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Active tuned mass dampers for control of in-plane vibrations of wind turbine blades

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ABSTRACT

This paper investigates the use of active tuned mass dampers (ATMDs) for the mitigation of in-plane vibrations in rotating wind turbine blades. The rotating wind turbine blades with tower interaction represent time-varying dynamical systems with periodically varying mass, stiffness, and damping matrices. The aim of this paper is to determine whether ATMDs could be used to reduce in-plane blade vibrations in wind turbines with better performance than compared with their passive counterparts. A Euler–Lagrangian wind turbine mathematical model based on energy formulation was developed for this purpose, which considers the structural dynamics of the system and the interaction between in-plane and out-of-plane vibrations. Also, the interaction between the blades and the tower including the tuned mass dampers is considered. The wind turbine with tuned mass dampers was subjected to gravity, centrifugal, and turbulent aerodynamic loadings. Investigations show promising results for the use of ATMDs in the vibration control of wind turbine blades. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: active tuned mass dampers; wind turbines; active control; LQR; in-plane vibration; vibration control

1. INTRODUCTION

The principal objective in the design of a wind turbine is to maximize the possible power output under specified atmospheric conditions. This has led to the development of larger wind turbines with increased rotor diameters of over 120 m. Although increased rotor diameters allow more of the available wind resource to be extracted for power generation, the increased blade lengths have also increased the flexibility of the blades, which has led to increased vibrations.

The blades are now a limiting factor in the design of even larger turbines. With large blades, the general assumptions made by most design codes of small deflections and the application of loads on the undeformed blade do not hold true. It has also been shown that large blade vibrations have a major influence on power production [1]. As a result, the reduction of the vibration of wind turbine blades has become an increasingly important area of research in the wind turbine industry.

In this paper, the control of in-plane blade vibrations (which are predominantly edgewise with some flapwise contribution) is considered, that is, reducing blade vibrations that occur in the blade rotation plane. In the in-plane direction, the modal damping is low due to low aerodynamic damping. Aeroelastic stability problems arising from edgewise vibrations were first noticed in the early 1990s [2]. Since then, problems caused by edgewise blade vibrations have been investigated by several researchers. Thomsen et al. investigated the problems caused by edgewise vibrations in stall-regulated wind
In some cases, the aerodynamic damping of the blade vibrations in the edgewise direction becomes negative. As a result, the sum of the structural damping and the aerodynamic damping can be less than zero. It was found that the risk of the occurrence of violent edgewise blade vibration is hard to predict and depends on numerous wind turbine parameters (airfoil characteristics, turbine natural frequencies, etc.). Thus, small changes in turbine design could lead to edgewise blade vibration problems. Hansen showed that edgewise blade vibration could cause aeroelastic instability in modern commercial wind turbines. Riziotis et al. presented an aeroelastic stability tool to consider the complete wind turbine configuration. The authors found that blades equipped with stall strips perform better concerning edgewise vibrations. The beneficial effect of the stall strips also reflects on the tower vibrations. However, this improved behavior incurs a power penalty. The use of stall strips to reduce edgewise blade vibrations has also been investigated by Petersen et al. The investigation demonstrated a significant improvement in the aerodynamic damping conditions when stall strips are mounted on the leading edge of wind turbine blades. However, the peak rotor power was also reduced considerably.

Some studies have considered structural control techniques to reduce blade vibrations. Although structural control has been an active area of research for over two decades, applying structural control techniques to wind turbines is a new and developing area of research. There are presently limited studies that focus on this area of research. Furthermore, the papers that are published in the literature generally focus on passive structural control techniques. Some of these studies recommend the use of active control in future studies.

In this paper, the use of active tuned mass dampers (ATMDs) to reduce in-plane vibrations in wind turbine blades is investigated. An active structural control strategy is used because there are some issues to consider when using traditional passive TMDs in large structural systems such as wind turbines. The mass ratio is the governing parameter that affects the performance of a passive TMD. If the primary structure is very large (e.g., a wind turbine blade), the TMD will inevitably have a small mass ratio. When the mass ratio is small, it makes it difficult to accurately tune the natural frequency of the damper to the natural frequency of the primary structure. Small deviations from the optimal tuning frequency result in poor control performance. Furthermore, if the mass ratio is small, the auxiliary system response may become very large. Therefore, free vibration of the TMD will continue for a long time after the excitation has been removed from the primary structure. Moreover, the main problem with using a passive TMD to dampen wind turbine blade motion is that there is no single dominant resonant frequency present consistently for the entire time history of the response. These problems can be mitigated by introducing an active control strategy.

Three MDOF models of wind turbines are developed in this paper. The first model is uncontrolled, the second model is a passively controlled wind turbine with TMDs, and the third model is an actively controlled wind turbine with ATMDs. These models are developed using the Euler–Lagrangian approach and lead to time-varying systems with the possibility of negative damping. Time-domain simulations are performed on the models using turbulent wind loading. TMDs are placed at strategic points on the blades and at the top of the tower, and active and passive control strategies are investigated. The results have indicated encouraging prospects for the use of ATMDs in the control of in-plane vibrations of wind turbine blades.

2. EULER–LAGRANGE EQUATIONS

Dynamic models of wind turbines are created using the Euler–Lagrange formulation. The wind turbine model consists of rotating pre-twisted blades modeled as continuous beams of variable mass and stiffness. The blades are coupled in the in-plane and out-of-plane directions. The blades are attached at the root to a large mass representing the tower/nacelle of the turbine. The model therefore accounts for coupling in the two directions of blade vibration and accounts for blade–tower interaction. A schematic of the blade vibration model is shown in Figure 1.

The in-plane and out-of-plane vibrations of the ith blade are modeled by two generalized DOFs, $q_{i,\text{in}}(t)$, and $q_{i,\text{out}}(t)$. The coupled in-plane and out-of-plane mode shapes, $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$, have been normalized at the tip so that $q_{i,\text{in}}(t)$ represents the in-plane tip displacement and $q_{i,\text{out}}(t)$ represents
the out-of-plane tip displacement. The structural twist of the blade is accounted for in the calculation of the in-plane and out-of-plane mode shapes. Therefore, these mode shapes each have both edgewise and flapwise components.

