. The opportunity for students to explore their ways of understanding can deepen their existing ways of thinking and ways of understanding, lead them to encounter new ways of understanding, and expose their deficient ways of understanding.
4. *Opportunity for students to predict*. The opportunity for students to predict, and then to check their prediction by plugging in numbers, can help students avoid their rushing into a procedure, to learn from their predictions, to abstract certain structures, and to cultivate a disposition to predict.
5. *Use of non-directive tasks*. Non-directive tasks are more likely to lead students to engage in exploring and predicting. In contrast, directive tasks such as “solve for *x*” and “simplify” tend to foster impulsive anticipation.

The Recency Effect and the Presence Effect

The Recency Effect and Presence Effect are indications that students prefer to “do” something rather than to “think” about the problem situation. The Recency Effect refers to the phenomenon in which a student applies a recently learned idea to a problem situation without checking its validity. The Presence Effect refers to the phenomenon in which the presence of a particular way of understanding prevents a student from considering alternative ideas.

7.2 Contribution to the Field of Mathematics Education

This research is novel in several ways: (a) it uses Harel's (2001) notion of mental acts to study students' problem solving, (b) it provides a preliminary framework for studying students' act of foreseeing/predicting in the context of problem solving, (c) it characterizes students' problem-solving behaviors in terms of their ways of thinking associated with foreseeing and predicting, and (d) it identifies common student interpretations of inequalities. Each of these points is discussed below.

Pioneering the Use of Mental Acts as a Means for Studying Students' Problem Solving

One contribution of this research to the field of mathematics education is the novel use of mental acts as an analytical tool to study students' problem-solving behaviors. This research is the first of its kind to use Harel's (2001, in press c) MA-WoU-WoT (Mental act - Way of Thinking - Way of Understanding) triad for studying students' problem solving in terms of acts of anticipating. As explained in Chapter 2, this triad allows researchers to study students' thinking at a fine-grained level. The act of problem solving involves various interrelated mental acts. By focusing on one mental act at a time and identifying the product and character of the act, researchers can examine and express students' thinking with greater clarity. Combining findings from various acts can provide a broader picture of students' problem solving while retaining detailed information.

The results obtained in this exploratory study of students' acts of foreseeing and predicting provide an existence "proof" of the viability of studying students' problem-solving by focusing on specific mental acts. These results could encourage other researchers to study students' problem solving by focusing on other mental acts and to identify ways of thinking associated with those acts.

Providing a Preliminary Framework for Studying Students' Mental Acts of Foreseeing and Predicting

This study synthesizes Piaget's (1967/1971) notion of anticipation, von Glasersfeld's (1998) categorization of anticipation, and Cobb's (1985) hierarchical levels of anticipation. Furthermore, as previously stated, it utilizes Harel's (2001) MA-WoU-WoT triad to study students' anticipations in the context of problem solving.

Based on von Glasersfeld's (1998) identification of three types of anticipation, I identified three aspects of anticipation: the regulatory aspect, the predictive aspect, and the volitive aspect. In this research, I found that the regulatory aspect of anticipation cannot be inferred from students' actions and statements, and I separated the act of anticipating a result from the act of anticipating an action. I call the former act "predicting" and the latter act "foreseeing". These two acts correspond to the predictive aspect and the volitive aspect of anticipation.

I found Cobb's (1985) notion of sub-context to be useful for conceptualizing students' anticipations at a broader level. I found that students' anticipations were largely influenced, as proposed by Cobb, by the sub-context in which they operated. In this research, three sub-contexts were identified: manipulating symbols, plugging in numbers, and reasoning with symbols. These sub-contexts provide educators with a broader perspective on students' improvement in relation to ways of thinking associated with foreseeing/predicting and ways of understanding inequalities/equations. For example, Talia's learning trajectory was depicted as transitions from one sub-context to another.

Using Harel's (2001) MA-WoU-WoT framework, I identified five categories for ways of thinking associated with foreseeing, three categories for ways of thinking associated with predicting, and five categories for ways of understanding inequalities/equations. These observation categories are still in the early stages of development and are subject to further refinement and modification.

Figure 7.1 depicts the hierarchy between sub-context and mental acts in the sense that one's acts of foreseeing, predicting, and interpreting depend on the sub-context in which one operates. This preliminary framework can be further extended to include other mental acts.

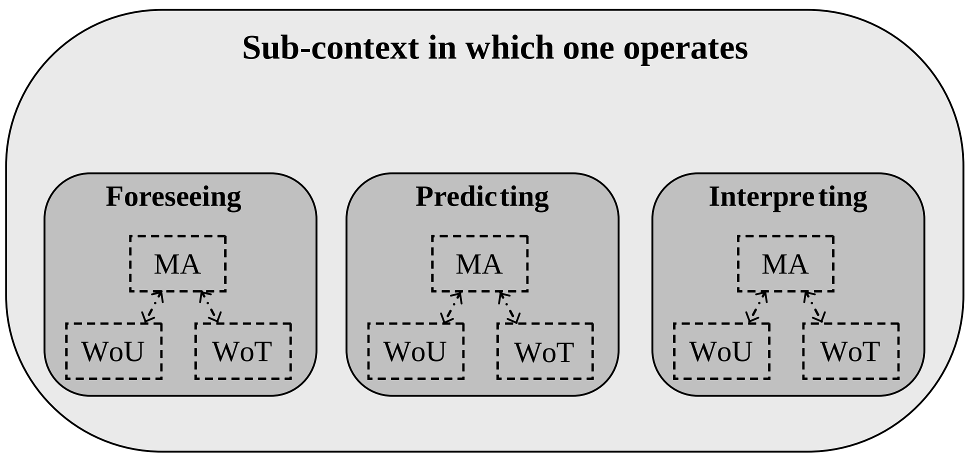


Figure 7.1: The dependence of mental acts on sub-context

Providing Categories that Characterizes Students' Problem-solving Behaviors

Two related phenomena are commonly observed in mathematics classrooms: students rush into a procedure without analyzing the problem situation, and students make a quick association without considering the basis for the association. This research provides explicit terms to characterize such behaviors: impulsive anticipation and association-based prediction. With names to communicate such undesirable behaviors, teachers may be more likely to recognize them and take measures to address them. Teachers can then specify or identify desirable ways of thinking that are more appropriate. For example, a teacher can establish comparison-based prediction as an interim instructional objective for a student who tends to engage in association-based prediction, while maintaining an intention for the student to develop coordination-based prediction.

In this research, a preliminary list of ways of thinking associated with foreseeing and ways of thinking associated with predicting was developed. This list could be used and refined by other researchers. This list can also be used by mathematics teachers for describing their students' problem-solving behaviors. This list is especially useful for teachers who adopt a curriculum that emphasizes problem-solving and reasoning because ways of thinking that are desirable can be set as cognitive objectives for their students.

Identifying Students' Ways of Understanding Inequalities/Equations

Previous research on algebraic inequalities has focused on student errors in solving inequalities (Tsamir et al., 1998; Tsamir & Almog, 2001; Tsamir & Bazzini, 2002), solutions of inequalities (Vaiyavutjamai & Clements, 2006; Tsamir & Bazzini, 2003), equivalent inequalities (Sfard & Linchevski, 1994b), algebraic-graphical connection (Garuti et al., 2001; Sackur, 2004), and equation-inequality connection (Kieran, 2004). There are virtually no published studies that have focused on students' interpretation of inequalities. Part of this research focuses on students' act of interpreting inequalities and equations. The categories for ways of understanding inequalities/equations that were identified in this research can be extended and refined by other researchers.

