Classes of graphs with minimum skew rank 4

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Abstract

The minimum skew rank of a simple graph $G$ is the smallest possible rank among all real skew-symmetric matrices whose $(i, j)$-entry is nonzero if and only if the edge joining $i$ and $j$ is in $G$. It is known that a graph has minimum skew rank 2 if and only if it consists of a complete multipartite graph and some isolated vertices. Some necessary conditions for a graph to have minimum skew rank 4 are established, and several families of graphs with minimum skew rank 4 are given. Linear algebraic techniques are developed to show that complements of trees and certain outerplanar graphs have minimum skew rank 4.

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1. Introduction

Let $A = [a_{i,j}]$ be an $n \times n$ skew-symmetric matrix over the reals. The graph, $G_A$, of $A$ has vertex set $\{1, 2, \ldots, n\}$ and edge set $\{(i, j): a_{i,j} \neq 0, 1 \leq i < j \leq n\}$. For a graph $G$ with vertex set $\{1, 2, \ldots, n\}$, we denote by $S^-(G)$ the set of all real skew-symmetric matrices whose graph is $G$. The minimum skew rank of the graph $G$ is denoted by $mr^-(G)$ and equals $\min\{\text{rank } A: A \in S^-(G)\}$.

The motivation behind the minimum skew rank of a graph is the minimum rank of a graph. For a survey of the minimum rank of graphs, see [4]. Spectral properties of skew-adjacency matrices of graphs are investigated in [1] and the characteristic polynomial of a skew-adjacency matrix associated...
to a weighted oriented graph is studied in [5]. The minimum skew rank of a graph is introduced and graphs with minimum skew rank 2 are characterized in [6]. The minimum skew rank of some families of graphs are studied in [2,3]. The problem of characterizing graphs with minimum skew rank 4 is in [6]. In this article we determine some necessary conditions for a graph to have minimum skew rank 4, and give several families of graphs with minimum skew rank 4.

We now establish some needed notation and terminology. We denote the complement of a graph $G$ by $G^c$. We write $uv$ to denote the edge joining vertices $u$ and $v$. The neighborhood of vertex $v$ in a graph $G$ is denoted by $N(v)$; that is $N(v) = \{u: uv \text{ is an edge}\}$. We write $N[v]$ for $\{v\} \cup N(v)$. Two vertices $u$ and $v$ of a graph $G$ are duplicate vertices if $N(u) = N(v)$ and $u \neq v$. In particular, duplicate vertices are not adjacent. We denote the degree of a vertex $v$ by $d(v)$. The vertex set of the graph $G$ is denoted by $V(G)$. If $S \subseteq V(G)$, then $G[S]$ is the induced subgraph of $G$ on the vertex set $S$ and $G - S$ is the induced subgraph of $G$ on the vertex set $V(G) - S$. If $S = \{v\}$, we write $G - v$ for $G - S$. We write $N(v)$ for $V(G) - N(v)$.

The complete graph on $n$ vertices is denoted by $K_n$. A matching of a graph $G$ is a collection of edges of $G$ such that no two edges have a common vertex. A perfect matching of $G$ is a matching with the property that each vertex of $G$ is incident to one of the edges of the matching.

A diamond is a graph on four vertices incident to one of the edges of the matching.

A co-diamond is the complement graph of a diamond. The union of $k$ vertex disjoint copies of the graph $H$ is denoted by $kH$. The graph obtained from $2K_3$ by inserting one additional edge is denoted by $2K_3 + e$. (See Fig. 1.)

A graph $G$ is $H$-free if $H$ is not an induced subgraph of $G$. It is known that $G$ is diamond-free if and only if $N(i)$ is a disjoint union of cliques for each vertex $i$ of $G$ [7]. This gives the following characterization of co-diamond-free graphs.

**Corollary 1.1.** The graph $G$ is co-diamond-free if and only if for each vertex $i$, $G - N[i]$ is a complete multipartite graph or a graph with no edges.

### 2. Preliminary results

In this section we state basic properties of minimum skew rank, and derive some preliminary results regarding graphs with minimum skew rank at most 4.

We begin by recalling an algebraic characterization of the rank of a skew-symmetric matrix and basic results from [6].

**Proposition 2.1.** For a real skew-symmetric matrix $A$ of order $n$, $\text{rank}(A) \leq 2k$ if and only if there exist $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{R}^n$ such that $A = \sum_{i=1}^k (x_i y_i^T - y_i x_i^T)$.

**Proposition 2.2.** Let $G$ be a graph of order $n$. Then

(a) $\text{mr}^-(G)$ is even.
(b) $\text{mr}^-(G) = 0$ if and only if $G$ has no edges.
(c) If $H$ is an induced subgraph of $G$, then $\text{mr}^-(H) \leq \text{mr}^-(G)$.
(d) If the connected components of $G$ are $G_1, \ldots, G_t$, then

$$\text{mr}^-(G) = \sum_{i=1}^t \text{mr}^-(G_i).$$

Graphs with minimum skew rank 2 are characterized by the following theorem in [6].
Theorem 2.3. \( \text{mr}^{-}(G) = 2 \) if and only if \( G \) is a union of a complete multipartite graph and some isolated vertices.

Note that, in the preceding theorem, some means an arbitrary number including zero.

Graphs whose minimum skew rank is maximal are characterized as follows in [6].

Theorem 2.4. Let \( G \) be a graph of order \( n \). Then \( \text{mr}^{-}(G) = n \) if and only if \( G \) has a unique perfect matching.

By parts (b) and (d) of Proposition 2.2, the presence of isolated vertices does not change the minimum skew rank of a graph. In the following theorem we will show that the presence of duplicate vertices has also no effect on the minimum skew rank of a graph.

Theorem 2.5. Let \( G \) be a graph with duplicate vertices \( u \) and \( v \). Then \( \text{mr}^{-}(G - u) = \text{mr}^{-}(G - v) = \text{mr}^{-}(G) \).

Proof. Without loss of generality, we assume \( u = 1 \) and \( v = 2 \). Suppose \( A \in S^{-}((G - 1) \cup K_{1}) \) with rank\( A = \text{mr}^{-}(G - 1) \). Then by Theorem 2.2, rank\( A = \text{mr}^{-}(G - 1) \). The matrix \( A \) has the form

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & x^{T} \\
0 & -x & B
\end{bmatrix},
\]

where \( G(A(\{1\})) = G - 1 \). Set

\[
A' = \begin{bmatrix}
0 & 0 & x^{T} \\
0 & 0 & x^{T} \\
-2 & -x & -B
\end{bmatrix}.
\]

Then \( G(A') = G \) and \( \text{rank}(A') = \text{rank}(A) = \text{mr}^{-}(G - 1) \). Thus \( \text{mr}^{-}(G) \leq \text{mr}^{-}(G - u) \). Since \( G - u \) is an induced subgraph of \( G \), equality follows by Proposition 2.2(c). As \( G - v \) is isomorphic to \( G - u \), we have \( \text{mr}^{-}(G - v) = \text{mr}^{-}(G) \).

Since the presence of duplicate vertices and isolated vertices does not change the minimum skew rank of a graph, we can focus on graphs with neither duplicate vertices nor isolated vertices.

Note that since \( \text{mr}^{-}(G) \leq |G| \), we have \( \text{mr}^{-}(G) \leq 4 \) for any graph \( G \) with at most 5 vertices. The following gives necessary conditions for a graph with at least 6 vertices and with neither duplicate vertices nor isolated vertices to have minimum skew rank at most 4.