In the following formulation, it has been assumed that the displacement at any point $x$ along the $i$th blade is given in terms of the fundamental mode shapes obtained using a finite element analysis and generalized coordinates such that

$$
\begin{align*}
      u_{i,in}(x,t) &= \phi_{in}(x)q_{i,in}(t) \\
      u_{i,out}(x,t) &= \phi_{out}(x)q_{i,out}(t)
\end{align*}
$$

(1)

The displacement of the tower in the rotor plane (side-to-side) is modeled by the DOF, $q_{n,in}(t)$, and the displacement of the tower out of the rotor plane (fore-aft) is modeled by the DOF, $q_{n,out}(t)$. The Euler–Lagrange equation is shown in equation 2.

$$
\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial \dot{q}_i} = Q_i
$$

(2)

where $T$ is the kinetic energy of the system, $V$ is the potential energy of the system, and $Q_i$ is the generalized loading for DOF $i$.

The kinetic and potential energies were calculated and substituted into equation 2 to obtain the equations of motion for the system. Equation 3 shows the potential energy term, $V$, calculated for the $i$th blade.

$$
\begin{align*}
V &= \frac{1}{2} \sum_{i=1}^{3} \int_0^L \left[ EI_{in}(x) \left( \frac{\partial \theta_{i,in}}{\partial x} \right)^2 + EI_{out}(x) \left( \frac{\partial \theta_{i,out}}{\partial x} \right)^2 \right] dx \\
&+ 2EI_{inout}(x) \left( \frac{\partial \theta_{i,in}}{\partial x} \right) \left( \frac{\partial \theta_{i,out}}{\partial x} \right) dx \\
&+ N(x) \left( \frac{\partial u_{i,in}}{\partial x} \right)^2 dx + N(x) \left( \frac{\partial u_{i,out}}{\partial x} \right)^2 dx \\
&+ G(x) \left( \frac{\partial \theta_{i,lin}}{\partial x} \right)^2 dx + G(x) \left( \frac{\partial \theta_{i,out}}{\partial x} \right)^2 dx + \frac{1}{2} k_{n,in}q_{n,in}^2 + \frac{1}{2} k_{n,out}q_{n,out}^2
\end{align*}
$$

(3)

where $L$ is the length of the blade and $E$ is the modulus of elasticity of the blade. The rotational speed of the rotor is denoted by $\Omega$ and is assumed to be constant in this study. The term $k_{n,in}$ is the in-plane stiffness of the tower/nacelle, and $k_{n,out}$ is the out-of-plane stiffness of the tower/nacelle. The rotation of the blade leads to a centrifugal stiffening effect, and $N(x)$ denotes the centrifugal force that gives rise to this stiffening. The gravity force acting on the blade will also contribute to the stiffness of the blade. This can be seen in Figure 2. The term $G(x)$ denotes the gravity force.
In equation 3,
\[
\theta_{i,\text{in}} = \frac{\partial u_{i,\text{in}}}{\partial x}
\]
\[
\theta_{i,\text{out}} = \frac{\partial u_{i,\text{out}}}{\partial x}
\]
(4)

\[
N(x) = \Omega^2 \int_x^L \mu(\tilde{\xi}) \tilde{\xi} d\tilde{\xi}
\]
(5)

and

\[
G(x) = -\frac{1}{2} g \cos \Psi_i \int_x^L \mu(\tilde{\xi}) d\tilde{\xi}
\]
(6)

where \(\mu(x)\) is the variable mass per unit length of the blade. The azimuthal angle of the \(i\)th blade is
\[
\Psi_i = \Omega t + \frac{2\pi}{3}(i - 1)
\]
(7)

The parameters \(I_{\text{in}}(x)\), \(I_{\text{out}}(x)\), and \(I_{\text{inout}}(x)\) are the second area moments of inertia and the second area products of inertia of a cross section of the blade, respectively. By using \(I_{\text{in}}(x)^*\) and \(I_{\text{out}}(x)^*\), the principal second area moments of the cross section, \(I_{\text{in}}(x)\), \(I_{\text{out}}(x)\), and \(I_{\text{inout}}(x)\), can be calculated by equation 8.

\[
I_{\text{in}}(x) = \frac{I_{\text{in}}(x)^* + I_{\text{out}}(x)^*}{2} + \frac{I_{\text{in}}(x)^* - I_{\text{out}}(x)^*}{2} \cos(2\beta)
\]

\[
I_{\text{out}}(x) = \frac{I_{\text{in}}(x)^* + I_{\text{out}}(x)^*}{2} - \frac{I_{\text{in}}(x)^* - I_{\text{out}}(x)^*}{2} \cos(2\beta)
\]

\[
I_{\text{inout}}(x) = \frac{I_{\text{in}}(x)^* - I_{\text{out}}(x)^*}{2} \sin(2\beta)
\]
(8)

where \(\beta\) is the pre-twist angle of a cross section of the blade with respect to the hub. Thus, \(\beta\) is zero at the tip and \(\beta\) is at a maximum value at the hub. The effect of the blade structural pre-twist is therefore accounted for in this model following Zhu [10].
The terms in equation 3 can alternatively be written as

\[
V = \frac{1}{2} \sum_{i=1}^{3} (k_{e,in} + k_{g,in} + k_{gr,in}) q_{i,in}^2 + \frac{1}{2} \sum_{i=1}^{3} (k_{e,out} + k_{g,out} + k_{gr,out}) q_{i,out}^2
\]

\[+ \sum_{i=1}^{3} k_{e,i} q_{i,in} q_{i,out} + \frac{1}{2} k_{n,in} q_{i,in}^2 + \frac{1}{2} k_{n,out} q_{i,out}^2 \tag{9}\]

where \(k_{e,in}\) and \(k_{e,out}\) represent the elastic stiffness in the in-plane and the out-of-plane directions, respectively, and are expressed as

\[
k_{e,in} = \int_{0}^{L} EI_{in}(x) \left( \frac{\partial^2 \phi_{in}(x)}{\partial x^2} \right)^2 \, dx
\]

\[
k_{e,out} = \int_{0}^{L} EI_{out}(x) \left( \frac{\partial^2 \phi_{out}(x)}{\partial x^2} \right)^2 \, dx
\]