7.3 Implications for Instruction on Middle/High-School Algebra

Based on the findings of this research and a review of the literature on the learning and teaching of algebra, I offer the following recommendations for instruction of algebra in general, and of algebraic inequalities and equations in particular: (a) incorporate ways of thinking as cognitive objectives, (b) build on students' ways of understanding and ways of thinking, (c) use non-directive tasks, (d) introduce algebraic inequalities prior to algebraic equations, and (e) strengthen students' connections between algebra and arithmetic.

Incorporate Ways of Thinking as Cognitive Objectives for Instruction

Students' problem-solving behaviors are influenced by their classroom experiences, which depend on teachers' actions. These actions are in turn influenced by the teachers' cognitive objectives for their students. Many teachers focus mainly on ways of understanding (e.g., facts, procedures, explanations, theorems, and proofs) and fail to effectively help students develop desirable ways of thinking. An exclusive focus on ways of understanding may even cause students to develop undesirable ways of thinking. For example, consider impulsive anticipation, like when Talia spontaneously multiplied out the factors in the inequality $(2x - 6)(x - 3) < 0$ without analyzing the problem situation. To help students abandon this impulsive way of thinking, teachers should have a clear image of alternative ways of thinking that they want their students to develop, such as explorative anticipation and analytic anticipation. Teachers should then set these desirable ways of thinking as instructional objectives in addition to the mathematical concepts (i.e., ways of understanding) specified in the standards.

However, according to Harel and Sowder (2005), teaching ways of thinking directly to students is unproductive. According to the Duality Principle (Harel, 2001, in press a), it is through the construction of ways of understanding that students develop ways of thinking; conversely, it is through the application of ways of thinking that students develop their ways of understanding. These ways of understanding may be deficient initially, but can be progressively refined towards those that are institutionalized (i.e., accepted as correct and useful by the mathematics community). Hence, the target ways of thinking and ways of understanding must complement each other, so that applying certain ways of thinking will lead to the development of certain ways of understanding, which may help to cultivate target ways of thinking. For example, recall Talia's trajectory from manipulating-symbols to reasoning-with-symbols; the plugging-in-number strategy (an instantiation of explorative anticipation) enabled Talia to encounter the making-one-factor-positive-and-one-factor-negative way of understanding. Repeated use of this way of understanding should reinforce the goal-oriented way of thinking and the reasoning-with-structure way of thinking (both of which are related to analytic anticipation). In summary, the Duality Principle should guide the cognitive objectives that instructors set.

Implementing instructional activities to achieve those objectives is just as important as incorporating desirable ways of thinking as cognitive objectives for instruction. The remaining four sub-sections describe factors that algebra teachers should consider when structuring their courses and when planning and implementing their lessons.

Build on Students' Ways of Understanding and Ways of Thinking

In order to help students address their undesirable ways of understanding and ways of thinking or advance their existing ones, teachers must be aware of students' current ways of understanding and thinking. In classroom situations, teachers may not be able to attend to the ways of understanding or ways of thinking of all students simultaneously, but they can be sensitive to those students with whom they are interacting. In a one-on-one situation, a teacher should focus on the learner's ways of understanding and ways of thinking, rather than focusing on how best to impart or explain her or his own ways of understanding to the learner.

An awareness of the learner's ways of understanding and ways of thinking can empower the teacher to interact effectively with the learner; for instance, the teacher can pose questions to help the learner encounter certain desirable ways of understanding. For example, as discussed in the section on *Attending to Meaning and Referents of Symbols* in Chapter 6, the question "Are you able to find the 'biggest' negative number that will make it ($\frac{x+7}{x+11} > 0$) true?" was posed after Talia had predicted that positive numbers and negative small numbers would make the inequality true. This question was based on my awareness of Talia's way of understanding (making both numerator and denominator positive would make the inequality true). The intent of this question was for Talia to encounter the notion of critical value in a meaningful manner. This query presented a need for Talia to search for the critical value. The posing of this question provides an example of the implementation of the Necessity Principle (Harel, 1998). To implement the Necessity Principle effectively, a teacher must know, to some extent, her or his students' ways of understanding and ways of thinking.

Use Well-designed Tasks

As mentioned in Chapter 3, task characteristics influence the way students think about a problem. For example, consider the following tasks: (a) Solve for x: $(2x - 6)(x - 3) < 0$; (b) Is there a value for x that will make the $(2x - 6)(x - 3) < 0$ true; and (c) Is $(2x - 6)(x - 3) < 0$ always true, sometimes true, or never true? The first task is considered a directive task in that it directs the student to do something. The second task may be interpreted by some students as a directive task to plug in numbers for x into the inequality. It may also be interpreted more literally by students as a question, which they can ponder. The third task is more likely to constitute a problem for students to solve. It may even be a source of puzzlement for some students. These three tasks serve different purposes. In general, the third task should be used first because it is more likely to intrigue students and lead them to explore different ways to solve the problem. The second task is useful for fostering the *plugging-in-numbers-to-notice-patterns-or-structure* way of thinking. The first task is useful for helping students to internalize and retain certain procedures. In summary, teachers must be sensitive to task characteristics and select, or create, tasks according to the needs of their students.

Algebra has traditionally been difficult for many students. This might be due to the prevalence of directive tasks and the lack of non-directive tasks. While directive tasks may help students develop efficiency, they tend to promote the non-referential symbolic way of thinking. This happens because students can manipulate symbols according to prescribed rules and obtain the correct answer without knowing the referents of those symbols or the underlying principles behind those rules. Comments such as, "What is there to think about when I already know how to do it?" are not uncommon among algebra students. For these students, directive tasks foster impulsive anticipation. Ultimately, mathematics educators want students to engage in interiorized anticipation, but most students will need to work with non-directive tasks prior to encountering directive tasks.

The importance of curricula being based on problems is discussed in Chapter 1. A problem-based curriculum tends to use non-directive tasks. A reason for preferring non-directive tasks is that they encourage students to explore and analyze the problem situation. The task should challenge students to apply and adapt their existing ways of understanding and ways of thinking. Through exploration and reflection, students can encounter new ways of understanding, expose and address their deficient ways of understanding and ways of thinking, cultivate desirable ways of thinking, and develop new ways of thinking.

To encourage prediction, a non-directive task should be phrased as a question with forced choices (e.g., yes or no; true or false; always true, sometimes true or never true; larger than, equal to, or smaller than). The pedagogical value of having students predict prior to performing is discussed in Chapter 1. Generally speaking, when students predict they are more likely to avoid impulsive application of a certain procedure, to focus on the big picture, and to attend to symbolic structure.

Sequencing of tasks is also important. Teachers should take into account the recent ways of understanding and ways of thinking that their students have learned or encountered. According to the Recency Effect, students have a tendency to apply recently learned ways of understanding. Hence, teachers can capitalize on such tendencies to create cognitive conflict in students through proper sequencing of tasks. For example, sequencing $2y + (4y - 9) \leq 0$ after $x(6x + 8) < 0$ led Talia to experience cognitive conflict: "Mmm, I thought why can't I set this ($2y$) and this part ($4y - 9$), both equal to 0. But then I'm adding, so I don't know, I'm not sure if I can do that." Tasks can also be sequenced to help students encounter new ways of understanding. For example, the use of $5(8x - 20) < 10(8x - 20)$ after the use of $5 + (8x - 20) < 10 + (8x - 20)$ allows students to encounter the *considering-negative-numbers* way of understanding and to develop the *considering-falsity* way of thinking.