Theorem 2.6. Let \( G \) be a graph of order \( n \geq 6 \) with neither duplicate vertices nor isolated vertices. If \( \text{mr}^{-}(G) \leq 4 \), then \( G \) is co-diamond-free and \( 2K_{3} + e \)-free.

Proof. We prove the contrapositive. Assume that \( G \) either has a co-diamond or a \( 2K_{3} + e \) as an induced subgraph. By Theorem 2.4 and Proposition 2.2(c), it suffices to show \( G \) contains an induced subgraph on 6 vertices with a unique perfect matching.

Since \( 2K_{3} + e \) has a unique perfect matching, we are done if \( G \) contains a \( 2K_{3} + e \).

Now suppose that \( G \) has an induced co-diamond, say on vertices 1, 2, 3 and 4. Without loss of generality, let 34 be an edge. Since 1 and 2 are not duplicate vertices, there is a vertex, say 5 which is adjacent to only one of 1 and 2. Without loss of generality we may assume that 15 is an edge and 25 is a non-edge. Since 2 is not an isolated vertex, there is a vertex, say 6 such that 26 is an edge. Let \( H \) be the subgraph of \( G \) induced by vertices 1, ..., 6. Since vertex 2 has degree 1 in \( H \) and vertex 1 has exactly one neighbor other than possibly vertex 6 in \( H \), every perfect matching on \( H \) contains the edge 26, 15 and 34. Thus \( H \) has a unique perfect matching. \( \square \)
3. Complete multipartite cover number of a graph

In this section, we give an upper bound on the minimum skew rank of a graph in terms of a combinatorial property and describe a family of graphs with minimum skew rank at most 4.

A complete multipartite cover of a graph $G$ is a set of complete multipartite subgraphs of $G$ such that each edge of $G$ belongs to at least one of them. The complete multipartite cover number of a graph $G$, denoted by $cm(G)$, is the minimum number of complete multipartite graphs in a complete multipartite cover of $G$. We denote a complete multipartite graph with $k$ parts by $K_{n_1,n_2,\ldots,n_k}$, where $n_i$ is the number of vertices in the $i$-th part.

The support of an $m \times n$ matrix $A = [a_{i,j}]$ is denoted by $Z(A)$ and is defined to be $\{(i, j): a_{i,j} \neq 0\}$. For a row or a column vector $v$, we simplify this to $Z(v) = \{i: v_i \neq 0\}$.

**Lemma 3.1.** Let $A_1, A_2, \ldots, A_k$ be $m \times n$ real matrices. Then there exist real numbers $\varepsilon_1, \ldots, \varepsilon_k$ such that $Z(\sum_{i=1}^k \varepsilon_i A_i) = \bigcup_{i=1}^k Z(A_i)$.

**Proof.** Set $Z = \bigcup_{i=1}^k Z(A_i)$ and for $(r, t) \in Z$, let $S_{r,t}$ be the subspace

$$S_{r,t} = \left\{ (\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k : \left( \sum_{i=1}^k \varepsilon_i A_i \right)_{r,t} = 0 \right\}.$$ 

Note that for each $(r, t) \in Z$, there is an $i$ such that $(A_i)_{r,t} \neq 0$, and hence $S_{r,t}$ is a proper subspace of $\mathbb{R}^k$.

For $(\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k$, we have $Z(\sum_{i=1}^k \varepsilon_i A_i) \subseteq Z$. Since $\mathbb{R}^k$ is not the union of finitely many proper subspaces of $\mathbb{R}^k$ (see, e.g., [9, Theorem 1.2]), there exists $(\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k$ which is not in any $S_{r,t}$. Hence for this $(\varepsilon_1, \ldots, \varepsilon_k)$, we have $(\sum_{i=1}^k \varepsilon_i A_i)_{r,t} \neq 0$ for all $(r, t) \in Z$, that is, $Z(\sum_{i=1}^k \varepsilon_i A_i) = Z$. □

**Theorem 3.2.** For each graph $G$, $mr^-(G) \leq 2cm(G)$. 

**Proof.** Suppose $K_{n_1,1,\ldots,n_k}$ form a complete multipartite cover of $G$. We may view each of these as a spanning subgraph of $G$. Thus we have a cover of $G$ consisting of subgraphs $G_1, \ldots, G_k$ where each $G_i$ is a union of a complete multipartite graph and isolated vertices. Since $mr^-(G_i) = 2$, then by Proposition 2.1, there exist $x_i, y_i \in \mathbb{R}^n$ such that $G_i(x_i y_i^T - y_i x_i^T) = G_i$ and rank($x_i y_i^T - y_i x_i^T$) = 2 for $i = 1, \ldots, k$. By Lemma 3.1, there exist real numbers $\varepsilon_1, \ldots, \varepsilon_k$ such that

$$Z(\sum_{i=1}^k \varepsilon_i (x_i y_i^T - y_i x_i^T)) = \bigcup_{i=1}^k Z(x_i y_i^T - y_i x_i^T).$$

Thus $G(\sum_{i=1}^k \varepsilon_i (x_i y_i^T - y_i x_i^T)) = G$ and

$$mr^-(G) \leq \text{rank} \left( \sum_{i=1}^k \varepsilon_i (x_i y_i^T - y_i x_i^T) \right) \leq 2k. \quad \Box$$

**Corollary 3.3.** If a graph $G$ has complete multipartite cover number 2, then $mr^-(G) = 4$.

**Proof.** Since $cm(G) > 1$, $G$ is not the union of a complete multipartite graph and isolated vertices. Thus, by Proposition 2.2, $mr^-(G)$ is even and greater than 2. The result now follows from Theorem 3.2. □

The converse of Corollary 3.3 is not true as $mr^-(C_6) = 4$ and $cm(C_6) = 3$.

We now describe a family $\Lambda$ of graphs whose complete multipartite cover number is at most 2.

$\Lambda$ consists of graphs $G$ whose vertex set is partitioned into 3 (possibly empty) sets $U, V$ and $W$ such that
(a) there is no edge joining a vertex \( u \in U \) and a vertex \( v \in V \),
(b) each vertex \( w \in W \) has at most one nonneighbor in \( U \) and at most one nonneighbor in \( V \),
(c) \( G[U], G[V] \) and \( G[W] \) are cliques, and
(d) \(|N(u) \cap N(v) \cap W| \leq 1\) for all \( u \in U \) and \( v \in V \).

This class of graphs is denoted by \( \Lambda \) because each \( G \in \Lambda \) can be thought of as roughly having a \( \Lambda \)-like shape. (See Fig. 2.)

\[
\begin{align*}
\text{Fig. 2. An example of a graph } G \text{ in } \Lambda.
\end{align*}
\]

**Theorem 3.4.** For each graph \( G \in \Lambda \), \( cm(G) \leq 2 \).

**Proof.** Let \( G \) be in \( \Lambda \). Set \( U = \{u_1, \ldots, u_m\}, V = \{v_1, \ldots, v_n\} \) and \( W = \{w_1, \ldots, w_p\} \). Let \( H_1 \) be the complete multipartite graph whose parts are

\[
\{u_i\} \cup (N(u_i) \cap W), \quad i = 1, \ldots, m,
\]

and let \( H_2 \) be the complete multipartite graph whose parts are

\[
\{v_i\} \cup (N(v_i) \cap W), \quad i = 1, \ldots, n,
\]

We claim that \( G = H_1 \cup H_2 \).