Similarly, \(k_{g,in}\) and \(k_{g,out}\) represent the geometric stiffness (arising out of centrifugal stiffening) in the in-plane and the out-of-plane directions, respectively, and are given by

\[
k_{g,in} = \Omega^2 \int_{0}^{L} \left( \int_{0}^{L} \mu(x) \xi d\xi \right) \left( \frac{\partial \phi_{in}(x)}{\partial x} \right)^2 \, dx
\]

\[
k_{g,out} = \Omega^2 \int_{0}^{L} \left( \int_{0}^{L} \mu(x) \xi d\xi \right) \left( \frac{\partial \phi_{out}(x)}{\partial x} \right)^2 \, dx
\]

and \(k_{gr,in}\) and \(k_{gr,out}\) are the stiffness terms arising out of the gravity force in the in-plane and the out-of-plane directions, respectively, and are given by

\[
k_{gr,in} = -\frac{1}{2} g \cos \Psi \int_{0}^{L} \left( \int_{0}^{L} \mu(x) \xi d\xi \right) \left( \frac{\partial \phi_{in}(x)}{\partial x} \right)^2 \, dx
\]

\[
k_{gr,out} = -\frac{1}{2} g \cos \Psi \int_{0}^{L} \left( \int_{0}^{L} \mu(x) \xi d\xi \right) \left( \frac{\partial \phi_{out}(x)}{\partial x} \right)^2 \, dx
\]

The stiffness term arising from the coupling of the in-plane and out-of-plane DOFs given by

\[
k_{c} = \int_{0}^{L} EI_{stout}(x) \left( \frac{\partial \phi_{in}(x)}{\partial x} \right) q_{i,in} \left( \frac{\partial \phi_{out}(x)}{\partial x} \right) q_{i,out} \tag{13}\]

The kinetic energy term, \(T\), calculated for the \(i\)th blade is shown in equation 14

\[
T = \frac{1}{2} \sum_{i=1}^{3} \int_{0}^{L} \mu(x) v_{i}^2 \, dx + \frac{1}{2} M_n q_{n,in}^2 + \frac{1}{2} M_n q_{n,out}^2 \tag{14}\]

where \(v_{i}\) is the absolute velocity of the \(i\)th blade, \(q_{n,in}\) is the absolute in-plane velocity of the tower/nacelle, \(q_{n,out}\) is the absolute out-of-plane velocity of the tower/nacelle, and \(M_n\) is the mass of the tower/nacelle.

The wind turbine model described in this section considers both in-plane (with primarily edgewise contribution) and out-of-plane blade vibrations (with primarily flapwise contribution) that are coupled. Because of the rotation of the blades, the model was formulated using a position vector from a stationary reference point. The position vector, \(r_i(t)\), of a given point on the \(i\)th blade at a distance of \(x\) from the hub has the following representation in the rotating coordinate system with the base unit vectors \((i(t), j(t), k(t))\) (Figure 1).

\[
r_i(t) = (x + q_{n,in} \sin \Psi) i(t)
\]

\[
+ (\phi_{in}(x) q_{i,in} + q_{n,in} \cos \Psi) j(t)
\]

\[
+ (\phi_{out}(x) q_{i,out} + q_{n,out}) k(t)
\]

The velocity vector of a point \(x\) along the blade, \(v_i\), is obtained by differentiating \(r_i\) with respect to time.
\[ v_i(t) = \dot{r}_i(t) = (\dot{q}_{n,in} \sin \Psi_i - \Omega \dot{\phi}_{in}(x) q_{i,in}) i(t) + (\Omega x + \phi_{in}(x) q_{i,in} + \dot{q}_{n,in} \cos \Psi_i) j(t) + (\phi_{out}(x) \dot{q}_{i,out} + \dot{q}_{n,out}) k(t) \] (16)

The expressions for potential and kinetic energies are substituted back into equation 2 to derive the equations of motion. The equations of motion are of the form

\[ [M(t)] \{ \ddot{q} \} + [C(t)] \{ \dot{q} \} + [K(t)] \{ q \} = \{ Q_l \} + \{ Q_g \} \] (17)

where \([M(t)]\), \([C(t)]\), and \([K(t)]\) are the time-dependent mass, damping and stiffness matrices, respectively. The details of these matrices are provided in the appendix. The term \([Q_l]\) is the aerodynamic loading vector, and \([Q_g]\) is the gravity loading vector, both of which are discussed later.

3. TUNED MASS DAMPERS FOR CONTROL OF IN-PLANE BLADE VIBRATION

Tuned mass dampers are now placed inside the blades and the nacelle to control the in-plane vibrations of the blades. The model is reformulated to incorporate TMDs within the blades and in the nacelle. The equations for the potential and kinetic energies are outlined.

The position vector of the damper in the \(i\)th blade, positioned at a location \(x_0\) along the blade, is given by

\[ r_{d,i}(t) = (x_0 + q_{n,in} \sin \Psi_i) i(t) + (\phi_{in}(x) q_{i,in} + q_{n,in} \cos \Psi_i + d_i + y_0) j(t) + (\phi_{out}(x) \dot{q}_{i,out} + q_{n,out}) k(t) \] (18)

where \(d_i\) is the in-plane motion of the damper relative to the blade and \(y_0\) is the reference position of the damper mass from the centerline of the blade along the longitudinal axis. The potential energy, \(V\), and the kinetic energy, \(T\), of the system are, respectively, given by

\[ V = \frac{1}{2} \sum_{i=1}^{3} \left( k_{e,in} + k_{g,in} + k_{gr,in} \right) q_{i,in}^2 + \frac{1}{2} \sum_{i=1}^{3} \left( k_{e,out} + k_{g,out} + k_{gr,out} \right) q_{i,out}^2 + \frac{1}{2} \sum_{i=1}^{3} k_e q_{i,in} q_{i,out} + \frac{1}{2} k_{g,in} q_{i,in}^2 + \frac{1}{2} k_{g,out} q_{i,out}^2 + \frac{1}{2} \sum_{i=1}^{3} k_{bd,i} d_i^2 + \frac{1}{2} k_{nd,in} d_{n,in}^2 \] (19)

\[ T = \frac{1}{2} \sum_{i=1}^{3} \int_0^L \mu(x) v_i^2 dx + \frac{1}{2} M_n q_{n,in}^2 + \frac{1}{2} M_n q_{n,out}^2 + \frac{1}{2} m_{bd} \sum_{i=1}^{3} v_{d,i}^2 + \frac{1}{2} m_{nd} (d_{n,in} + q_{n,in})^2 + \frac{1}{2} m_{nd} q_{n,out}^2 \] (20)

where \(m_{bd}\) is the mass of the blade dampers, \(m_{nd}\) is the mass of the nacelle dampers, \(k_{bd}\) is the stiffness of the blade dampers, and \(k_{nd}\) is the stiffness of the nacelle dampers. The motion of the in-plane nacelle damper is \(q_{n,in}\), and the motion of the out-of-plane nacelle damper is \(d_{n,out}\).

