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Adaptive Kriging controller design for hypersonic flight vehicle via back-stepping

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Abstract: In this study, the adaptive Kriging controller is investigated for the longitudinal dynamics of a generic hypersonic flight vehicle. For the altitude subsystem, the dynamics are transformed into the strict-feedback form where the back-stepping scheme is used. The velocity subsystem is transformed into the output-feedback form. Considering the non-linearity of the dynamics, the nominal feedback is included in the controller while Kriging system is used to estimate the uncertainty, which is described as the realisations of Gaussian random functions. To eliminate the infinite increase of the data size, the recursive Kriging algorithm is adopted in this study. Under the proposed controller, the almost surely bounded stability analysis is presented. The simulation study compared with neural back-stepping control is presented to show the effectiveness of the proposed control approach.

1 Introduction

The success of NASA’s X-43A experimental airplane in flight testing has affirmed the feasibility of hypersonic flight vehicles (HVs). HVs are intended to present a reliable and more cost efficient way to access space. It has great value in military and civil applications. Different from traditional flight vehicles, the longitudinal model of HV is known to be unstable, non-minimum phase with respect to the regulated output and affected by significant model uncertainty. Therefore HVs are extremely sensitive to changes in atmospheric conditions as well as physical and aerodynamic parameters. To improve the safety and reliability, for the controller design for hypersonic aircraft, it must guarantee stability for the system and provide a satisfied control performance [1–3].

Based on different modelling analysis, many methods have been applied on HV control. The sliding mode control [4] and the approximate feedback linearisation approach [5] are proposed through the input–output linearisation using Lie derivative notation. By linearising the model at the trim state, the adaptive control [6] and robust control [7] are investigated. In [29], the control structure combines the inputs from the pilot model, baseline controller and adaptive controller. The sequential loop closure controller design [8, 9] is based on the decomposition of the equations into functional subsystems. The method followed combined robust adaptive dynamic inversion with back-stepping arguments to obtain control architecture. As described in [10, 11], the altitude subsystem can be transformed into the strict-feedback form and the back-stepping design [12] is adopted. Owing to the system uncertainty, fuzzy logic system [10] and neural network (NN) [7] are used to compensate the unknown non-linearities and modelling errors. With the same problem, by system transformation, high gain observer is taken to estimate the newly defined variables where NN is used to approximate the lumped uncertainty [13].

Now with the development of hardware, computer control is drawing more and more attention. However, there exist fewer works in the literatures for discrete control of HV. This is due to the difficulty of intractable Lyapunov design. In this paper, the HV model is expanded by Taylor method. The discrete model is transformed into the strict-feedback form by proper assumptions. Theoretical results on the control of discrete strict-feedback system can be found in [14, 15]. Robust estimation formulation [14] is conducted with the linear controller and neural adaptive control by non-linear identifiers. In [15], the overparameterisation problem is overcome by using the prediction errors. However, considering the peculiar features of HV, it is difficult to obtain the prior information of the parameters for the linear estimation [14, 15]. In [16], the general NN control with predictor function is presented for strict-feedback system. Owing to the non-linearity and the coupling, the dynamics information into consideration for controller design to provide the good control performance.

In this paper we designed the controller step by step with the back-stepping scheme. The key idea of back-stepping is to start with a system which is stabilisable with a known feedback law for a known Lyapunov function, and then to add to its input an integrator [12]. For the special HV control, we take consideration of the nominal non-linearity for feedback design. In [17],
the discrete controller is investigated on HFV. However, the method did not analyse the system uncertainty. With NN approximation, the back-stepping design is presented in [18]. However in most researches, the structures of NNs and fuzzy logic systems are fixed in advance and determined only by trial and prior knowledge. Kriging is one of the non-parametric multivariate interpolation algorithms, developed from theories of spatial statistics and random functions [19]. Compared with other estimators, it can obtain the best unbiased estimates and evaluate the reliability of the estimations based on the hypothesis of regionalised variables. It has been applied for model approximation [20–22] and controller design [23–25, 30]. One Kriging estimator [23, 24] is applied to design an adaptive predictive controller for Gaussian process-based non-linear models. This method, however, requires online handling of a great deal of sample data, and the stability is difficult to analyse. To deal with this problem, we will adopt the recursive form of the Kriging estimator [25] with only a finite number of data to be recorded. Furthermore, the closed system stability through a back-stepping design with Kriging estimation will be analysed.