Note that if \( W \) is empty, then \( H_1 = G[U] \) and \( H_2 = G[V] \) and consequently \( G = H_1 \cup H_2 \). So assume that \( W \) is non-empty. Now if \( U \) is empty, then \( H_1 \) does not exist and \( H_2 = G[V] \) and consequently \( G = H_1 \cup H_2 \). Similarly if \( V \) is empty, \( G = H_1 \cup H_2 \).

Assume that each of \( U, V \) and \( W \) is non-empty. Since no \( v_i \in V \) is in a part of \( H_1 \) and no \( u_j \in U \) is in a part of \( H_2 \), \( H_1 \cup H_2 \) has no edge of the form \( u_j v_i \) with \( u_j \in U \) and \( v_i \in V \). Note the only non-edges of the form \( u_i w_j \) in \( H_1 \cup H_2 \) correspond to non-edges of \( G \). Similarly, the only non-edges of the form \( v_i w_j \) in \( H_1 \cup H_2 \) correspond to non-edges of \( G \).

Now consider two distinct vertices \( w_i \) and \( w_j \) in \( W \). If \( w_i \) and \( w_j \) are not adjacent in \( H_1 \cup H_2 \), then there exist \( u_k \in U \) and \( v_l \in V \) such that \(|N(u_k) \cap N(v_l) \cap W| \geq 2\), which by assumption never occurs in \( G \). Thus \((H_1 \cup H_2)[W]\) is a clique. Also by construction, \( H_1[U] \) is a clique and \( H_2[V] \) is a clique. Hence \( G = H_1 \cup H_2 \). \( \square \)

**Theorem 3.5.** If \( G \) is a graph of order \( n \geq 6 \) with neither duplicate vertices nor isolated vertices and \( cm(G) \leq 2 \), then \( G \in \Lambda \).

**Proof.** Let \( G \) be a graph with \( cm(G) \leq 2 \) and with neither duplicate vertices nor isolated vertices. Suppose the complete multipartite graphs \( H \) and \( K \) form a complete multipartite cover of \( G \). Set \( W = V(H) \cap V(K), U = V(H) \setminus W \) and \( V = V(K) \setminus W \).

By construction of \( U \) and \( V \), there is no edge joining a vertex \( u \in U \) and a vertex \( v \in V \).

All the nonneighbors of a vertex of \( W \) in \( U \) belong to the same part of \( H \) and have the same neighbors in \( G \). Since \( G \) has no duplicate vertices, each vertex \( w \in W \) has at most one nonneighbor in \( U \). Similarly each vertex \( w \in W \) has at most one nonneighbor in \( V \). If two vertices of \( U \) are not adjacent, then they belong to the same part of \( H \) and have the same neighbor in \( G \). Since \( G \) has no
duplicate vertices, $G[U]$ is a clique. Similarly $G[V]$ is a clique. Now if two vertices of $W$ are not adjacent, then they belong to the same part of $H$ and the same part of $K$ and consequently they have the same neighbors in $G$. Since $G$ has no duplicate vertices, $G[W]$ is a clique. Since $cm(G) \leq 2$, by Theorem 3.2 and Theorem 2.6, $G$ is co-diamond-free. Since $G[W]$ is a clique, and $G$ is co-diamond-free, $|N(u) \cap N(v) \cap W| \leq 1$ for all $u \in U$ and $v \in V$. □

4. Some linear algebraic techniques

In this section we develop some linear algebraic techniques and show their immediate applications to minimum skew rank. These results also will be used in the next section.

The Sherman-Morrison formula [8, Equation 3.8.2] deals with inversion of a rank one perturbation of a skew-symmetric matrix to decrease by 2. We denote the column space of a matrix $A$ by $CS(A)$.

**Theorem 4.1.** Let $A$ be an $n \times n$ real matrix. If

$$\text{rank}(A + xy^T - yx^T) = \text{rank}(A) - 2,$$

then there exists $u, v \in \mathbb{R}^n$ such that

$$x = Au, \quad y = Av, \quad u^T Av = -1, \quad v^T Au = 1, \quad u^T Au = 0, \quad v^T Av = 0.$$ 

The converse holds when $A$ is skew-symmetric.

**Proof.** Let $B = A + xy^T - yx^T$. Then $A = B - xy^T + yx^T$. Consequently

$$CS(A) = CS(B - xy^T + yx^T) \subseteq CS(B) + CS(-yx^T + xy^T)$$

(4.1)

and $\text{rank}(A) \leq \text{rank}(B) + 2$. Furthermore, if $x$ and $y$ are linearly independent, then $\text{rank}(B) = \text{rank}(A) - 2$ if and only if $CS(B) \cap \text{span}(x, y) = \{0\}$.

First assume $\text{rank}(B) = \text{rank}(A) - 2$. Then $x$ and $y$ are necessarily linearly independent. Otherwise $xy^T - yx^T = 0$ and consequently $B = A$ giving us the contradiction $\text{rank}(B) = \text{rank}(A)$. Then $CS(B) \cap \text{span}(x, y) = \{0\}$ and by (4.1), $CS(A) = CS(B) + \text{span}(x, y)$. Thus $x, y \in CS(A)$ and consequently $x = Au$ and $y = Av$ for some $u, v \in \mathbb{R}^n$. Now $A = B - xy^T + yx^T = B - xy^T A^T + yu^T A^T$. Furthermore, $y = Av = Bv - x(v^T A) + y^T (u^T A)$. So $(v^T A) x + (1 - u^T A^T) y = Bv \in CS(B)$. Since $CS(B) \cap \text{span}(x, y) = \{0\}$, $(v^T A) x + (1 - u^T A^T) y = 0$. Since $x$ and $y$ are linearly independent, $v^T A^T y = 0$ and $u^T A^T v = 1$. Similarly, $u^T A^T u = 0$ and $v^T A^T u = -1$.

Conversely assume that $A$ is skew-symmetric and $x = Au, y = Av$ and $u^T A^T v = v^T A v = 1$, and $u^T A^T Au = v^T A v = 0$ for some $u, v \in \mathbb{R}^n$. First of all, note that $x$ and $y$ are linearly independent. Otherwise, $1 = v^T Au$ is a multiple of $v^T A$, which is zero. Since $x$ and $y$ are linearly independent, $\text{rank}(B) = \text{rank}(A) - 2$ if and only if $CS(B) \cap \text{span}(x, y) = \{0\}$. Thus it suffices to show that $CS(B) \cap \text{span}(x, y) = \{0\}$. Suppose $Bz \in CS(B) \cap \text{span}(x, y)$ for some vector $z$. Then $Az = Bz - xy^T z + yx^T z = cx + dy = cAu + dAv$ for some $c, d$. Then $u^T Az = cu^T Au + du^T Av$ and $v^T Az = cv^T Au + dv^T Av$. Since $-u^T Av = v^T Au = 1$ and $u^T Au = v^T Av = 0$, $d = -u^T Az$ and $c = v^T Az$. So $Az = (v^T Az)x - (u^T Az)y$. Since $A$ is skew-symmetric, $Az = (v^T Az)x - (u^T Az)y = -(v^T A^T z)x + (u^T A^T z)y = (u^T z)y = xy^T z + yx^T z$. Thus $Bz = Az + xy^T z - yx^T z = 0$. □

Using the preceding lemma, we have the following result about a skew-realization of a graph $G$ with minimum skew rank 4.