Substituting the terms for the kinetic and potential energy into equation 2 yields the equations of motion for the passively controlled 12-DOF system. There are two DOFs for each blade (in-plane and out-of-plane tip displacement), two DOFs for the nacelle/tower (in-plane and out-of-plane displacement), one DOF for each blade damper (displacement relative to the blade), and one DOF for the nacelle damper (in-plane displacement).

The equations of motion for the passively controlled system also lead to a time-varying system with mass, stiffness, and damping matrices given in the appendix with the vector \([q] = \{ q_{1,in} \quad d_1 \quad q_{2,in} \quad d_2 \quad q_{3,in} \quad d_3 \quad q_{n,in} \quad d_n \quad q_{1,out} \quad q_{2,out} \quad q_{3,out} \quad q_{n,out} \}^T\).
4. ACTIVE TUNED MASS DAMPERS FOR CONTROL OF IN-PLANE BLADE VIBRATION

Active control of the in-plane blade vibration is now investigated. An ATMD system is proposed to reduce blade vibration and reduce the motion of the blade damper. A schematic demonstrating the new model is shown in Figure 3. The addition of actuators to the mass dampers used in the passive TMD case changes the equations of motion. The equations of motion for the new system are outlined as follows

\[
[M(t)]\ddot{\mathbf{q}} + [C(t)]\dot{\mathbf{q}} + [K(t)]\mathbf{q} = \{\mathbf{Q}_l\} + \{\mathbf{Q}_g\} + \{\mathbf{U}_m\} \tag{21}
\]

where \([M(t)],[C(t)],[K(t)],[\mathbf{Q}_l]\), and \([\mathbf{Q}_g]\) are unchanged from the case with passive TMDs. The generalized active control force vector \(\{\mathbf{U}_m\}\) appears as an additional vector on the right hand side of equation 21.

The principle of virtual work is again used to obtain the generalized control force vector \(\{\mathbf{U}_m\}\). The virtual work is expressed as

\[
\delta w = \sum_{i=1}^{3} (\delta d_i + \delta q_{i,\text{in}}\phi_{n}(x_0))u_{ai} - \sum_{i=1}^{3} \delta q_{i,\text{in}}\phi_{n}(x_0)u_{ai} + (\delta d_n + \delta q_{n,\text{in}})u_{an} - \delta q_{n,\text{in}}u_{an} \tag{22}
\]

Equation 22 leads to the generalized control force vector \(\{\mathbf{U}_m\}\)

\[
\{\mathbf{U}_m\} = \begin{bmatrix} \delta w \\ \delta \mathbf{q} \end{bmatrix} = [\mathbf{B}]\{\mathbf{U}_a\} \tag{23}
\]

where the control influence matrix, \([\mathbf{B}]\), is given by

\[
[\mathbf{B}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{24}
\]

The vector of the actuator control forces \(\{\mathbf{U}_a\}\) is given by
\( \left\{ U_a \right\} = \begin{bmatrix} u_{a1} \\ u_{a2} \\ u_{a3} \\ u_{an} \end{bmatrix} \)  

(25)

where \( u_{a1}, u_{a2}, \) and \( u_{a3} \) are the control forces on the actuators connected to the blade dampers for the first, second, and third blades, respectively, and \( u_{an} \) is the control force on the actuator connected to the in-plane damper located at the top of the tower.

The equations of motion are represented in state-space form

\[
\left\{ \dot{q} \right\} = [A] \left\{ q \right\} + \{ Q_l \} + \{ Q_g \} + [B] \left\{ U_a \right\}
\]

(26)

where \( \left\{ \dot{q} \right\} = \begin{bmatrix} \{ q \} \\ \{ \dot{q} \} \end{bmatrix} \), \([A] = \begin{bmatrix} 0_{12 \times 12} & I_{12 \times 12} \\ -M^{-1}K & -M^{-1}C \end{bmatrix}\), \( \{ Q_l \} = \begin{bmatrix} 0_{12 \times 1} \end{bmatrix} \), \( \{ Q_g \} = \begin{bmatrix} 0_{12 \times 1} \\ M^{-1}Q_g \end{bmatrix} \), and \([B] = \begin{bmatrix} 0_{12 \times 4} \\ M^{-1}B \end{bmatrix}\).

An appropriate control scheme is required to implement the active vibration control strategy. In this paper, an optimal control scheme is chosen to obtain the control forces on the actuators, \( \{ U_a \} \). The linear-quadratic regulator (LQR) is used in this paper as it is a well-known method for optimal control. The LQR is a feedback controller used to operate a dynamic system at minimum cost [11]. The linear state feedback is given by

\[
\left\{ U_a \right\} = [G_{LQR}] \left\{ \dot{q} \right\}
\]

(27)

where \([G_{LQR}]\) is the feedback gain and \( \left\{ \dot{q} \right\} \) is the state vector. The value for \([G_{LQR}]\) is optimized such that the following performance criterion is minimized

\[
J = \min_{U_a} \int_{t_0}^{t_f} \left[ \{ q \}^T [Q] \{ q \} + \{ U_a \}^T [R] \{ U_a \} \right] dt
\]

(28)

where \([Q]\) and \([R]\) are weighting matrices for the response and the control force, respectively. Appropriate choice of the weighting matrices is crucial when performing LQR control. In this study, the LQR cost function consists of applying weights to individual state and input values separately (i.e., all non-diagonal \([Q]\) and \([R]\) matrix elements are set to zero). Importance is placed on maintaining small values for blade displacements; thus, their weighting factors are chosen to be large. Additionally, because each TMD has a limited range of motion, similar importance is placed on maintaining small values for the displacements of the TMDs, and their weighting factors are chosen to be large as well.

After the gain matrix is obtained by the LQR method, the control input is then calculated and incorporated into the equations of motion.

5. LOADING

5.1. Aerodynamic loading

Models based on the corrected blade element momentum (BEM) theory have been adopted to determine the aerodynamic load the rotor is subjected to. The loads calculated are quasi-static aerodynamic loads with homogeneous isotropic turbulence. The loads are based on the aerodynamic properties of the blade section airfoils, the geometric characteristics of the rotor, the wind speed, and the rotational speed of the blades.