Introduce Algebraic Inequalities prior to Algebraic Equations

In traditional curricula, a substantial amount of time is dedicated to learning and practicing techniques for solving different types of equations, often beginning with one-step linear equations, such as $x + 3 = 10$ and $5x = 18$, and progressing to equations involving logarithms, such as $\log(2x) + \log(x + 1) = 2\log(x - 3)$. Algebraic inequalities are often taught as add-ons to equations, resulting in many students treating inequalities as equations and making inappropriate analogies between the two (Tsamir & Almog, 2001; Tsamir et al., 1998). In a problem-based curriculum, algebraic inequalities should be introduced with, if not prior to, algebraic equations. The advantage of introducing inequalities first is that students are more likely to attend to the variable attribute of a literal symbol while solving inequalities. This is because the solution to a single-variable inequality is a range of numerical values, whereas the solution to a single-variable equation is usually a specific number. As a result, inequalities tend to foster the *letter-as-a-variable* way of understanding whereas equations tend to foster the *letter-as-an-*

unknown way of understanding. Students are more likely to engage in coordination-based prediction with the former and comparison-based prediction with the latter.

Strengthen Students' Arithmetic-Algebra Connection

According to Lee and Wheeler (1989), the difficulties students face in algebra are partly a result of their inability to relate algebra to arithmetic: "Students behaved as though algebra were a closed system untroubled by arithmetic" (p. 46). This phenomenon is observed in Talia's responses to items Pre-S1 and Pre-S2. The improvement Talia showed from the pre-interview to the post-interview was mainly due to her change from operating in the sub-context of manipulating symbols non-referentially to operating in the sub-context of reasoning with symbols. This change involved an intermediate stage during which Talia operated in the sub-context of working with numbers. This suggests that to help algebra students who reason in a non-referential symbolic manner, teachers should spend time helping them strengthen their arithmetic-algebra connection instead of reviewing procedures and rules that were taught in a previous course.

Ideally, a pre-algebra course should focus primarily on grounding students' algebraic experience with quantitative relationships as well as with numbers. Current reform efforts, such as Mathematics in Context (Romberg & Lange 1998) and Realistic Mathematics Education (Freudenthal, 1991), tend to emphasize the former, in which the referents for literal symbols are quantities such as distance, time, weight, cost, and number of items. Balacheff (2001) and Kirshner (2001) highlight that using contextually rich activities does not help students to develop deductive rigor in their algebraic reasoning. In fact, over-reliance on contextualized situations may even create an obstacle to students developing such rigor in more advanced algebra courses.

A complementary approach would be to use numbers as referents for literal symbols. By effectively using well-designed tasks, this approach can help students develop *plugging-in-numbers-to-notice-patterns-or-structure* strategy and promote reasoning-with-structure way of thinking. For example, tasks such as "Is $5x + 6 < 5x + 18$ always true, sometimes true, or never true?" can help students abstract certain structural relationships in linear equations of the form $Ax + B < Cx + D$, such as realizing that if $A \neq C$, then the inequality must be sometimes true. When students' learning of algebraic inequalities and equations is not grounded in numbers, they are more likely to engage in association-based reasoning, such as associating the disappearance of x with the inequality having no solutions.

In conclusion, algebra teachers should use a problem-based curriculum in which both numbers and quantities are referents for literal symbols. Teachers must help students develop a strong arithmetic-algebra connection through effective use of non-directive tasks. Finally, teachers should be sensitive to students' ways of understanding and ways of thinking, and they should strive to help their students develop desirable ways of understanding and ways of thinking.

7.4 Limitations of this Research

The observation categories developed during this study are based on a very small sample size and concern a single domain in mathematics. They are therefore neither robust nor exhaustive. In fact, they are still in the early stages of development. More research is required to refine and extend these categories.

The categories developed for ways of thinking associated with foreseeing and ways of thinking associated with predicting in this research are considered local in the sense that they are descriptors to characterize an occurrence of an act, although other occurrences are taken into consideration in the analysis of any particular occurrence. Strictly speaking, the categories developed in this research are not considered ways of thinking, because terms such as *analytic anticipation* and *coordination-based prediction* do not suggest the image of a conceptual tool, although they do characterize a student's mental act of anticipating. Terms such as *analytic anticipative scheme* and *coordination-based predictive scheme* suggest the general character that underlies students' acts of foreseeing/predicting. These can be conceived of as conceptual tools that students can apply to solve problems through the act of foreseeing or predicting.

As explained in a footnote in Chapter 4, I chose to use terms like analytic anticipation and coordination-based prediction instead of analytic anticipative scheme and coordination-based predictive scheme. The use of the term anticipative/predictive scheme entails a greater inference about the way students foresee/predict in general, rather than in a specific instance. There are many factors that could contribute to a student's foresight/prediction. I could not determine whether a student's foresight/prediction was due to the student's anticipative/predictive scheme (especially when I did not know what they were yet) or other circumstantial factors. Nevertheless, creating categories that characterize students' foresight/prediction is necessary prior to creating categories for ways of thinking in terms of schemes. The next step would be to "convert" these observation categories into anticipative schemes and predictive schemes. The data collected and transcribed in this research could be used for such a purpose.

Harel and Sowder (1998) have used proof schemes to characterize students' mental act of proving, which includes two sub-processes: ascertaining and persuading. Similarly, foresight and prediction are actually two aspects of anticipation—volitive and predictive, respectively. Therefore, anticipative schemes and predictive schemes can be merged into one class. If conceiving ways of thinking in terms of schemes proves to be viable for the mental acts of foreseeing and predicting, then schemes are probably appropriate for characterizing other mental acts.

7.5 Directions for Future Research

Studies to refine and extend the categories developed in this research. Similar research could be conducted in the same domain with students in a lower grade level such as students who are taking Pre-algebra or Algebra 1. To increase the robustness of these categories, research on other domains in mathematics may be necessary.

Research to extend the existing framework. In terms of Cobb's hierarchical levels of anticipation, this research focuses on the most specific level of anticipation, which includes foresight of actions and prediction of results, within a heuristically constrained sub-context. Subsequent research could focus on students' anticipation of heuristics and their movement between sub-contexts.

Research on other mental acts in the context of problem solving. As mentioned earlier, the act of problem solving encompasses many interrelated mental acts. An extension to this research would be to study students' mental acts of exploring and analyzing. In this research, I found that students who have an element of doubt (i.e., consider the "falsity" of their way of understanding) tend to demonstrate more sophisticated ways of thinking. It could be worthwhile to study the mental acts of exploring and analyzing, examining effects of doubt

Research on task characteristics. The non-directive tasks used in this research are appropriate for eliciting students' foresights and predictions. Further research should explore how task characteristics influence students' foresights and predictions. The effect of task characteristics on students' learning could also be investigated. In addition, the effectiveness of using non-directive tasks for pedagogical purposes should be studied.

Classroom studies. This research offers several recommendations for improving the teaching of inequalities and equations. Based on these recommendations and the findings in this research, a curriculum on algebraic inequalities and equations could be developed for pre-algebra. Subsequently, an action research study could be conducted to examine the effectiveness of such a curriculum.

7.6 Conclusion

This dissertation pioneers an investigation on the viability of characterizing students' problem-solving behaviors based on their acts' of anticipation. It combines multiple perspectives: Piaget's (1967/1971) notion of anticipation, von Glasersfeld's (1998) three general kinds of anticipation, Cobb's (1985) hierarchical levels of anticipation, and Harel's (2001, in press a, in press c) notions of mental act, way of understanding, and way of thinking. It differentiates between two types of anticipating acts: foreseeing an action and predicting a result.