This paper is organised as follows. Section 2 describes the longitudinal dynamics of a generic HFV. The strict-feedback form is formulated and the discrete analysis model is obtained by first-order Taylor expansion in Section 3. The brief description of stochastic boundedness and Kriging estimators are explained in Sections 4 and 5. Section 6 presents the adaptive Kriging controller design and the stability analysis. The simulation result is included in Section 7. Section 8 presents several comments and final remarks.

The main contributions of the paper lie in:

1. Hypersonic flight controller is designed via back-stepping method and Kriging estimation. The method is compared with NN method to show the effectiveness of the controller.
2. A stochastic Lyapunov function is introduced for system stability analysis and Kriging control design by a back-stepping scheme.

## 2 Hypersonic vehicle modelling

The control-oriented model of the longitudinal dynamics of a generic hypersonic aircraft considered in this study is given by Wang and Stengel [26]. This model is comprised of five state variables \( X = [V, h, \alpha, \gamma, q]^T \) and two control inputs \( U = [\delta_e, \beta]^T \), where \( V \) is the velocity, \( \gamma \) is the flight path angle, \( h \) is the altitude, \( \alpha \) is the attack angle, \( q \) is the pitch rate, \( \delta_e \) is elevator deflection and \( \beta \) is the throttle setting. The velocity subsystem \( \dot{V} \) is mainly related to throttle \( T \) and elevator deflection \( \delta_e \) (3) can be neglected because it is generally much smaller than \( L \).

Assumption 1: Since \( \gamma \) is quite small during the gliding phase, we can take \( \sin \gamma \approx \gamma \) in (2) for simplification. The thrust term \( T \sin \alpha \) in (3) can be neglected because it is generally much smaller than \( L \).

Remark 1: Assumption 1 is made according to the characteristics of the dynamics of HFV. Especially in [9], the flight path angle is set in \([-3^\circ, 3^\circ]\).

### 3 Strict-feedback formulation and discrete-time model

#### 3.1 Strict-feedback formulation

From (1) to (5), the velocity is mainly related to throttle setting whereas the rate change of altitude is mainly related to the elevator deflection. So the dynamics can be decoupled into two functional subsystems. Given the tracking reference \( V_d \) and \( h_d \), we design the velocity and altitude controller separately.

(\( A \) Velocity subsystem: The velocity subsystem (1) can be rewritten as follows

\[
\dot{V} = f_v + g_v u_v
\]

\[
u_v = \beta
\]

\[
y_v = V
\]

where \( f_v = -(D/m + \mu \sin \gamma/r^2) + \bar{q}S \times 0.0224 \cos \alpha/m \), \( g_v = \bar{q}S \times 0.00336 \cos \alpha/m \) if \( \beta > 1 \). Otherwise \( f_v = -(D/m + \mu \sin \gamma/r^2) \), \( g_v = \bar{q}S \times 0.02576 \cos \alpha/m \).

(B) Altitude subsystem: Define \( \theta = \alpha + \gamma \), \( X_4 = [x_1, x_2, x_3, x_4, x_5]^T \), \( x_1 = h \), \( x_2 = \gamma \), \( x_3 = \theta \), \( x_4 = q \), \( u_4 = \delta_e \). With Assumption 1, the dynamics of (2)–(5) can be written as

\[
\dot{x}_4 = f + g u_4
\]

\[
\nu_4 = \delta_e
\]

\[
y_4 = h
\]
the strict-feedback form
\[
\begin{align*}
\dot{x}_1 &= V \sin x_2 \simeq V x_2 = f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(\dot{x}_1, x_2, \dot{x}_3) + g_3(x_1, x_2, \dot{x}_3)x_4 \\
\dot{x}_4 &= f_4(x_1, x_2, x_3, x_4) + g_4(x_1, x_2, x_3, x_4)u_d
\end{align*}
\]