The rotor blade is assumed to be discretized into \( N \) elements. These elements are analyzed in sections along their length, and the forces are summed over all sections to obtain the total force on the rotor [12]. BEM accounts for a steady aerodynamic effect. Changes of the angle of attack are instantly felt in the aerodynamic loads, and therefore, the time scale for adjustment of the non-stationary flow is assumed to be small compared with the fundamental eigenfrequency of the blade. The effects of a discrete number of blades and far-field effects when the turbine is heavily loaded are not considered.
The BEM method couples momentum theory with blade element theory, the local events taking place at each blade element.

To describe the BEM algorithm for calculation of the aerodynamic wind loads, the following parameters are defined

\[
V_{rel}(x,t) = \sqrt{(V_0(1 - a) + V')^2 + \Omega^2 r^2(1 + a')^2}
\]  
\[
\Phi(x,t) = \arctan\left(\frac{(1 - a)V_0 + V'}{(1 + a')\Omega r}\right)
\]  
\[
\varphi(x,t) = \Phi(x,t) - \varpi(t) - \kappa(x)
\]

where \(V_{rel}\) is the relative wind speed and \(V_0\) is the instantaneous wind speed incorporating the mean wind speed and the vertical wind shear.

The term \(V'\) represents the stochastic (turbulent) component of the wind flow on the rotor plane and has been added to the steady wind field impacting on the rotor. The rotational sampled turbulence spectra are non-homogeneous in nature. However, for simplicity, an isotropic, homogeneous turbulence has been assumed over the rotor field, corresponding to the turbulence represented at the hub height for illustration of the application of the controllers in the paper. A 1-D fully coherent turbulence has been generated compatible with Kaimal spectra with parameters as in Murtagh et al. [13]. The time series of turbulence has been generated following the digital simulation algorithm with random phases as proposed by Shinozuka and Jan [14].

The terms \(a\) and \(a'\) are the axial and tangential induction factors, respectively, calculated by the BEM method, \(\Phi\) is the flow angle, \(\varphi\) is the instantaneous local angle of attack, \(\varpi\) is the pitch angle, and \(\kappa\) is the local pre-twist of the blade (Figure 4).

The lift and drag forces per unit length of the blade, \(p_L\) and \(p_D\), respectively, are calculated by the BEM method using an iterative scheme as described by Hansen [12]. They are calculated as

\[
p_L = \frac{1}{2} \rho V_{rel}^2(x,t)c(x)C_l(\varphi)
\]
\[
p_D = \frac{1}{2} \rho V_{rel}^2(x,t)c(x)C_d(\varphi)
\]

where \(\rho\) is the density of air, \(c(x)\) is the chord length, and \(C_l(\varphi)\) and \(C_d(\varphi)\) are the drag and lift coefficients, respectively, which depend on the local angle of attack.

The normal and tangential coefficients, \(C_N(\varphi)\) and \(C_T(\varphi)\), are determined by

\[
\begin{bmatrix}
C_N(\varphi) \\
C_T(\varphi)
\end{bmatrix} =
\begin{bmatrix}
\cos(\Phi) & \sin(\Phi) \\
\sin(\Phi) & -\cos(\Phi)
\end{bmatrix}
\begin{bmatrix}
C_l(\varphi) \\
C_d(\varphi)
\end{bmatrix}
\]

The aerodynamic loads normal to and tangential to the rotor plane (corresponding to the aerodynamic loads in the out-of-plane and in-plane directions, respectively) are given by

\[
\begin{bmatrix}
p_L \\
p_D
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Omega r(1 + a') \\
V_0(1 - a)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varphi \\
\varpi + \kappa
\end{bmatrix}
\]

\[
c(x)
\]

Figure 4. Airfoil for use in blade element momentum theory.

$$p_N(x,t) = p_t(x,t)\cos(\Phi) + p_l(x,t)\sin(\Phi) \quad (34)$$

$$p_T(x,t) = p_l(x,t)\sin(\Phi) - p_t(x,t)\cos(\Phi) \quad (35)$$

Prandtl’s tip loss factor and Glauert correction have been applied to improve the accuracy of the aerodynamic load model.

### 5.2. Virtual work to determine generalized aerodynamic loads

Virtual work was used to determine the generalized loads on the blades and the nacelle. The total virtual work carried out is given by

$$\delta w = \sum_{i=1}^{3} (\delta q_{i,in}P_{i,in} + \delta q_{i,out}P_{i,out} + \delta q_{n,in}P_{n,in} + \delta q_{n,out}P_{n,out}) \quad (36)$$

where $P_{i,out}$ and $P_{i,in}$ are calculated for the $i$th blade by integrating equations 34 and 35 along the blade length and considering the appropriate mode shape

$$P_{i,out} = \int_0^L p_N(x,t)\phi_{out}(x)dx \quad (37)$$

$$P_{i,in} = \int_0^L p_T(x,t)\phi_{in}(x)dx$$

and $P_{n,out}$ and $P_{n,in}$ are given by

$$P_{n,out} = \sum_{i=1}^{3} P_{i,out}(x,t) \quad (38)$$

$$P_{n,in} = \sum_{i=1}^{3} \int_0^L P_{T,i}(x,t)\cos(\Psi_i)$$

The generalized loads for each generalized DOF corresponding to the in-plane vibration of the blades are given by

$$Q_{i,in} = \frac{\delta w}{\delta q_{i,in}} = P_{i,in}; i = 1, 2, 3 \quad (39)$$

The generalized loads for each generalized DOF corresponding to the out-of-plane vibration of the blades are given by

$$Q_{i,out} = \frac{\delta w}{\delta q_{i,out}} = P_{i,out}; i = 1, 2, 3 \quad (40)$$

The generalized load for the generalized DOF corresponding to the in-plane vibration of the nacelle is given by

$$Q_{n,in} = \frac{\delta w}{\delta q_{n,in}} = P_{n,in} \quad (41)$$

The generalized load for the generalized DOF corresponding to the out-of-plane vibration of the nacelle is given by

$$Q_{n,out} = \frac{\delta w}{\delta q_{n,out}} = P_{n,out} \quad (42)$$

It may be noted that the in-plane and out-of-plane aerodynamic loads are coupled for the model used in this paper.