One objective of the research was to identify and categorize students' ways of thinking associated with foreseeing and predicting. Five ways of thinking associated with foreseeing were identified. These ways of thinking allow educators to communicate the way a student approaches a problem, such as whether he or she hastily applies a hastily applies a procedure (impulsive anticipation), is tenacious in his or her way of understanding (tenacious anticipation), explores different ideas (explorative anticipation), analyzes the problem situation and identifies a goal (analytic anticipation), or spontaneously applies well-established ways of understanding (interiorized anticipation). Having made these categories explicit, mathematics teachers can design and implement instructional activities that aim to help students progress from being impulsive to being analytic and from being tenacious to being explorative.

Three ways of thinking associated with predicting were identified. These ways of thinking allow educators to communicate the bases underlying students' predictions, whether a prediction is based on an association, based on a comparison, or based on some coordination. Such distinctions can help teachers to be more explicit about their goal of advancing students from association-based prediction to coordination-based prediction.

The second objective concerned the relationship between students' ways of thinking associated with foreseeing/predicting and their ways of understanding algebraic inequalities/equations. The relationship between the desirability of students' ways of thinking associated with predicting/foreseeing and the sophistication in their ways of understanding inequalities/equations suggests that we, as teachers, should attend to students' ways of thinking associated with predicting/foreseeing while helping students to develop sophisticated ways of understanding inequalities/equations, and vice versa. This recommendation is in keeping with Harel's (in press a) call to incorporate desirable ways of thinking and sophisticated ways of understanding as cognitive objectives for instruction: “In designing, developing, and implementing mathematics curricula, ways of thinking and ways of understanding must be the ultimate cognitive objectives, and they must be addressed simultaneously, for each affects the other.”

The third objective involved investigating the potential for advancing students’ ways of thinking through an instructional intervention informed by Harel’s (2001, in press a) DNR-based instruction. One learner, Talia, demonstrated substantial improvement after five problem-solving sessions. In the pre-interview, she demonstrated impulsive anticipation and association-based prediction while operating in the sub-context of manipulating symbols. In contrast, in the post-interview, she demonstrated analytic anticipation and coordination-based prediction while operating in the sub-context of reasoning with symbols. The improvement from manipulating symbols non-referentially to reasoning with symbols involved an intermediate stage in which Talia operated in the sub-context of working with numbers. This observation underscores the importance of providing opportunities to explore algebraic expressions and symbolic structures by using numbers as referents for literal symbols.

APPENDIX A: WRITTEN INSTRUMENT

A Short Activity prior to Administering the Written Assessment

In order to give the students an idea of what the survey was about, I conducted a pre-assessment activity. This activity served two purposes (a) to let the students experience the fun of seeing other students' strategies, and (b) to communicate to them the research focuses on their thoughts and reasoning rather than on their answers.

I handed out the activity sheet (considered as the first item of the survey on students' reasoning) and gave them 5-10 minutes to solve the problem. We then had a whole-class discussion to elicit their reasoning. "Let me list down all the initial responses you have and then take a quick poll. OK, what are some of the initial responses you have?" After listing all the initial responses I asked them to share their final solutions and what caused them to change their mind. I listed down all the strategies that are shared. Finally, I took a poll on (a) who use what strategy in their initial response, and (b) who use what strategy in their final solution. Having taken the poll, I thanked them for their participation in this part of the activity and handed out the other items in the survey (i.e., the written assessment itself). In one particular class, from which only four students turned in the written assessments, the students were uninterested and the activity did not get carried out in the manner described above.

Name: ___________________________ Current Math Class: _________________

A SURVEY ON STUDENTS' REASONING

The objective of this survey is to find out the way students think as they solve problems in algebra, and <u>not</u> to determine whether they solve the problems correctly. Please write down your initial response. Then write your further thoughts and your reasoning in detail. The more you describe your thought process, the more accurate the information on the way students think I can have. Thank you for sharing your thinking.

Note: Please use a pen. If you use a pencil do not erase what you have written. You may draw lines to cross them out.

Simplify the following expression: $n - 2n + 3n - 4n + 5n - 6n + \ldots - 96n + 97n - 98n + 99n$

Your Initial Response:

Your Further Thoughts:

The Written Assessment

The worksheet for the each of the five items in the written assessment is the same as that used in the pre-assessment activity. The five items are as follows.

1. Is there an even integer for k that will satisfy $10k + 5 > k + 1000$?

2. d and p are positive even integers less than 25. Is $d + 2p + 3p + 4d > 245$ always true, sometimes true, or never true?

3. Given that m and n are odd integers where $m > n$. Can $m – 10$ ever be equal to $10 – n$?

4. Is the following statement always true, sometimes true, or never true?
   $2(4x – 56) = 3(4x – 56)$

5. Is there an odd integer for n that will satisfy $(n – 2)(n – 8) + 10 < 0$?

APPENDIX B: INTERVIEW PROTOCOL

Opening Statements

First of all, I want thank you for doing this interview with me. My name is Kien Lim. I am a graduate student at UCSD and SDSU. In order for me to graduate, I have to conduct a research study. This interview is part of my study. Thank you for helping me.

The main purpose of this interview is for me **to understand how you think**, and not to find out whether you can do this or do that. So, instead of focusing on the correctness of your answer, we will focus on the reasoning in your answer. For example, I will ask you "why do you think your answer is correct" or "what is it that cause you to be unsure of your answer." Is that OK?

I am most interested in **the way you think** as you solve algebra problems. I will give you some problems in algebra to solve. I like you to say out what goes on in your head as you are solving each problem. I am interested in your thought processes. I will prompt you to continue thinking out loud at times. Is that OK with you?

This interview will last about 60 minutes. If you need to use a calculator, let me know and I'll do the computation for you. That way, it's easier for me to follow your thought process.

Do you have any questions before we start?

Closing Statements

Thank you very much for participation. Here are some gift certificates. Which one do you like?

In the consent form, you indicated that you are interested to participate in the tutoring sessions. If you are selected to participate in the tutoring sessions, I will inform you when the tutoring sessions would be. I foresee the first session will be in May. I need some time to study your reasoning so that I can design problems for the tutoring sessions to advance your reasoning.

<u>Guidelines for Conducting the Interview</u>

Prompts for during think out loud (especially if the student has been silent for more than 10 seconds)

- Please share with me what you are thinking.
- What are you thinking?

Questions to get students to talk about his level of conviction

- On a scale of 1 to 10, how confident are you that your answer is correct?
- Please explain why you are _____ confident.
- (If the confident level is less than 10) Can you make it a 10?

Prompts to get students to say more

- Can you show me what you mean?
- Can you tell me what you mean by ____ (saying that)?
- Can you show me how you got ____ (that answer)?
- What does ___ (the problem statement, the expression, etc.) mean to you?
- Please tell me more about ____ (what you have just said/written).

Prompts to get students to share his thought process in retrospect

- What were you thinking when you were ______ (doing that)?
- Did you have an idea what you were going to get before you started _____ (doing that)?
- When you saw this problem, what was the first thing that came to your mind?
- Do you expect to obtain _____ (this result)?
- What is the difference between the way you think now and the way you think earlier?

Prompts that focus on students' rationale

- Why do you think it is so?
- Why do you ______ (do that)?

Potential impromptu questions

- What if I change part of the question, from ____ to ____.
- Do you think you can find another value that makes the statement true?
- Do you think you can find a value that makes the statement false?
- If your classmate says _____, will you believe him?