**Lemma 4.2.** Let $G$ be a graph without isolated vertices and with minimum skew rank 4. Then there is a matrix $A = xy^T - yx^T + wz^T - zw^T$ in $S^-(G)$ such that $\text{rank}(A) = mr^-(G)$ and $x$ is the all ones vector.
Proof. Let $A' \in S(G)$ such that $\text{rank}(A') = 4$. Since $G$ has no isolated vertices, each column $A'_j$ of $A'$ has at least one nonzero entry. By Lemma 3.1, we can find $u = [\varepsilon_1, \ldots, \varepsilon_n]^T \in \mathbb{R}^n$ such that $x' = A'u = \sum_{i=1}^{n} \varepsilon_i A'_i$ has all entries nonzero. Choose vector $v$ such that $u^T A' v = 1$, and let $y = A' v$. Then by Theorem 4.1, $\text{rank}(A - xy^T + yx^T) = \text{rank}(A) - 2 = 2$. By Proposition 2.1, there are $w', z' \in \mathbb{R}^n$ such that $A = x'y^T - y'x^T + w'z^T - z'w^T$. Thus $A = x'y^T - y'x^T + w'z^T - z'w^T$. Now let $D = \text{diag}(1/x'_1, \ldots, 1/x'_n)$. Then $A := DA'D = D(x'y^T - y'x^T + w'z^T - z'w^T)D = xy^T - yx^T + wz^T - zw^T$, where $x = Dx' = 1$, $y = Dy'$, $w = Dw'$ and $z = Dz'$. Since $A'_{i,j} = x_i x_j A_{i,j}$ for all $i, j$, $G(A) = G(A') = G$. □

Let $G_1$ and $G_2$ be graphs with disjoint vertex sets. The join of $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is the graph obtained by inserting an edge between each vertex of $G_1$ and each vertex of $G_2$.

Theorem 4.3. Let $G_1$ and $G_2$ be two graphs without isolated vertices such that $\text{mr}^-(G_1) \leq 4$ and $\text{mr}^-(G_2) = 4$. Then $\text{mr}^-(G_1 \vee G_2) = 4$.

Proof. First suppose $G_1$ and $G_2$ have no isolated vertices and $\text{mr}^-(G_1) = \text{mr}^-(G_2) = 4$. Then Lemma 4.2 implies that there are vectors $u, x$ and $y$ such that $G(1u^T - u1^T + xy^T - yx^T) = G_1$. Similarly there are vectors $u', x'$ and $y'$ such that $G(1u'^T - u'1^T + x'y'^T - y'x'^T) = G_2$. Set

$$U = \begin{bmatrix} u \\ u' \end{bmatrix}, \quad X = \begin{bmatrix} x \\ x' \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y \\ y' \end{bmatrix}. $$

It suffices to show that $G(1U^T - U1^T + XY^T - YX^T) = G_1 \vee G_2$ for some choice of $u$ and $u'$. We will perturb $u$ and $u'$ such that $1u'^T - u^T + xy^T - yx^T$ has all nonzero entries and keep the facts that $G(1u^T - u1^T + xy^T - yx^T) = G_1$ and $G(1u'^T - u'1^T + x'y'^T - y'x'^T) = G_2$.

Note that since $1(u+c1)^T - (u+c1)1^T = 1u^T - u1^T$ and $1(u'+d1)^T - (u'+d1)1^T = 1u'^T - u'1^T$, we have $G(1(u+c1)^T - (u+c1)1^T + xy^T - yx^T) = G_1$ and $G(1(u'+d1)^T - (u'+d1)1^T + x'y'^T - y'x'^T) = G_2$ for any real numbers $c$ and $d$. We claim that each entry of $1(u'+d1)^T - (u+c1)1^T + xy^T - yx^T$ is nonzero for some real numbers $c$ and $d$. If not, then

$$\bigcup_{1 \leq i \leq |V(G_1)| \atop 1 \leq j \leq |V(G_2)|} S_{i,j} = \mathbb{R}^2,$$

where

$$S_{i,j} = \{(c, d) \in \mathbb{R}^2 : (1(u'+d1)^T - (u+c1)1^T + xy^T - yx^T)_{i,j} = 0 \} = \{(c, d) \in \mathbb{R}^2 : d-c + (u'_j - u_i + x_i(y'_j - y_i(x'_j)) = 0 \}.$$

Since the union of a finite number of affine subspaces $S_{i,j}$ of $\mathbb{R}^2$ is $\mathbb{R}^2$ (a consequence of [9, Theorem 1.2]), one of them is $\mathbb{R}^2$. But $S_{i,j} = \{(c, d) \in \mathbb{R}^2 : d-c + (u'_j - u_i + x_i(y'_j - y_i(x'_j)) = 0 \} \neq \mathbb{R}^2$ for all $i$ and $j$.

Thus for some choice of $c$ and $d$, $G(1u^T - u1^T + XY^T - YX^T) = G_1 \vee G_2$. Therefore $\text{mr}^-(G_1 \vee G_2) \leq 4$ and equality follows since $G_1 \vee G_2$ has induced subgraph $G_1$ with $\text{mr}^-(G_1) = 4$.

Now suppose $G_1$ and $G_2$ have no isolated vertices and $\text{mr}^-(G_1) = 2$ and $\text{mr}^-(G_2) = 4$. Then Lemma 4.2 implies that there are vectors $u', x'$ and $y'$ such that $G(1u'^T - u'1^T + x'y'^T - y'x'^T) = G_2$. Since $\text{mr}^-(G_1) = 2$ and $G_1$ has no isolated vertices, by Theorem 2.3, $G_1$ is a complete multipartite graph. Then there is a vector $u$ such that $G(1u^T - u1^T) = G_1$ (see the proof of Theorem 2.1 in [6]). Now setting

$$U = \begin{bmatrix} u \\ u' \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ x' \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 \\ y' \end{bmatrix}$$

and following similar arguments we have $G(1U^T - U1^T + XY^T - YX^T) = G_1 \vee G_2$. Then $\text{mr}^-(G_1 \vee G_2) \leq 4$ and equality follows since $G_1 \vee G_2$ has induced subgraph $G_2$ with $\text{mr}^-(G_2) = 4$. □
A vertex of a graph of order \( n \) is **dominating** if its degree is \( n - 1 \). So if a graph has no dominating vertex, then the complement of the graph has no isolated vertices. Using **Theorem 4.3** and induction on the number of connected components of a graph \( G \), we have the following corollary.

**Corollary 4.4.** For \( i = 1, 2, \ldots, k \), let \( G_i \) be a graph with no isolated vertex. If \( \text{mr}^-(G_i) \leq 4 \) for each \( i \), then \( \text{mr}^-(G_1 \lor G_2 \lor \cdots \lor G_k) \leq 4 \).

Note that under the hypotheses of **Corollary 4.4** the complement of \( G = G_1 \lor \cdots \lor G_k \) will be the disjoint union of \( H_1, \ldots, H_k \), where \( H_i \) is the complement of \( G_i \). Thus **Corollary 4.4** is equivalent to the following.

If the connected components of the graph \( H \) are \( H_1, H_2, \ldots, H_k \), no \( H_i \) has a dominating vertex and \( \text{mr}^-(H_i^c) \leq 4 \) for each \( i \), then \( \text{mr}^-(H^c) \leq 4 \).

The conclusion of **Theorem 4.3** is not valid if either \( G_1 \) or \( G_2 \) has isolated vertices as we see in the following observation.

**Observation 4.5.** Let \( \text{mr}^-(G_1) = 4 \) and \( \text{mr}^-(G_2) = 4 \). If one of \( G_1 \) and \( G_2 \) has isolated vertices, then either \( G_1 \lor G_2 \) has duplicate vertices or \( \text{mr}^-(G_1 \lor G_2) \geq 6 \).