### 5.3. Gravity loading

The loading effect of gravity has been considered. The gravity force causes a harmonic in-plane load on the blade. The gravity force acting on the blade is illustrated by Figure 2. Resolving the gravity force leads to two forces, one normal to the blade:
and one along the blade:

\[ dG_N = \mu(x)dxg \sin \Psi_i \]  

(43)

\[ dG_A = \mu(x)dxg \cos \Psi_i \]  

(44)

where \( \mu(x) \) is the mass per unit length of the blade and \( g \) is the acceleration due to gravity, 9.81 m/s\(^2\).

The total virtual work done by the gravity force normal to the blade, \( \delta w_g \), is obtained as

\[ \delta w_g = \sum_{i=1}^{3} \int_0^L (g \mu(x)dx \sin(\Psi_i))(\delta q_{i,\text{in}} \phi(x) + \delta q_{n,\text{in}} \cos \Psi_i) \]  

(45)

Therefore, the load on blade \( i \) due to gravity is given by

\[ Q_{i,g} = \frac{\delta w_g}{\delta q_{i,\text{in}}} = \int_0^L (\mu(x)dx)g \sin(\Psi_i) = m_1 g \sin(\Psi_i) \]  

(46)

The load on the nacelle is given by

\[ Q_{n,g} = \frac{\delta w_g}{\delta q_{n,\text{in}}} = \sum_{i=1}^{3} \int_0^L (\mu(x)dx)g \sin(\Psi_i) \cos(\Psi_i) = m_0 g \sum_{i=1}^{3} \sin(\Psi_i) \cos(\Psi_i) = 0 \]  

(47)

6. NUMERICAL SIMULATIONS

Numerical simulations have been performed to illustrate the control strategies developed. The National Renewable Energy Laboratory (NREL) offshore 5-MW baseline wind turbine [15] has been used to develop and test the model. The details of this turbine are provided in Table 1. The proposed model and control strategies were implemented in MATLAB (The MathWorks Inc., MA, USA).

The blade considered is the LM61.5 P2 (manufactured by LM Wind Power, LM Wind Power Group, Kolding, Denmark), which is 61.5 m long and has a total mass of 17,740 kg. Because the radius of the hub is 1.5 m, the total rotor radius is 63 m. The in-plane and out-of-plane mode shapes considered are shown in Figure 5. These have been computed from blade structural data by using the Modes [16] finite element code. Modes performs eigenvalue analysis to compute mode shapes and frequencies for wind turbine blades.

Table I. Properties of NREL 5-MW baseline Horizontal Axis Wind Turbine (HAWT) [15].

<table>
<thead>
<tr>
<th>NREL 5-MW baseline wind turbine properties</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Basic description</strong></td>
<td></td>
</tr>
<tr>
<td>Max. rated power</td>
<td>5 MW</td>
</tr>
<tr>
<td>Rotor orientation, configuration</td>
<td>Upwind, three blades</td>
</tr>
<tr>
<td>Rotor diameter</td>
<td>126 m</td>
</tr>
<tr>
<td>Hub height</td>
<td>90 m</td>
</tr>
<tr>
<td>Cut-in, rated, cut-out wind speed</td>
<td>3 m/s, 11.4 m/s, 25 m/s</td>
</tr>
<tr>
<td>Cut-in, rated rotor speed</td>
<td>6.9 rpm, 12.1 rpm</td>
</tr>
<tr>
<td><strong>Blade (LM 61.5 P2)</strong></td>
<td></td>
</tr>
<tr>
<td>Length</td>
<td>61.5 m</td>
</tr>
<tr>
<td>Overall (integrated) mass</td>
<td>17 740 kg</td>
</tr>
<tr>
<td>Second mass moment of inertia</td>
<td>11 776 kg m(^2)</td>
</tr>
<tr>
<td>First in-plane mode natural frequency</td>
<td>1.0606 Hz</td>
</tr>
<tr>
<td>First out-of-plane mode natural frequency</td>
<td>0.6767 Hz</td>
</tr>
<tr>
<td>Structural-damping ratio (all modes)</td>
<td>0.48%</td>
</tr>
<tr>
<td><strong>Hub + nacelle</strong></td>
<td></td>
</tr>
<tr>
<td>Hub diameter</td>
<td>3 m</td>
</tr>
<tr>
<td>Hub mass</td>
<td>56 780 kg</td>
</tr>
<tr>
<td>Nacelle mass</td>
<td>240 000 kg</td>
</tr>
<tr>
<td><strong>Tower</strong></td>
<td></td>
</tr>
<tr>
<td>Height above ground</td>
<td>87.6 m</td>
</tr>
<tr>
<td>Overall (integrated) mass</td>
<td>347 460 kg</td>
</tr>
<tr>
<td>First fore-aft mode natural frequency</td>
<td>0.324 Hz</td>
</tr>
<tr>
<td>First side-to-side mode natural frequency</td>
<td>0.312 Hz</td>
</tr>
<tr>
<td>Structural-damping ratio (all modes)</td>
<td>1%</td>
</tr>
</tbody>
</table>
The blades are modeled as cantilever beams fixed at the hub. The mode shapes must therefore have zero deflection and slope at the hub. The following sixth-order polynomials have been obtained from Modes [16], to be used as admissible shape functions:

\[
\begin{align*}
\phi_{in}(\bar{x}) &= -0.6893\bar{x}^6 + 2.3738\bar{x}^5 - 3.6043\bar{x}^4 + 2.5737\bar{x}^3 + 0.3461\bar{x}^2 \\
\phi_{out}(\bar{x}) &= -2.4766\bar{x}^5 + 5.1976\bar{x}^4 - 3.4820\bar{x}^3 + 1.7085\bar{x}^2 + 0.0525\bar{x}^2
\end{align*}
\] (48)

where \( \bar{x} = x/L \). The mode shapes are normalized such that \( \phi_{in}(1) = 1 \) and \( \phi_{out}(1) = 1 \).

In all of the actively controlled simulations, the following weighting matrices have been used for the LQR algorithm:

\[
\begin{bmatrix}
Q \\
R
\end{bmatrix} = \begin{bmatrix}
\alpha & I_{24} \\
\beta & I_{4}\times 4
\end{bmatrix}
\] (49)

where \( \alpha = 1 \) and \( \beta = 10^{-10} \).