(7)
\[
\begin{align*}
u_d &= \delta_4 \\
y_d &= x_1
\end{align*}
\]
where \( f_1 = 0, \quad g_1 = V, \quad f_2 = -(\mu - V^2\tau) \cos y/(V^2\tau^2 - 0.6203q\Sigma y/(mV)), \quad g_2 = 0.6203q\Sigma y/(mV), \quad f_3 = 0, \quad g_3 = 1, \quad f_4 = \tilde{q}\tilde{S}[C_I(\alpha) + C_M(q) - 0.0292\alpha]/I_{\alpha}^2, \quad g_4 = 0.0292\tilde{q}\tilde{S}/I_{\alpha}^2.
\]

Assumption 2: \( f_i \) and \( g_i \) are unknown smooth functions and can be decomposed into the nominal part \( f_{IN}, \ g_{IN} \) and the unknown part \( \Delta f_i, \ \Delta g_i \). There exist constants \( \tilde{g}_i \geq g_i \geq \bar{g}_i \geq 0 \) such that \( \bar{g}_i \geq g_i \geq \bar{g}_i > 0 \) and \( \bar{g}_i \geq g_{IN} > 0 \), \( i = 2, 4, V \).

Remark 2:
(a) The nominal part will be considered as the feedback item for the controller design in Section 6.
(b) Although some items are constants such as \( f_i \) and \( f_j \), the expression is remained to make (7) and the future control design in Section 6 more explicit.

3.2 Discrete-time model
By first-order Taylor expansion with sample time \( T_s \), systems (6)–(7) can be approximated by a discrete-time model as
\[
V(k+1) = V(k) + T_s[f_i(k) + g_i(k)u_V(k)]
\]
(8)
\[
x_1(k+1) = x_1(k) + T_s[f_1(k) + g_1(k)x_2(k)]
\]
(9)
\[
x_2(k+1) = x_2(k) + T_s[f_2(k) + g_2(k)x_3(k)]
\]
\[
x_3(k+1) = x_3(k) + T_s[f_3(k) + g_3(k)x_4(k)]
\]
\[
x_4(k+1) = x_4(k) + T_s[f_4(k) + g_4(k)u_d(k)]
\]

Remark 3: System (8) and (9) are in strict-feedback form. The first-order Taylor expansion is just for analysis and controller design. The control inputs will be applied on the real system (1)–(5).

4 Stochastic boundedness
The following two concepts for the boundedness of stochastic process are given.

Definition 1 [27]: The stochastic process \( \{\xi(n), n \geq n_0\} \) is said to be almost surely (a.s.) bounded if
\[
\sup_{n \geq n_0} \|\xi(n)\| < \infty \quad \text{a.s.}
\]

Definition 2 [28]: The stochastic process \( \{\xi(n), n \geq n_0\} \) is said to be exponentially bounded in mean square if there are real numbers \( 0 \leq \alpha \leq 1, \ k_1 \geq 0 \) and \( k_2 > 0 \) such that
\[
E[\|\xi(n)\|^2] < k_1 + k_2\alpha^n, \quad n \geq n_0
\]