**Proof.** First suppose that \( \text{mr}^-(G_1) = \text{mr}^-(G_2) = 4 \) and \( G_1 \lor G_2 \) has no duplicate vertices. Without loss of generality suppose that \( G_1 \) has an isolated vertex \( v \). We will show that \( \text{mr}^-(G_1 \lor G_2) \geq 6 \). Suppose to the contrary that \( G_1 \lor G_2 \) has no duplicate vertices and \( \text{mr}^-(G_1 \lor G_2) < 4 \). Since \( G_2 \) is an induced subgraph of \( G_1 \lor G_2 \) and \( \text{mr}^-(G_2) = 4 \), by **Proposition 2.2(c)**, \( \text{mr}^-(G_1 \lor G_2) = 4 \). Since \( \text{mr}^-(G_1 \lor G_2) = 4 \) and \( G_1 \lor G_2 \) has neither isolated vertices nor duplicate vertices, by **Theorem 2.6** and **Corollary 1.1**, \( G_1 \lor G_2 - N[v] = G_1 - v \) is a complete multipartite graph or has no edges. Then by **Theorem 2.3**, we are led to the contradiction that \( \text{mr}^-(G_1) \leq 2 \).

The following lemma asserts that appending a vertex of degree \( n - 1 \) to a graph of order \( n \) with minimum skew rank 4 does not change the minimum skew rank. A more general version of this lemma is given in **Corollary 4.11**.

**Lemma 4.6.** Let \( G \) be a graph of order \( n \) with neither isolated vertices nor duplicate vertices. Let \( H \) be the graph obtained from \( G \) by appending a vertex of degree \( n - 1 \). If \( \text{mr}^-(G) = 4 \), then \( \text{mr}^-(H) = 4 \).

**Proof.** Assume that \( \text{mr}^-(G) = 4 \) and \( H \) is the graph obtained from \( G \) by appending a vertex \( v \) of degree \( n - 1 \). Since \( \text{mr}^-(G) = 4 \) and \( G \) has no isolated vertex, by **Lemma 4.2**, there are vectors \( u, x \) and \( y \in \mathbb{R}^n \) such that \( G(1u^T - a1^T + xy^T - yx^T) = G \). We take \( v = n + 1 \) and \( N(v) = \{1, \ldots, n - 1\} \). It suffices to find numbers \( a, b \) and \( c \) such that \( G(1U^T - U1^T + XY^T - YX^T) = H \) where

\[
U = \begin{bmatrix} u \\ a \end{bmatrix}, \quad X = \begin{bmatrix} x \\ b \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y \\ c \end{bmatrix}.
\]

If there are such numbers \( a, b, c \), then \( a - u_n + x_nc - y_nb = 0 \) and \( a - u_i + x_ic - y_ib \neq 0 \) for \( i = 1, \ldots, n - 1 \). That means \((a, b, c)\) lies on the plane \( z - u_n + x_ny = 0 \) and not on the plane \( z - u_i + x_iy = 0 \) for \( i = 1, \ldots, n - 1 \). Since \( \mathbb{R} \) is an infinite field, we can always find such \((a, b, c)\) if the plane \( z - u_n + x_ny = 0 \) is not the same with \( z - u_i + x_iy = 0 \) for all \( i = 1, \ldots, n - 1 \). If the plane \( z - u_n + x_ny = 0 \) is the same with \( z - u_i + x_iy = 0 \) for some \( i = 1, \ldots, n - 1 \), then \( x_n = x_i, y_n = y_i \) and \( u_n = u_i \). Then \( N(n) = N(i) \) in \( G \) contrary to \( G \) not having any duplicate vertices.

Now we develop a second technique using the left null space of a skew-symmetric matrix. We denote the null space, left null space and row space of a matrix \( A \) by \( \text{NS}(A) \), \( \text{LNS}(A) \), and \( \text{RS}(A) \).
respectively. We can recall from linear algebra that \( \text{LNS}(A) = \text{NS}(A^T) \), \( \text{CS}(A) = \text{LNS}(A)^\perp \) and \( \text{RS}(A) = \text{NS}(A)^\perp \). We denote column \( i \) of a matrix \( A \) by \( A_i \). A basis matrix of the left null space of \( A \) is a matrix whose rows form a basis of \( \text{LNS}(A) \).

A collection \( \{ N_i : i \in I \} \) of vectors is a minimally dependent set of vectors if it is a linearly dependent set and for each \( j \in I \), \( \{ N_i : i \neq j, i \in I \} \) is a linearly independent set of vectors.

**Lemma 4.7.** Let \( N \) be a matrix with columns \( N_1, \ldots, N_n \). Then there exists a vector \( w \) with support \( Z \) with \( w \in \text{NS}(N) \) if and only if \( Z \) is the union of index sets of minimally dependent sets of columns of \( N \).

**Proof.** First suppose \( w \in \text{NS}(N) \) and has support \( Z \). Thus

\[
\sum_{i \in Z} w_i N_i = \begin{bmatrix} 0 \\
\vdots \\
0 \end{bmatrix}
\]

and consequently for each \( j \in Z \), \( N_j \in \text{span}\{ N_i : i \neq j, i \in Z \} \). Consider \( j \in Z \). It suffices to show that there is a minimally dependent set of columns of \( N \) whose indices contain \( j \) and is a subset of \( Z \). There is some \( I \subseteq Z - \{j \} \) such that \( \{ N_i : i \in I \} \) is a basis of \( \text{span}\{ N_i : i \neq j, i \in Z \} \). Then \( N_j \in \text{span}\{ N_i : i \neq j, i \in Z \} = \text{span}\{ N_i : i \in I \} \) and consequently \( N_j = \sum_{i \in I} c_i N_i \) for some unique scalars \( c_i \). Let \( I' = \{ i : c_i \neq 0, i \in I \} \). Then \( N_j \in \text{span}\{ N_i : i \in I' \} \) and for each \( k \in I' \), \( N_j \notin \text{span}\{ N_i : i \in I' - \{k \} \} \). Thus \( \{ N_i : i \in I' \cup \{j \} \} \) is a minimally dependent set of columns of \( N \) whose indices contain \( j \) and is a subset of \( Z \).

Conversely suppose that \( Z \) is a union of indices of minimally dependent columns of \( N \). Suppose \( I \subseteq Z \) and \( \{ N_i : i \in I \} \) is a minimally dependent set of vectors of \( N \). Then

\[
\sum_{i \in I} y_i N_i = \begin{bmatrix} 0 \\
\vdots \\
0 \end{bmatrix}
\]

for some nonzero \( y_i \). Let \( x \) be an \( n \times 1 \) vector where \( x_i = y_i \) if \( i \in I \) and \( x_i = 0 \) otherwise. Then

\[
Nx = \begin{bmatrix} 0 \\
\vdots \\
0 \end{bmatrix}
\]

and \( x \in \text{NS}(N) \). So for each \( I \subseteq Z \) for which \( \{ N_i : i \in I \} \) is a minimally dependent set of vectors of \( N \), there is \( x_T \in \text{NS}(N) \) with \( Z(x_T) = I \). Now if \( Z = \bigcup_{i=1}^k I_i \), then by Lemma 3.1, we can find numbers \( \varepsilon_1, \ldots, \varepsilon_k \) such that

\[
w = \sum_{i=1}^k \varepsilon_i x_{I_i}
\]

is in \( \text{NS}(N) \). \( \square \)