APPENDIX C: TASKS FOR CLINICAL INTERVIEWS

Single-variable Tasks

S1. Is there a value for x that will make the following statement true?

$$(6x - 8 - 15x) + 12 > (6x - 8 - 15x) + 6$$

S2. Is there a value for x that will make the following statement true?

$$(2x - 6)(x - 3) < 0$$

S3. Is there a value for x that will make the following statement true?

$$1.2x + 3456 < 7 + 8.9x$$

S4. Is there a value for x that will make the following statement true?

$$3(2x - 9) = 6(2x - 9)$$

S5. Is the following statement always true, sometimes true, or never true?

$$(x + 1) + (x + 2) + (x + 3) + \ldots + (x + 99) + (x + 100) < 100x$$

S6. Consider these two inequalities: $3(4x - 10) > 0$ and $6(4x - 10) > 0$.
Is there a value for x that will make one of them true but will make the other false?

Two-variable Tasks

T1. Given that $5a = b + 5$, which is larger: a or b?

T2. p and q are odd integers between 20 and 50.
For these values, is $5p - q > 2p + 15$ always true, sometimes true or never true?

T3. Given that m is greater than n, can $m - 14$ ever be equal to $7 - n$?

T4. Consider $(x + 1)(2k - 7) = 3(2k - 7)$.
Is there a value for k that makes this equation true for all values of x?

T5. Consider $2k + 9 < (x - 6)^2 + 5$.
Is there a value for k that makes this inequality true for all values of x?

Extra Tasks if Time Allows

E1. Consider $(k - 3)^2 + 5 \leq (x + 1)^2 + 5$.
Is there a value for k that makes this inequality true for all values of x?

E2. Is the following statement always true, sometimes true, or never true?

$$x + 2x + 3x + \ldots + 100x = x^2$$

E3. Given that $0 < x < 50$ and $50 < y < 100$, can $(x - 1)(y + 1) = (x + 1)(y - 1)$ ever be true?

E4. Given that $x > 10$, is there a value for x that will make $x(7 - x) > 3(x + 1)$ true?

E5. Is there a value for x that will make the following statement true?

$$5(2x + 4) > 3(2x + 4) + 4x$$

APPENDIX D: TASKS USED IN THE TEACHING INTERVENTION

Teaching Intervention for Talia

Tasks Used in Teaching Episode 1 for Talia

TE1-$_{T}$N1 "Here is the first problem." $\dfrac{x-5}{x-10} < 0$

TE1-$_{T}$N2 Is $x(6x + 8) < 0$ always true, sometimes true, or never true?

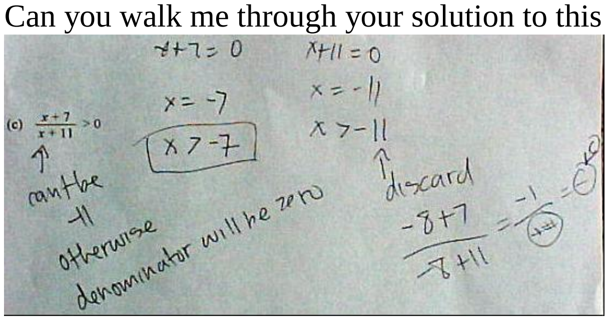
TE1-$_{T}$N2b What is the solution set for $x(6x + 8) < 0$?

(The Code TE1-$_{T}$N1 can be read as Teaching Episode 1 for Talia, New-task 1 and the Code TE2-$_{T}$R1 can be read as Teaching Episode 2 for Talia Repeat-a-task-did-in-homework 1)

Tasks Used in Teaching Episode 2 for Talia

TE2-$_{T}$R1 Is there a value of x that makes $\dfrac{x+7}{x+11} > 0$ true?

TE2-$_{T}$R1b Can you walk me through your solution to this (homework) problem?



TE2-$_{T}$R2 Is there a value of y that makes $2y + (4y - 9) \leq 0$ true?

TE2-$_{T}$R2b Can you walk me through your solution to this (homework) problem?

TE2-$_{T}$N3 Is the following statement always true, sometimes true, or never true?

$$5 + (8x - 20) < 10 + (8x - 20)$$

TE2-$_{T}$N4a When I plug $x = 127$ into $5 + (8x - 20) < 10 + (8x - 20)$, the output value for the function on its LHS is 1001. What is the output value for the function on its RHS?

TE2-$_T$N4b When I plug $x = 99$ into $5 + (8x - 20) < 10 + (8x - 20)$, the output value for the function on its RHS is 782. What is the output value for the function on its LHS?

Tasks Used in Teaching Episode 3 for Talia

TE3-$_T$N1 Consider these two inequalities: $x < 1$ and $8x + 3 > 8 + 3x$. Is there a value for x that will make one of them true but will make the other false?

TE3-$_T$N2 Consider these two inequalities: $\frac{3x - 10}{x + 30} < 0$ and $3x - 10 < 0$.
Is there a value for x that will make one of them true but will make the other false?

TE3-$_T$N3 Consider these two inequalities: $6x + 15 < 0$ and $8x + 20 < 0$.
Is there a value for x that will make one of them true but will make the other false?

TE3-$_T$N4 Is the following statement always true, sometimes true, or never true?

$$5(8x - 20) < 10(8x - 20)$$

TE3-$_T$N5a When I plug $x = 61$ into $5(8x - 20) < 10(8x - 20)$, the output value for the function on its LHS is 2340. What is the output value for the function on its RHS?

TE3-$_T$N5b When I plug $x = -53$ into $5(8x - 20) < 10(8x - 20)$, the output value for the function on its LHS is –2220. What is the output value for the function on its RHS?

TE3-$_T$N6 Is the following statement always true, sometimes true, or never true?

$$5 + (8x - 20) < 10 + (16x - 40)$$

Tasks Used in Teaching Episode 4 for Talia

TE4-$_T$N1 Given that $x > 10$, is $3x(500 - 2x) < 30(500 - 2x)$ always true, sometimes true, or never true?

TE4-$_T$N2 What is the solution set for $3x(500 - 2x) < 30(500 - 2x)$?

TE4-$_T$N3a Plugging $x = 127$ into $4x - 20 > 3x - 20$, we get 361 for the right hand side. What is the value on the left hand side?

TE4-$_T$N3b Plugging x = 8.01 into $4x - 20 > 3x - 20$, we get $12.04 > 4.03$. What will we get if plug in x = 16.02.

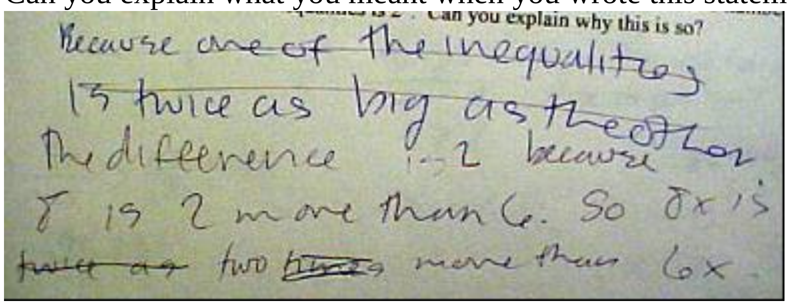
TE4-$_T$N3c Plugging x = 9.11 into $4x - 20 > 3x - 20$, we get $16.44 > 7.33$. What will we get if we plug in x = 10.11.

TE4-$_T$N3d What if we plug in x = 19.11.

<u>Tasks Used in Teaching Episode 5 for Talia</u>

TE5-$_T$R1 Jimmy says that if x is a whole number then the output of the function $6x + 15$ is always odd. Do you agree with him?

TE5-$_T$R2 Can you explain what you meant when you wrote this statement?



TE5-$_T$R3 Given $y_1 = 6x + 15$ and $y_2 = 8x + 20$. Find the function $\dfrac{y_2}{y_1}$.

What do you expect to get if you were to solve $\dfrac{8x+20}{6x+15} = \dfrac{4}{3}$?