6.1. Simulation results

Simulations were performed to determine the effectiveness of the proposed control strategy. In all simulations, the rotational speed of the turbine, \( \Omega \), is 12.1 rpm. This is the rated rotor speed for the turbine. The first simulation performed considered a turbine operating at rated speed (12 m/s) with a turbulence intensity of 25%. The load applied to the rotor is shown in Figure 8. The passive control strategy is investigated first. TMDs with a mass ratio of 3% are tuned to the fundamental in-plane blade frequency and placed at a location 75% along the blade length. At this location, a damper stroke of 3 m can be incorporated inside the blade. Figure 6 shows the modest performance of the passive control strategy. Figure 7 shows that the active control strategy performs significantly better than the passive control strategy. A window of 100 s from each 10-min simulation is shown for illustration. The passively controlled blade achieves peak-to-peak reductions of 22% when compared with the uncontrolled blade.
uncontrolled blade and again does not reduce the peak response of the blade. The active TMDs achieve peak-to-peak reductions of 53% compared with the uncontrolled blade and 42% when compared with the passively controlled blade. The active TMDs also reduce the peak response of the uncontrolled blades by 24%. The aerodynamic loads applied to the blades and the control force required for the actively controlled blade are given in Figure 11.

For the last simulation, a parked rotor case is investigated. When wind speeds exceed the cut-out speed, the turbine is shut down and parked. However, it is important that the stationary turbine can

Figure 6. (a) Comparison of uncontrolled system and passively controlled system, $U = 12$ m/s and $I = 25\%$. (b) Zoomed in 100-s window, passively controlled, $U = 12$ m/s and $I = 25\%$. TMDs, tuned mass dampers.

Figure 7. (a) Comparison of uncontrolled system and actively controlled system, $U = 12$ m/s and $I = 25\%$. (b) Zoomed in 100-s window, actively controlled, $U = 12$ m/s and $I = 25\%$. LQR, linear quadratic regulator.
sustain the loads from the wind without failure or damage. In this case, the wind speed is taken as 25 m/s, and the turbulence intensity is again 30%. The passive control strategy used is the same as the previous case. A window of 100 s from each 10-min simulation is shown in Figures 12 and 13 for illustration. The actively controlled system again performs markedly better than the passive control strategy. The passively controlled blade achieves peak-to-peak reductions of 29% when compared with the uncontrolled blade and again does not reduce the peak response of the blade. The active TMDs achieve peak-to-peak reductions of 82% compared with the uncontrolled blade and 53% when compared with the passively controlled blade. The active TMDs also reduce the peak response of the uncontrolled blades by 39%. The aerodynamic loads applied to the blades and the control force required for the actively controlled blade are given in Figure 14.

Figure 8. (a) Blade aerodynamic loading, $U = 12$ m/s and $I = 25\%$. (b) Control force, $U = 12$ m/s and $I = 25\%$.

Figure 9. (a) Comparison of uncontrolled system and passively controlled system, $U = 12$ m/s and $I = 30\%$. (b) Zoomed in 100-s window, passively controlled, $U = 12$ m/s and $I = 30\%$. TMDs, tuned mass dampers.
In each of the aforementioned cases, the damper stroke required to achieve the reductions can be easily accommodated within the blade width. For the purpose of illustration, the damper displacement required for an actively controlled blade with rated condition and 30% turbulence is given in Figure 15.

6.2. Analysis of simulation results

As the turbulence intensity is increased, it is noticed that the control strategy is more effective. At 30% turbulence intensity, there is reduction in the peak value of vibration of 24% compared with the uncontrolled case. At this level of turbulence, there are also reductions in the peak-to-peak values of vibration of up to 53%.

Figure 10. (a) Comparison of uncontrolled system and actively controlled system, $U = 12$ m/s and $I = 30\%$. (b) Zoomed in 100-s window, actively controlled, $U = 12$ m/s and $I = 30\%$. LQR, linear quadratic regulator.

Figure 11. (a) Blade aerodynamic loading, $U = 12$ m/s and $I = 30\%$. (b) Control force, $U = 12$ m/s and $I = 30\%$.

In each of the aforementioned cases, the damper stroke required to achieve the reductions can be easily accommodated within the blade width. For the purpose of illustration, the damper displacement required for an actively controlled blade with rated condition and 30% turbulence is given in Figure 15.

6.2. Analysis of simulation results

As the turbulence intensity is increased, it is noticed that the control strategy is more effective. At 30% turbulence intensity, there is reduction in the peak value of vibration of 24% compared with the uncontrolled case. At this level of turbulence, there are also reductions in the peak-to-peak values of vibration of up to 53%.
7. CONCLUSIONS

In this study, the use of active TMDs for in-plane vibration control in wind turbine blades was investigated. The wind turbine was modeled as a time-varying MDOF system under turbulent wind loading. The coupled MDOF model developed focused on the structural dynamics of the turbine including the interaction between the blades and the tower and the coupling between the in-plane
and out-of-plane blade vibrations. Four mass dampers were added to the model, one attached to each blade, and one at the top of the tower to control the response of each component. An active control strategy was developed using the LQR to control the displacement response of the blades, and this was compared with the passively controlled system (passive TMDs) and the uncontrolled system (no dampers). Numerical simulations were carried out to determine the effectiveness of the active control strategy for various turbulence levels as compared with passive and uncontrolled cases. It was found that the active control strategy used achieved greater response reductions than the passive TMDs in general. This study has also shown that an ATMD control strategy is feasible for a wind turbine blade particularly for higher turbulent loadings with enough room for stroke within the blade.

Figure 14. (a) Blade aerodynamic loading, out-of-plane $U = 25$ m/s and $I = 30\%$. (b) Blade aerodynamic loading, in-plane $U = 25$ m/s and $I = 30\%$. (c) Control force, $U = 25$ m/s and $I = 30\%$.