**Theorem 4.8.** Let \( A \) be a real skew-symmetric matrix of order \( n \geq 2 \). Let \( N \) be a basis matrix of the left null space of \( A \). Let

\[
A' = \begin{bmatrix} A & w^T \\
-w^T & 0 \end{bmatrix}
\]

for some vector \( w \) with support \( Z \). If \( \text{rank}(A') = \text{rank}(A) \), then \( Z \) is a union of indices of minimally dependent columns of \( N \). Conversely if \( Z \) is a union of indices of minimally dependent columns of \( N \), then there exists a vector \( w \) with support \( Z \) such that

\[
\text{rank} \left( \begin{bmatrix} A & w^T \\
-w^T & 0 \end{bmatrix} \right) = \text{rank}(A).
\]

**Proof.** Assume that \( \text{rank}(A') = \text{rank}(A) \). Then \( w \) is in \( \text{CS}(A) \). Since \( \text{CS}(A) = \text{LNS}(A)^\perp = \text{RS}(N)^\perp = \text{NS}(N) \), \( w \) is in \( \text{NS}(N) \). By Lemma 4.7, \( Z \) is a union of indices of minimally dependent columns of \( N \).
Conversely suppose that \( Z \) is a union of indices of minimally dependent columns of \( N \). By Lemma 4.7, there exists a vector \( w \) with support \( Z \) such that \( w \) is in \( NS(N) = CS(A) \). Thus
\[
\text{rank} \left( \begin{bmatrix} A & w \\ -w^T & 0 \end{bmatrix} \right) = \text{rank}(A). \quad \square
\]

Theorem 4.8 gives us a way to adjoin a new vertex \( v \) and some edges from \( v \) to a graph \( G \) without increasing the minimum skew rank.

**Corollary 4.9.** Let \( G \) be a graph on \( n \) vertices. Let \( A \in S^- (G) \) with \( \text{rank}(A) = \text{mr}^-(G) \) and \( N \) be a basis matrix of the left null space of \( A \). Let \( G + v \) be a graph obtained by adjoining a new vertex \( v \) and some edges from \( v \) to \( G \). If \( N(v) \) is a union of indices of minimally dependent columns of \( N \), then \( \text{mr}^-(G + v) = \text{mr}^- (G) \).

**Example 4.10.** Consider
\[
\begin{bmatrix}
3 & 2 \\
1 & 4
\end{bmatrix}
\]

Let
\[
A = \begin{bmatrix}
0 & 0 & 4 & 4 \\
0 & 0 & 4 & 4 \\
-4 & -4 & 0 & 4 \\
-4 & -4 & -4 & 0
\end{bmatrix} \in S^- (G).
\]

Then it can be verified that \( \text{rank}(A) = \text{mr}^- (G) = 2 \) and that
\[
N = \begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & -1 & 1
\end{bmatrix}
\]
is a basis matrix of the left null space of \( A \). The indices of minimally dependent sets of columns of \( N \) are \( \{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\} \). The set of all possible union of indices of minimally dependent columns of \( N \) is
\[
\{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}.
\]
If we adjoin vertex 5 to \( G \), where \( N(5) \) is one of \( \{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}, \{1, 2, 3, 4\} \), then \( \text{mr}^- (G + 5) = \text{mr}^- (G) \).

**Corollary 4.11.** Let \( G \) be a graph on \( n \) vertices with neither isolated vertices nor duplicated vertices. Let \( G + v \) be the graph obtained by adjoining a new vertex \( v \) and some edges from \( v \) to \( G \). Then the following hold:
(a) if \( d(v) = n \) or \( n - 1 \) in \( G + v \), then \( mr^{-}(G + v) = mr^{-}(G) \), and
(b) if \( d(v) = n - 2 \) in \( G + v \) where \( V(G + v) - N[v] = \{x, y\} \) and \( G + v \) has neither a clique nor a coclique on vertices \( x, y \) and \( j \) for all \( j \neq x, y, v \), then \( mr^{-}(G + v) = mr^{-}(G) \).

**Proof.** Without loss of generality let \( v \) be vertex \( n + 1 \) of \( G + v \). Let

\[
A' = \begin{bmatrix}
A & \text{w}^T \\
-w^T & 0
\end{bmatrix} \in S^-(G + v)
\]

with \( \text{rank}(A') = mr^{-}(G + v) \) and \( N \) be a basis matrix of the left null space of \( A \).

(a) First suppose \( d(n + 1) = n \). Then \( N(n + 1) = \{1, \ldots, n\} \). We claim that \( \{1, \ldots, n\} \) is a union of indices of minimally dependent columns of \( N \). Otherwise for some \( j \in \{1, \ldots, n\} \), \( N_j \) is not in \( \text{span}(N_i : i \neq j, i = 1, \ldots, n) \). Then

\[
\text{rank}\left(\begin{bmatrix} N \\ e_j^T \end{bmatrix}\right) = 1 + \text{rank}(N(:, \{j\})) = \text{rank}(N).
\]

Thus \( e_j^T \in \text{RS}(N) = \text{LNS}(A) \). This implies that the \( j \)-th row of \( A \) is \( e_j^T A = [0, 0, \ldots, 0] \) which contradicts the assumption that \( G \) has no isolated vertices.

Next suppose \( d(n + 1) = n - 1 \). Without loss of generality we can take \( N(n + 1) = \{1, \ldots, n - 1\} \). We claim that \( \{1, \ldots, n - 1\} \) is the union of indices of columns of some sets of minimally dependent columns of \( N \). Otherwise for some \( j \in \{1, \ldots, n - 1\} \), \( N_j \) is not in \( \text{span}(N_i : i \neq j, i = 1, \ldots, n - 1) \). Then

\[
\text{rank}\left(\begin{bmatrix} N(:, \{j\}) \\ e_j^T \end{bmatrix}\right) = 1 + \text{rank}(N(:, \{j, n\})) = \text{rank}(N(:, \{j\})).
\]

Thus \( e_j^T \in \text{RS}(N(:, \{j\})) \) and consequently \( e_j^T + ae_j^T \in \text{RS}(N) \) for some \( a \). If \( a = 0 \), then \( e_j^T \in \text{RS}(N) = \text{LNS}(A) \) implies that the \( j \)-th row of \( A \) is \( e_j^T A = [0, 0, \ldots, 0] \) which contradicts that \( G \) has no isolated vertices. If \( a \neq 0 \), then \( e_j^T + ae_j^T \in \text{RS}(N) = \text{LNS}(A) \) implies that the \( j \)-th row and \( n \)-th row of \( A \) are multiples of each other which contradicts that \( G \) has no duplicate vertices. Thus the claim is established and by Corollary 4.9, \( mr^{-}(G + v) = mr^{-}(G) \).