TE5-$_T$N4 Is the following statement always true, sometimes true, or never true?

$$(x-3)^2 + 1 > 0$$

TE5-$_T$N5 Is the following statement always true, sometimes true, or never true?

$$(x + 2 + 3x) + 4 + (5x + 6 + 7x) < 2(8x + 9)$$

Teaching Intervention for Chela

Tasks Used in Teaching Episode 1 for Chela

TE1-$_C$N1 (This is the first item) $\frac{x-5}{x-10}<0$

TE1-$_C$N2 (This is the next item) $x(6x+8)<0$

TE1-$_C$N2b What is the solution set for $x(6x+8)<0$?

TE1-$_C$N3 Is the following statement always true, sometimes true, or never true?
$(x-5)(x-10)<0$

TE1-$_C$N4 Is $(x+3)(x+6)<0$ always true, sometimes true, or never true?

TE1-$_C$N5. Is there a value of *x* that makes $(2x-4)(x-12)<0$ true?

TE1-$_C$N6. Find the solution set for $(2x-12)(x-4)<0$.

Tasks Used in Teaching Episode 2 for Chela

TE2-$_C$R1 Solve for *x*: $(2x+6)(x+1)<0$

TE2-$_C$N2 Consider these two inequalities: $6x + 15 < 0$ and $8x + 20 < 0$. Is there a value for *x* that will make one of them true but will make the other false?

TE2-$_C$N2a Consider $2x + 7 > 0$ and $4x + 14 > 0$.
Is there a value for *x* that will make 1T1F?

TE2-$_C$N2b Solve for *x*: $6x + 15 < 0$

TE2-$_C$N2c Solve for *x*: $8x + 20 < 0$

Tasks Used in Teaching Episode 3 for Chela

TE3-$_C$N1 Is the following statement always true, sometimes true, or never true?
$5+(8x-20)<10+(8x-20)$

(a) When I plug x = 127 into $5 + (8x - 20) < 10 + (8x - 20)$, the output value for the function on its left-hand side is 1001. What is the output value for the function on its right-hand side?

(b) When I plug x = 99 into $5 + (8x - 20) < 10 + (8x - 20)$, the output value for the function on its right-hand side is 782. What is the output value for the function on its left-hand side?

TE3-$_{C}$N2 Is the following statement always true, sometimes true, or never true?

$$(4x - 20)(3x - 20) > 0$$

(a) What is the solution set for $(4x - 20)(3x - 20) > 0$?

(b) What is the solution set for $(4x - 20)(3x - 20) < 0$?

(c) What is the solution set for $3x - 20 < 0$?

(d) What is the solution set for $4x - 20 < 0$?

TE3-$_{C}$R3 Consider these two inequalities: $4x – 20 > 0$ and $3x – 20 > 0$. Is there a value for x that will make one of them true but will make the other false?

TE3-$_{C}$N4 (a) When I plug x = 8.01 into $4x - 20 > 0$ and $3x - 20 > 0$, I get 12.04 for $4x - 20$ and 4.03 for $3x - 20$. What do you think the output of each function will be if I were to plug x = 16.02?

(b) When I plug x = 9.11 into $4x - 20 > 0$ and $3x - 20 > 0$, I get 16.44 for $4x - 20$ and 7.33 for $3x - 20$. What do you think the output of each function will be if I were to plug x = 10.11?

(c) When x = 8.01, I get $4(8.01) - 20 = 12.04$. Can you predict the value for $4(9.01) - 20$?

<u>Tasks Used in Teaching Episode 4 for Chela</u>

TE4-$_{C}$R1 Find the solution set for $\dfrac{2x - 4}{x - 12} \geq 0$.

(a) Consider these two inequalities: $\dfrac{2x - 4}{x - 12} \geq 0$ and $x \geq \dfrac{1}{6}$.

Is there a value for x that will make one of them true but will make the other false?

(b) What is the solution set for $\dfrac{2x - 4}{x - 12} < 0$?

(c) Find the solution set for $\dfrac{2x - 6}{x - 10} \geq 0$.

TE4-$_C$N2 Is the following statement always true, sometimes true, or never true?

$$(7x - 31 + 5x) - 43 < (7x - 31 + 5x) - 18$$

(a) When we plug $x = 37$ into $(7x - 31 + 5x) - 43 < (7x - 31 + 5x) - 18$, what is the output value for the function on its left-hand side? What about the output value for the function on its right-hand side?

(b) Plugging x = 37, you found $(7x - 31 + 5x) - 43$ to be 370. Can you predict the value of $(7x - 31 + 5x) - 43$ for $x = 38$?

<u>Tasks Used in Teaching Episode 5 for Chela</u>

TE5-$_C$N1 Is the following statement always true, sometimes true, or never true?

$$5(8x - 20) < 10(8x - 20)$$

(a) What is the solution set for it?

(b) When I plug $x = -53$ into $5(8x - 20) < 10(8x - 20)$, the output value for the function on its left-hand side is –2220. What is the output value for the function on its right-hand side?

(c) Plugging $x = 88$ into $5(8x - 20) < 10(8x - 20)$, we get $3420 < 6840$. What is the value of $5(8x - 20)$ when $x = 89$?

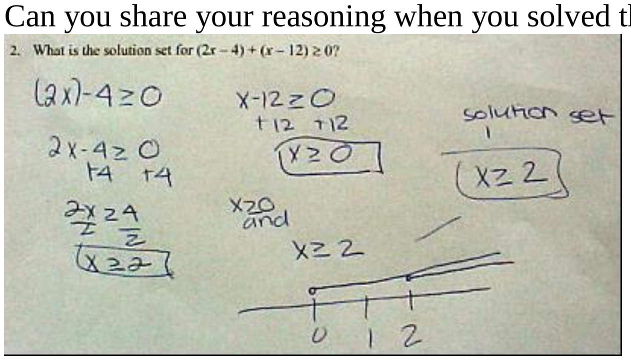
TE5-$_C$N2 Is the following statement always true, sometimes true, or never true?

$$(x + 2 + 3x) + 4 + (5x + 6 + 7x) < 2(8x + 9)$$

TE5-$_C$R3a What is the solution set for $(2x - 4) + (x - 12) \geq 0$?

TE5-$_C$R3b Consider these two inequalities: $(2x - 4) + (x - 12) \geq 0$ and $x \geq 2$. Is there a value for x that will make one of them true but will make the other false?

TE5-$_C$R3c Can you share your reasoning when you solved this problem at home?



TE5-$_C$N4 Given that $x > 10$, is $3x(500 - 2x) < 30(500 - 2x)$ always true, sometimes true, or never true?

Teaching Intervention for Vito

<u>Tasks Used in Teaching Episode 1 for Vito</u>

TE1-$_V$N1 "Here is the first problem for you." $\frac{x-5}{x-10} < 0$

TE1-$_V$N2 "What can you say about this inequality?" $9 - 2x > 90 - 20x$

TE1-$_V$N3 Is the following statement always true, sometimes true, or never true?
$6x + 18 > 6x + 3$

TE1-$_V$N4 Is the following statement always true, sometimes true, or never true?
$5x - 10 < 5x - 20$

TE1-$_V$N5a Is the following statement always true, sometimes true, or never true?
$2x + 2222 < 8x + 88$

TE1-$_V$N5b Suppose we want to make $2x + 2222 < 8x + 88$ never true. Is it possible to change one of the numbers (2, 2222, 8, 88) so as to make it never true?

TE1-$_V$N6 Is the following statement always true, sometimes true, or never true?
$6x < 54321 + x$

<u>Tasks Used in Teaching Episode 2 for Vito</u>

TE2-$_V$R1 Went over homework items, and discussed the rules he had created for determining whether an inequality of the form $Ax + B < Cx + D$ is always true, sometimes true, or never true.