Figure 15. Blade damper stroke, $U = 12$ m/s and $I = 30\%$. TMD, tuned mass damper.
Appendix A: System matrices for uncontrolled and controlled cases

Uncontrolled matrices

\[
[M] = \begin{bmatrix}
m_{2,\text{in}} & 0 & 0 & a_1 & 0 & 0 & 0 & 0 \\
0 & m_{2,\text{in}} & 0 & a_2 & 0 & 0 & 0 & 0 \\
0 & 0 & m_{2,\text{in}} & a_3 & 0 & 0 & 0 & 0 \\
a_1 & a_2 & a_3 & N_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m_{2,\text{out}} & 0 & 0 & b_1 \\
0 & 0 & 0 & 0 & 0 & m_{2,\text{out}} & b_2 & b_3 \\
0 & 0 & 0 & 0 & b_1 & b_2 & b_3 & N_m \\
\end{bmatrix}
\] (50)

where

\[a_i = m_{1,\text{in}} \cos \Psi_i\]
\[b_i = m_{1,\text{out}} \cos \Psi_i\]
\[N_m = 3m_0 + M_n\]

and

\[m_0 = \int_0^L \mu(x)dx\]
\[m_{1,\text{in}} = \int_0^L \mu(x)\phi_{\text{in}}(x)dx\]
\[m_{2,\text{in}} = \int_0^L \mu(x)(\phi_{\text{in}}(x))^2dx\]
\[m_{1,\text{out}} = \int_0^L \mu(x)\phi_{\text{out}}(x)dx\]
\[m_{2,\text{out}} = \int_0^L \mu(x)(\phi_{\text{out}}(x))^2dx\]

\[
[C] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] (51)

where

\[d_i = -2\Omega m_{1,\text{in}} \sin \Psi_i\]

\[
[K] = \begin{bmatrix}
k_{b,\text{in}} & 0 & 0 & 0 & k_c & 0 & 0 & 0 \\
0 & k_{b,\text{in}} & 0 & 0 & 0 & k_c & 0 & 0 \\
0 & 0 & k_{b,\text{in}} & 0 & 0 & 0 & k_c & 0 \\
c_1 & c_2 & c_3 & k_{n,\text{in}} & 0 & 0 & 0 & 0 \\
k_c & 0 & 0 & 0 & k_{b,\text{out}} & 0 & 0 & 0 \\
k_c & 0 & 0 & 0 & 0 & k_{b,\text{out}} & 0 & 0 \\
k_c & 0 & 0 & 0 & 0 & 0 & k_{b,\text{out}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{n,\text{out}} \\
\end{bmatrix}
\] (52)

where

\[k_{b,\text{in}} = k_{e,\text{in}} + k_{g,\text{in}} - k_{gr,\text{in}} - \Omega^2 m_{2,\text{in}}\]
\[k_{b,\text{out}} = k_{e,\text{out}} + k_{g,\text{out}} - k_{gr,\text{out}}\]
\[c_i = -\Omega^2 m_{1,\text{in}} \cos \Psi_i\]

Controlled matrices

\[
[M] = \begin{bmatrix}
t_{1,\text{in}} & v & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\
v & m_{bd} & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t_{2,\text{in}} & v & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & v & m_{bd} & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_{2,\text{in}} & v & a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & v & m_{bd} & b_3 & 0 & 0 & 0 & 0 & 0 \\
a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & N_m & m_{nd} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{nd} & m_{nd} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{1,\text{out}} & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{1,\text{out}} & 0 & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & c & c & N_m \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & c & c & N_m
\end{bmatrix}
\] (53)

where

\[
a_i = m_{1,\text{in}} \cos \Psi_i + m_{bd} \phi_{\text{in}}(x_0) \cos \Psi_i
\]
\[
b_i = m_{bd} \cos \Psi_i
\]
\[
t_{i,\text{in}} = m_{2,\text{in}} + m_{bd} \phi_{\text{in}}(x_0)^2
\]
\[
t_{i,\text{out}} = m_{2,\text{out}} + m_{bd} \phi_{\text{out}}(x_0)^2
\]
\[
v = m_{bd} \phi_{\text{in}}(x_0)
\]
\[
c = m_{1,\text{out}} + m_{bd} \phi_{\text{out}}(x_0)
\]
\[
N_m = 3m_0 + 3m_{bd} + 2m_{nd} + M_n
\]

\[
[C] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_1 & e_1 & d_2 & e_2 & d_3 & e_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (54)

where

\[
d_i = -2\Omega m_{1,\text{in}} \sin \Psi_i - 2\Omega m_{bd} \phi_{\text{in}}(x_0) \sin \Psi_i
\]
\[
e_i = -2\Omega m_{bd} \sin \Psi_i
\]

\[
[K] = \begin{bmatrix}
k_{b,\text{in}} & k_d & 0 & 0 & 0 & 0 & 0 & k_c & 0 & 0 & 0 & 0 \\
k_d & k_{bt} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{b,\text{in}} & k_d & 0 & 0 & 0 & 0 & k_c & 0 & 0 & 0 \\
0 & 0 & k_d & k_{bt} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{b,\text{in}} & k_d & 0 & 0 & 0 & 0 & k_c & 0 \\
0 & 0 & 0 & 0 & k_d & k_{bt} & 0 & 0 & 0 & 0 & 0 & 0 \\
f_1 & g_1 & f_2 & g_2 & f_3 & g_3 & k_n,\text{in} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k_{ns,\text{in}} & 0 & 0 & 0 & 0 & 0 \\
k_c & 0 & 0 & 0 & 0 & 0 & 0 & k_{b,\text{out}} & 0 & 0 & 0 \\
k_c & 0 & 0 & 0 & 0 & 0 & k_{b,\text{out}} & 0 & 0 & 0 & 0 & 0 \\
k_c & 0 & 0 & 0 & k_c & 0 & 0 & 0 & 0 & k_{b,\text{out}} & 0 \\
k_c & 0 & 0 & 0 & 0 & k_{b,\text{out}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_c & 0 & 0 & 0 & 0 & 0 & k_{n,\text{out}} & 0 \\
0 & 0 & 0 & 0 & 0 & k_{n,\text{out}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (55)

where
\[
k_{b, in} = k_{e, in} + k_{g, in} - k_{gr, in} - \Omega^2 m_{2, in} - \Omega^2 m_{bd} \Phi_{in}(x_0)^2
\]

\[
k_d = -\Omega^2 m_{bd} \Phi_{in}(x_0)
\]

\[
k_{dt} = -\Omega^2 m_{bd} + k_{bd}
\]

\[
f_i = -\Omega^2 m_{1, in} \cos \Psi_i - \Omega^2 m_{bd} \Phi_{in}(x_0) \cos \Psi_i
\]

\[
g_i = -\Omega^2 m_{bd} \cos \Psi_i
\]

REFERENCES