(b) Suppose \( d(n + 1) = n - 2 \). Without loss of generality we can take \( N(n + 1) = \{1, \ldots, n - 2\} \). We claim that \( \{1, \ldots, n - 2\} \) is a union of indices of minimally dependent columns of \( N \). Otherwise for some \( j \in \{1, \ldots, n - 2\} \), \( N_j \) is not in \( \text{span}(N_i : i \neq j, i = 1, \ldots, n - 2) \). Then

\[
\text{rank}\left(\begin{bmatrix} N(:, \{n - 1, n\}) \\ e_j^T \end{bmatrix}\right) = 1 + \text{rank}(N(:, \{j, n - 1, n\})) = \text{rank}(N(:, \{j, n - 1, n\})).
\]

Thus \( e_j^T \in \text{RS}(N(:, \{n - 1, n\})) \) and \( e_j^T + ae_{n-1}^T + be_n^T \in \text{RS}(N) \) for some \( a \) and \( b \). If \( a = 0 \) and \( b = 0 \), then \( e_j^T \in \text{RS}(N) = \text{LNS}(A) \) implies that the \( j \)-th row of \( A \) is \( e_j^T A = [0, 0, \ldots, 0] \) which contradicts that \( G \) has no isolated vertices. If only one of \( a \) and \( b \) is zero, say \( a = 0 \) and \( b \neq 0 \), then \( e_j^T + be_n^T \in \text{RS}(N) = \text{LNS}(A) \) implies that the \( j \)-th row and \( n \)-th row of \( A \) are multiples of each other which contradicts that \( G \) has no duplicate vertices. If \( a \neq 0 \) and \( b \neq 0 \), then \( e_j^T + ae_{n-1}^T + be_n^T \in \text{RS}(N) = \text{LNS}(A) \) implies that

\[
A_j + aA_{n-1} + bA_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

Since \( a \neq 0 \) and \( b \neq 0 \), off-diagonal entries of \( A([j, n-1, n]) \) are all zero or all nonzero. Consequently \( G(A) = G \) has either a clique or a coclique on vertices \( j, n - 1 \) and \( n \) which contradicts our assumption that \( G + v \) has neither a clique nor a coclique on vertices \( x, y \) and \( j \) for all \( j \neq x, y, v \) where \( V(G + v) - N[v] = \{x, y\} \). Thus the claim is established and by Corollary 4.9, \( mr^{-}(G + v) = mr^{-}(G) \).

Now we introduce another technique which will play a crucial role in Section 5.
**Lemma 4.12.** Let $G$ be a graph on $n$ vertices with neither duplicated vertices nor isolated vertices and let $A \in S^-(G)$ with $\text{rank}(A) \geq 4$. Then there exists a vector $x \in \text{CS}(A)$ such that $x_i \neq 0$ for $i = 1, 2, \ldots, n-1$, $x_n = 0$, and $x$ is not in the span of any two columns of $A$.

**Proof.** Let $V = \text{CS}(A) \cap e_n^\perp$. Since $n$ is not isolated, $V$ has dimension at least 3. Consider the following subspaces: $e_i^\perp \cap V$ ($i = 1, 2, \ldots, n-1$), and span$(A_i, A_j) \cap V$ ($i \neq j$). By considering dimensions, span$(A_i, A_j) \neq V$ for all $i \neq j$. Also, $e_i^\perp \cap V \neq V$ for all $i \neq n$, for otherwise $V \subseteq e_i^\perp$ implying $e_i \in \text{CS}(A)^\perp + \text{span}(e_n)$ which leads to the contradiction that either vertex $i$ is isolated or is a duplicate of vertex $n$. Since $V$ is not the union of finitely many proper subspaces of $V$ [9, Theorem 1.2], the union of these sets is not all of $V$. This implies the existence of a vector $x$ with the desired properties. □

**Theorem 4.13.** Let $G$ be a graph on vertices $1, 2, \ldots, n$ with neither duplicated vertices nor isolated vertices, and let $H$ be a graph obtained from $G$ by appending two new vertices $n+1$ and $n+2$ and edges $(n+1)j$ ($j = 1, 2, \ldots, n-1$), $(n+2)n$, and $(n+2)j$ ($j = 1, 2, \ldots, n-2$). Then if $\text{mr}^-(G) \geq 4$, then $\text{mr}^-(H) = \text{mr}^-(G)$. If $\text{mr}^-(G) \leq 2$, then $\text{mr}^-(H) \leq 4$.

**Proof.** First assume that $\text{mr}^-(G) \geq 4$. Let $A \in S^-(G)$ with $\text{rank}(A) = \text{mr}^-(G)$. Let $x$ be a nonzero vector guaranteed by Lemma 4.12 and let

$$B = \begin{bmatrix} A & x \\ -x^T & 0 \end{bmatrix}. $$

Since $x \in \text{CS}(A)$, $\text{rank}(B) = \text{rank}(A)$. We claim that there exists a vector $y \in \text{CS}(B)$ such that $y_i \neq 0$ for $i = 1, 2, \ldots, n-2, n$, $y_{n+1} = 0$, and $y_{n-1} = 0$. To see this let $W = \text{CS}(B) \cap e_{n-1}^\perp \cap e_{n+1}^\perp$ and assume to the contrary that $W$ is the union of the sets: $e_i^\perp \cap W$ ($i = 1, 2, \ldots, n-2, n$). Since the union of a finite number of subspaces $e_i^\perp \cap W$ of $W$ is $W$, one of them is $W$ [9, Theorem 1.2]. This implies that there exists an $i \in \{1, 2, \ldots, n-2, n\}$ such that $e_i^\perp \cap W = W$, which implies that $W \subseteq e_i^\perp$. Then $e_i \in \text{CS}(B)^\perp + \text{span}(e_{n-1}) + \text{span}(e_{n+1})$. Then rows $i, n-1$ and $n+1$ of $B$ are linearly dependent. Since $B^T = -B$, we are led to the contradiction that either $G$ has duplicate vertices $i$ and $n-1$ or $x$ is a linear combination of columns $i$ and $n-1$ of $A$.

Now consider

$$C = \begin{bmatrix} B & y \\ -y^T & 0 \end{bmatrix}. $$

Then $C \in S^-(H)$ and $\text{rank}(C) = \text{rank}(B)$. Then $\text{mr}^-(H) \leq 2$ and the equality holds since $H$ has the induced subgraph $G$.

Next assume that $\text{mr}^-(G) \leq 2$. Then $\text{cm}(H) \leq 2$, and consequently by Theorem 3.2, $\text{mr}^-(H) \leq 4$. □

We can immediately apply Theorem 4.13 to find minimum skew rank of the complement of a cycle as follows.

**Corollary 4.14.** If $G$ is the complement of a cycle of length $n+2$ where $n \geq 2$, then $\text{mr}^-(G) = 4$.

**Proof.** Let $P$ be a path of order $n \geq 2$ with edges

$$(n-1)1, 12, 23, \ldots, (n-3)(n-2), (n-2)n. $$

By Theorem 5.2, we have $\text{mr}^-(P^c) \leq 4$. By applying Theorem 4.13, we see that $\text{mr}^-(G) = \text{mr}^-(P^c) \leq 4$ and the equality holds as $G$ is not a complete multipartite graph. □
5. Minimum skew rank of complement of bipartite graphs, trees and outerplanar graphs

In general, the complement of a bipartite graph does not have minimum skew rank 4. The following theorem gives sufficient conditions for the complement of a bipartite graph to have minimum skew rank 4. If $G$ is a bipartite graph that is a spanning subgraph of $K_{m,n}$, then we denote the graph obtained from $K_{m,n}$ by deleting all the edges of $G$ by $K_{m,n} - E(G)$.

**Theorem 5.1.** Let $G$ be a bipartite graph that is a spanning subgraph of $K_{m,n}$, $m, n \geq 3$ with no duplicate vertices. If $mr^-(K_{m,n} - E(G)) = 4$, then $mr^-(G^c) = 4$.