TE2-$_V$N2 Consider these two inequalities: $5x + 10 > x + 5000$ and $4x > 4990$. Is there a value for x that will make one of them true but will make the other false?

TE2-$_V$N3 Consider these two inequalities: $3x - 10 > 0$ and $4x - 10 > 0$. Is there a value for x that will make one of them true but will make the other false?

<u>Tasks Used in Teaching Episode 3 for Vito</u>

TE3-$_V$R1 Went over the activity where I solve his 10 SAN-T (sometimes, always, never true) problems.

TE3-$_V$R2 Consider $2(x - 123) > 3x + 111 - 3x + 222$ and $2x - 246 > 333$.

TE3-$_V$F3 Consider these two inequalities $3x - 10 > 0$ and $4x - 10 > 0$.
Find all the values of x that make one of them true but the other false?

Tasks Used in Teaching Episode 4 for Vito

TE4-$_V$N1 What does the following mean to you? "Solve for x: $\frac{4x - 10}{2x - 10} > 0$"

TE4-$_V$N2 Do you agree with this statement? "An inequality is a proposition whose truth-value ('true' or 'false') depends on the input-value of x."

TE4-$_V$N3 Is the following statement always true, sometimes true, or never true?

$$x + 2 + 3x + 4 + 5x + 6 + 7x < 2(8x + 9)$$

TE4-$_V$F4 Consider these two inequalities: $5x + 10 > x + 5000$ and $4x > 4990$. Is there a value for x that will make one of them true but will make the other false?

TE4-$_V$N5 Consider these two inequalities $7x + 45 > 2x + 5$ and $5x + 45 > 5$.
Find all the values of x that make one of them true but the other false?

TE4-$_V$N6 Definition: The *solution set* of an inequality refers to the collection of all the values of x that makes the inequality true.
What is the solution set for $5x + 10 > x + 5000$?

TE4-$_V$N6 Solve for x: $5x + 10 > x + 5000$

Tasks Used in Teaching Episode 5 for Vito

TE5-$_V$R1 Consider these two inequalities: $4x > 4990$ and $4x + 10 > 5000$.

TE5-$_V$R2 Consider these two inequalities: $7x + 45 > 2x + 5$ and $5x + 45 > 5$.

TE5-$_V$R3 Write an inequality that has the same solution set as, but looks different from $2x - 10 > 50 - x$.

TE5-$_V$R4 Consider these two inequalities: $2x + 7 > 0$ and $3x - 5 < 0$.

TE5-$_V$R5 Consider $5x + 10 > x + 5000$ and $4x > 4990$.
(i) Explain those results/patterns.
(ii) Try to explain why the two inequalities always have same true/false results?

TE5-$_{V}$N6 Given that $x > 10$, is $3x(500 – 2x) < 30(500 – 2x)$ always true, sometimes true, or never true?

TE5-$_{V}$N7 Is the following statement always true, sometimes true, or never true?

$$\frac{4x - 10}{2x - 10} < 0$$

Teaching Intervention for Ali

Tasks Used in Teaching Episode 1 for Ali

$TE1\text{-}_AN1$ (Here is the first problem for you) $\frac{x-5}{x-10} < 0$

$TE1\text{-}_AN2$ (What can you say about this inequality?) $9 - 2x > 90 - 20x$

$TE1\text{-}_AN3$ Is the following statement always true, sometimes true, or never true?
$6x + 18 > 6x + 3$

$TE1\text{-}_AN4$ Is the following statement always true, sometimes true, or never true?
$5x - 10 < 5x - 20$

$TE1\text{-}_AN5$ Is the following statement always true, sometimes true, or never true?
$2x + 2222 < 8x + 88$

$TE1\text{-}_AN5b$ Suppose we want to make $2x + 2222 < 8x + 88$ never true. Is it possible to change one of the numbers (2, 2222, 8, 88) so as to make it never true?

$TE1\text{-}_AN6$ Is the following statement always true, sometimes true, or never true?
$6x < 54321 + x$

$TE1\text{-}_AN6b$ Suppose we want to make $6x < 54321 + x$ always true. Is it possible to change one of the numbers (6, 54321, 1) so as to make it always true?

Tasks Used in Teaching Episode 2 for Ali

$TE2\text{-}_AN1$ Redoing the 10 SAN-T items in homework

$TE2\text{-}_AN2$ Do you agree with this statement?
"An inequality is a proposition whose truth-value ('true' or 'false') depends on the input-value of *x*."

$TE2\text{-}_AN3$ Consider these two inequalities: $5x + 10 > x + 5000$ and $4x > 4990$. Is there a value for *x* that will make one of them true but will make the other false?

$TE2\text{-}_AN3b$ Consider these two inequalities: $5x + 10 > x + 5000$ and $3x > 4990$.

$TE2\text{-}_AN3c$ Consider these two inequalities: $4x > 4990$ and $3x > 4990$.

Tasks Used in Teaching Episode 3 for Ali

TE3-$_A$N1. Consider these two inequalities: $6x - 20 > 0$ and $8x - 20 > 0$. Is there a value for x that will make one of them true but will make the other false?

TE3-$_A$N2. Consider these two inequalities: $4x > 4990$ and $3x > 4990$. Is there a value for x that will make one of them true but will make the other false?

TE3-$_A$N3. Consider these two inequalities: $5x + 10 > x + 5000$ and $4x > 4990$. Is there a value for x that will make one of them true but will make the other false?

Tasks Used in Teaching Episode 4 for Ali

TE4-$_A$N1 What does the following mean to you?

"Solve for x: $\dfrac{4x - 10}{2x - 10} > 0$"

TE4-$_A$R2 Follow up: $5x + 10 > x + 5000$ and $4x > 4990$

TE4-$_A$N3 Consider: $2x + 7 > 0$ and $3x + 15 > 0$

TE4-$_A$N4 Definition: The *solution set* of an inequality refers to the collection of all the values of x that makes the inequality true.
What is the solution set for $7x - 735 > 0$?

TE4-$_A$N5 What is the solution set for $5x + 10 > x + 5000$?

TE4-$_A$N5b Solve for x: $5x + 10 > x + 5000$

Tasks Used in Teaching Episode 5 for Ali

TE5-$_A$R1 Solve for x: $5x + 10 > x + 5000$

TE5-$_A$N2 You have worked on these two problems:

1. Find the solution set for $5x + 10 > x + 5000$
2. Solve for x: $5x + 10 > x + 5000$

In what ways are these two problems the same?
In what ways are they different?

TE5-$_A$N3. Are $5x + 10 > x + 5000$ and $4x > 4990$ the same, or are they different?

TE5-$_A$N4. Given that $x > 10$, is $3x(500 - 2x) < 30(500 - 2x)$ always true, sometimes true, or never true?

APPENDIX E: RECRUITMENT SCRIPT

Script for introducing myself and the research

Good morning/afternoon to all of you. I am Kien Lim, a joint doctoral student in Mathematics and Science Education at UCSD and SDSU. To graduate with a PhD degree, I have to conduct a research related to mathematic education. I choose to investigate high school students' algebraic thinking, because I find the way students think as they solve math problems very intriguing. By understanding the way students think, we as mathematics educators can learn how to teach algebra in a way that is more aligned with the way students think. In my research, I seek to understand how you think as you solve problems in algebra. The goal is not to find out whether you can do this or do that, but to understand your thought process.