**Proof.** Suppose that $mr^-(K_{m,n} - E(G)) = 4$. Let $A \in S^-(K_{m,n} - E(G))$ such that

$$A = \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}$$

and rank$(A) = 4$, where $B \in \mathbb{R}^{m \times n}$. Then rank$(B) = 2$ and $B = x_1y_1^T + x_2y_2^T$ for some $m \times 1$ vectors $x_1$ and $x_2$ and $n \times 1$ vectors $y_1$ and $y_2$, where $x_1$ and $x_2$ are linearly independent and $y_1$ and $y_2$ are linearly independent. Setting $X = [x_1 \ x_2]$ and $Y = [y_1 \ y_2]$, we see that $B = XY^T$. Let $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Consider

$$C = \begin{bmatrix} XSX^T & B \\ -B^T & YSY^T \end{bmatrix}.$$ 

Clearly $C$ is skew-symmetric. Since $G$ has no duplicate vertices, no two rows of $B = XY^T$ are multiples of each other. In particular, no two rows of $X$ are multiples of each other and no two rows of $Y$ are multiples of each other. If for $i \neq j$, $(XSX^T)_{ij} = (x_1)_i(y_2)_j - (x_2)_i(y_1)_j = 0$, then rows $i$ and $j$ of $X$ are multiples of each other. Thus all off-diagonal entries of $XSX^T$ are nonzero. Similarly all off-diagonal entries of $YSY^T$ are nonzero. So $C \in S^-(G^c)$. Now notice that

$$C = \begin{bmatrix} x_1x_2^T - x_2x_1^T & x_1y_1^T + x_2y_2^T \\ -y_1x_2^T - y_2x_1^T & y_1y_2^T - y_2y_1^T \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ -y_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2^T & y_2^T \\ x_1^T & y_1^T \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ -y_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ x_1 & y_1 \end{bmatrix} = \begin{bmatrix} x_2 & y_2 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2^T & y_2^T \\ x_1^T & y_1^T \end{bmatrix}.$$ 

By Proposition 2.1, rank$(C) \leq 4$ and consequently $mr^-(G^c) \leq 4$. Since $G^c$ is not a complete multipartite graph, $mr^-(G^c) = 4$. $\square$

Note that under the hypothesis of the preceding theorem, if $mr^-(G^c) = 4$ and $A = \begin{bmatrix} C & B \\ -B^T & D \end{bmatrix}$ is $S^-(G^c)$, where rank$(A) = 4$ and $B \in \mathbb{R}^{m \times n}$, then rank$(B) \leq 4$ and consequently $A' = \begin{bmatrix} C & B \\ -B^T & 0 \end{bmatrix}$ has rank at most 8. Since $A' \in S^-(K_{m,n} - E(G))$, $mr^-(K_{m,n} - E(G)) \leq 8$.

By Theorem 2.6, being co-diamond-free and $2k_3 + e$-free is a necessary condition for a graph with neither duplicate vertices nor isolated vertices to have minimum skew rank at most 4. We will show that this is in fact sufficient for some families of graphs. Note that if $G$ is the complement of a tree, then $G$ is co-diamond-free and $2k_3 + e$-free. We now show that $mr^-(G) \leq 4$ for such graphs.

**Theorem 5.2.** Let $G$ be a graph whose complement is a tree. Then $mr^-(G) \leq 4$.

**Proof.** We will prove by induction on $|G|$ that if $G^c$ is a tree then $mr^-(G) \leq 4$. Let $n = |G|$. We know that $mr^-(G) \leq 4$ for all graphs $G$ of order $n \leq 5$. So the statement is true for $n \leq 5$.

Assume $n \geq 6$ and proceed by induction. Consider a graph $G$ on $n + 1$ vertices $1, \ldots, n, n + 1$ such that $G^c$ is a tree. If $G^c = K_{1,n}$, then $G = K_n \cup K_1$ and by Theorem 2.3, $mr^-(G) = 2$. Assume
Theorem 2.4 and Proposition 2.2(c), let \( G' = G - (n + 1) \). Since \( n + 1 \) is a pendant vertex of the tree \( G' \), \( G' \) is a tree and \( mr_1(G') \leq 4 \) by the induction hypothesis. Since \( G' \) is a tree, \( G' \) has no duplicate vertices. If \( G' \) has an isolated vertex, then \( G' = K_{1,n-1} \) and by Corollary 3.3, \( mr_1(G) = 4 \). Otherwise by Corollary 4.11, \( mr_1(G) = mr_1(G + (n + 1)) \leq 4 \). \( \square \)

A graph is outerplanar if it can be drawn in the plane so that no edges cross, and all vertices belong to the unbounded face.

**Theorem 5.3.** Let \( G \) be a graph whose complement is a 2-connected outerplanar graph.

(a) If \( G \) is co-diamond-free, then \( mr_1(G) \leq 4 \).

(b) If \( G \) has at least 6 vertices and has an induced co-diamond, then \( mr_1(G) \geq 6 \).

**Proof.** (a) We prove by induction on \( n = |G| \) that if \( G' \) is a diamond-free 2-connected outerplanar graph, then \( mr_1(G) \leq 4 \). We know that \( mr_1(G) \leq 4 \) for all graphs \( G \) of order \( n \leq 5 \). So the statement is true for \( n \leq 5 \). Assume \( n \geq 6 \) and proceed by induction. Consider a graph \( G \) on \( n + 2 \) vertices \( 1, 2, \ldots, n + 2 \) so that \( G' \) is a diamond-free 2-connected outerplanar graph. Since outerplanar graph \( G' \) is 2-connected, the minimum degree of \( G' \) is 2. Since \( G' \) is 2-connected outerplanar, the outer face of \( G' \) is a Hamiltonian cycle. Then either \( G' \) has a triangle containing a vertex of degree 2 or \( G' \) has two adjacent vertices of degree 2 without a common neighbor.

Case 1. \( G' \) has a triangle containing a vertex \( v \) of degree 2.

Since \( G' \) is diamond-free, \( G - v \) has neither isolated vertices nor duplicate vertices. Then by Corollary 4.11, \( mr_1(G) = mr_1(G - v) \leq 4 \).

Case 2. \( G' \) has two adjacent vertices of degree 2 without a common neighbor.

Without loss of generality let \( N(n + 2) = \{ n + 1, n - 1 \} \) and \( N(n + 1) = \{ n + 2, n \} \) in \( G' \). Since \( G' \) is 2-connected, \( G - \{ n + 2, n + 1 \} \) has neither isolated vertices nor duplicate vertices. Then by Theorem 4.13, \( mr_1(G) = mr_1(G - \{ n + 2, n + 1 \}) \leq 4 \).

(b) Since \( G' \) is a 2-connected outerplanar graph with a diamond, \( G \) contains an induced subgraph \( H \) on 6 vertices including vertices of the diamond which has a unique perfect matching. Then by Theorem 2.4 and Proposition 2.2(c), \( mr_1(G) = m r_1(H) = 6 \). \( \square \)

**Corollary 5.4.** Let \( H \) be a graph whose complement is obtained from a diamond-free 2-connected outerplanar graph \( G \) by one of the following two graph operations, then \( mr_1(H) \leq 4 \):

(a) Appending a pendant edge or a pendant triangle.

(b) Appending two new vertices \( u \) and \( w \) and edges \( uw, uv, uv_1 \) and \( wv_j \) for any two distinct vertices \( v_i \) and \( v_j \) of \( G \).

**Proof.** By Corollary 4.11, the graph operation (a) does not change minimum skew rank and by Theorem 4.13, the graph operation (b) does not change minimum skew rank. \( \square \)

**Example 5.5.** The following graph \( G \) is diamond-free and 2-connected. \( H_1 \) and \( H_2 \) are obtained from \( G \) by appending a pendant edge and a pendant triangle respectively. \( H_3 \) is obtained by operation (b) of Corollary 5.4 on \( G \). Then \( mr_1(H_1), mr_1(H_2) \) and \( mr_1(H_3) \) are all at most 4.
References