Today, I will be asking you to fill out a survey on your algebraic reasoning in place of your regular K-BAC activity. You have the option of not doing this survey. In that case, your teacher would want you to work on your K-BAC worksheet. Your participation in this survey is voluntary and will not affect your grades one way or the other. I want to assure you that what you write in the survey is strictly confidential and will not be shared with your teachers. Those of you who decide to work on the survey may also be invited to participate in interviews or tutoring sessions that will occur during the semester.

Part 1 of this study consists of a one-hour interview conducted during your advisory period. I will interview about twenty 11^{th} graders. The purpose of the interview is for me to understand the way you think as you work on algebra problems involving equations and inequalities. You will be asked to explain your thinking.

Part 2 of the study consists of five to eight tutoring sessions. These are one-on-one sessions in which you will be challenged to solve problems in algebra that promote algebraic thinking. Each session lasts about 60 minutes and is conducted once a week during your advisory period. The purpose of the tutoring sessions is to understand how students learn and how to help students improve their thinking as they solve problems.

Your choice to participate in Part 1 or Part 2 of the study is completely voluntary. What you say during the interview or tutoring sessions will not affect your math grade one way or the other. Your math teacher will not be present at the interview/tutoring sessions. He or she will not see any videotapes of the sessions.

I am handing out two copies of the student assent form. One copy is for you to sign and turn in next week and the other is for you to keep. (Hand out two copies of the student assent form and give them a few minutes to read.)

You can choose to participate in just Part 1 of the study, both Part 1 and Part 2 of the study, or neither. Please indicate on the assent form whether you are interested to participate or not. Please talk to your parents about this study before you decide. I am also handing out two copies of the parent consent form. Please have your parent sign one copy and keep the other copy. (Hand out two copies of parental consent forms to each student.) Please turn in the signed parent consent form together with your assent form next week.

Taking part in this study is completely up to you. No one will be upset if you don't want to participate. If you decide to participate, you can still change your mind and stop any time you want. Do you have questions for me?

Script for introducing the written assessment "A Survey on Students' Reasoning"

The objective of this survey is to find out the way you think as you solve math problems, and not to determine whether you solve them correctly. It is very important that you write down your thoughts as you have them. First write down your initial response and then describe your further thoughts and reasoning in detail. The more you describe your thought process, the more accurate the information on the way students think I can have.

First, we will do just the first item of the survey. We will have a whole class-discussion on your initial response and subsequent solutions to that item. (I will hand out the first item of the written assessment. They will have about 5 minutes to solve the problem.)

Let me list down all the initial responses you have and then take a quick poll. OK, what are some of the initial responses you have? (After eliciting all the initial responses I will ask them to share their final solutions. I will list down all the strategies/solutions that are shared. I will then take a poll on who use what strategy in their initial response, and who use what strategy in their final solution. I will collect their first item and hand out the rest of the items in the survey.)

Now, I like you to do the same, that is, describe your initial response and subsequent thoughts in details, for the remaining items in this survey. Due to time constraint, we will not have whole-class discussions. So it is important that you write down your thinking in as much details as possible. Please don't worry about not completing all the problems. Your detailed descriptions of your thought processes are more valuable than your completion of the survey.

APPENDIX F: TALIA'S WRITTEN COMMENTS ON HER EXPERIENCE IN THE TEACHING INTERVENTION

Summative Report on Your Participation in the Tutoring Sessions (At the end of the Teaching Intervention)

What problem-solving strategies or ideas have you learned from your participation in this research? Please be as specific as possible.

Throughout this research I grasped several problem solving concepts that will really benefit me not only in math but also in other subjects. First of all, I learned that I have to reason and understand the problem before I begin my search for the solution. Many times I begin the problem without knowing what the question is asking me. I learned how to define my terms such as inequality and figure out what they mean. I also learned how to think in a goal oriented manner where you analyze the conditions of a problem and then try to think of a way to reach those conditions with different steps. It really ! he! lps when I think very general about a problem and then work towards specific methods. The goal oriented thinking, helped me find ways to make the general proposition/ problem true and then find more detailed answers.

I looked at inequalities as if they were a comparison of two functions. By using visuals of scales I learned how to compare different functions. I also looked at them as different propositions and I tried to find ways to make them true or false by thinking in terms of positive numbers and negative numbers. Several times I learned new ways to solve inequalities; however I learned that just because I found a new tool to help me, does not mean I should use it right away without considering the whole problem. My method of approaching math used to be very automatic. I tended to rush through a problem witho! ut analyzing what the question was asking me. I made this discovery in my last session when I solved for x in the problem 2x + 5 = even though I did not have to.

It is really important to comprehend the reason for certain patterns in math. In my last session I discovered that I had a lot of trouble understanding why even plus odd equals odd. By using reasoning though I found a different way to think about even and odd numbers. I saw even numbers in pairs and odd numbers as non-paired. By looking at the underlining reasons for certain patterns, I understood the problems better. The goal oriented thinking was frustrating at times because I reasoned very slowly, however, in the end it turned out to be very helpful.

Describe the mathematical concepts or ideas that you have learned from your participation in this research.

Through this research I learned numerous mathematical concepts ranging from problem reading strategies to technical skills. In the beginning of session one I noticed that my mistakes for my first homework assignment were practically the same mistakes done over and over. The question asked me to determine whether an inequality was always true, sometimes true or never true by analyzing it. I usually got one solution set (this is a set of numbers that makes the inequality true) however my mistake was that I continued to forget about the negative numbers in the solution set. In the first session I really tried to break that habit of mine and focus on finding all the solution sets possible.

By solving the inequality, I get the critical points of the function. I learned that a critical point is the point where the graph changes from negative to positive outputs. Once I made this discovery I learned how to use tables to test regions around my critical points. These tables reminded me of guess and check tables however, occasionally they were difficult to understand because I chose numbers arbitrarily. Another important idea I learned is to use visuals to help illustrate the problem for you. These concepts strengthened my knowledge and strength on math.

A Report on Your Mathematics Experience
(Almost a year later)

Looking back at the mathematics classes you took this year (12th grade) and last year (11th grade), do you notice any change in (a) the way you learn mathematics, and (b) the way you solve mathematics problems? Please be specific in your descriptions (i.e., provide examples to support your comment).

In eleventh grade and twelve grades, I noticed that I started asking myself more questions about how to go about a problem. I started thinking in terms of signs (positive or negative) whenever I was asked to solve for an inequality. Though in pre calculus and calculus I hardly dealt with inequalities, I was able to utilize the skills I learned in the SAT's. I was able to do the inequalities in the SAT's very easily because instead of thinking in terms of numbers I thought in terms of signs and greater vs. smaller quantities. I remember in learning about pre calculus, I took a lot of notes and made a lot of interactions with examples that the teacher provided so that I would learn the material better. I would put things in my own words. In solving problems for both calculus and pre calculus I worked in small groups and used reasoning skills to derive my answers. Since we dealt mostly with derivatives, I was able to see which derivative approach worked best to find the derivative. Before I used to add and subtract with my hands but now I think of numbers in groups and I try to add them or subtract them by comparing them to each other. I am still working on this last technique.

A year has passed since you participated in the "Students' Reasoning in Algebra" Project. What are some important things that you have learned from your participation? Has your participation change the way you learn mathematics and do mathematics? If yes, how? If not, why not? P/ease substantiate your points by proving examples, whenever possible.

Through the "Students' Reasoning in Algebra" project, I learned that math is reasoning and using logic. One of the most important lessons I learned though was to try different approaches to solve problems. I also learned that it is important to review the material that you learn so that it sticks to you. The project did help me and by the end of the whole project I think I was able to reason better. I found out that my thought process was slow and that this could be a problem for me in the future but I also found that I was able to get through the training alright. In conclusion, I learned how to ask questions when solving a problem and how to guide myself through a problem by using reasoning skills.

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