5 Kriging estimators
Given a domain \( D \in \mathbb{R}^d \), a random function \( Y \) is a collection of random variables \( \{Y(x) x \in D\} \) that can be characterised by the set of all its \( k \)-variate cumulative distributions functions for any number \( k \)
\[
F(x^1, \ldots, x^k) = \Pr[Y(x^1) \leq y^1 \ldots Y(x^k) \leq y^k]
\]
We denote by \( m(x) \) and \( \sigma(x, x') \) the mean function and covariance function of \( Y \)
\[
m(x) = E[Y(x)]
\]
\[
\sigma(x, x') = E[(Y(x) - m(x))(Y(x') - m(x'))]
\]
Further, if it is a joint multivariate Gaussian distribution, then \( Y \) is a Gaussian random function. It is easy to see that a Gaussian random function can be specified by its mean function and covariance function as
\[
(Y(x^1) \ldots Y(x^k)) \sim N((m(x^1) \ldots m(x^k))^T, \Sigma)
\]
(10)
where \( N(., .) \) denotes the Gaussian distribution and \( \Sigma \in \mathbb{R}^{k \times k} \) is the covariance matrix, with \( \Sigma_y = \sigma(x', x') \).
Suppose that the function \( y \) is a realisation of \( Y \), and the values of the variable \( y(x) \) are known at \( k \) points \( x^1, \ldots, x^k \). For a new point \( x' \), the value of \( y(x') \) can be estimated by simple Kriging estimator [19] as
\[
y_{SK}(x') = m(x') + \Sigma_{y}^T\Sigma^{-1}(y - \mathbf{m})
\]
(11)
where
\[
\Sigma_y = (\sigma(x^1, x^0) \ldots \sigma(x^k, x^0))^T
\]
\[
y = (y(x^1) \ldots y(x^k))^T
\]
\[
\mathbf{m} = (m(x^1), \ldots, m(x^k))^T
\]
If \( m(x) \equiv 0 \), (11) can be simplified as
\[
y_{SK}(x^0) = \Sigma_{y}^T\Sigma^{-1}y
\]
(12)
According to [25], the variance is bounded as
\[
\sigma_{SK}^2(x^0) \leq \min_{1 \leq i \leq k} E[(Y(x^i) - Y(x^0))^2]
\]
To avoid the infinite data size, we take the recursive Kriging estimation [25] with the following assumption.

Assumption 3: Each unknown function \( d_i \) is the realisation of a random function \( D_i \) with covariance \( \sigma_i \). The time correlation of \( D_i \), is local, that is
\[
\sigma_i(\theta_i(k_1), \theta_i(k_2)) = 0, \quad |k_1 - k_2| > M
\]
where \( M \) is the positive integer.
The related covariance function is defined as
\[
\sigma_1(\theta(k_1), \theta(k_2)) = C_1^0(C_1^0(\theta(k_1) - \theta(k_2)))^2 + \tau_0 I_{|k_1 - k_2|}
\]
with
\[
C_1^0(h) = \exp\left( -\frac{h^2}{h_0^2} \right)
\]
\[
C_1^1(h) = \begin{cases} 
1 - \frac{r^2}{(M + 1)^2} + \frac{35}{4} \frac{r^3}{(M + 1)^3} - \frac{7}{2} \frac{r^5}{(M + 1)^5} & \text{if } r \leq M \\
0 & \text{otherwise}
\end{cases}
\]
where \(h_0^2\) and \(C_1^0\) are design constants.

6 Discrete control design

6.1 Adaptive Kriging control via back-steping for the altitude subsystem

The desired controller can be obtained by performing the following back-stepping design procedures. The following errors are defined
\[
z_1(k) = x_1(k) - x_{ad}(k)
\]
\[
z_2(k) = x_2(k) - x_{ad}(k)
\]
\[
z_3(k) = x_3(k) - x_{ad}(k)
\]
\[
z_4(k) = x_4(k) - x_{ad}(k)
\]
where \(x_{ad}(k) = h_d(k)\) and \(x_{ad}(k), x_{ad}(k), x_{ad}(k)\) are the virtual control inputs to be designed.

To make the expression for Kriging estimation explicit, we have the following definitions: \(X_i(k) = [x_1(k), \ldots, x_i(k)]^T\), \(\Delta X_i(k) = [\Delta x_1(k), \ldots, \Delta x_i(k)]^T\) with \(\Delta x_i(k) = x_i(k) - x_i(k-1)\) and \(X_{ad}(k+i) = [x_{ad}(k), \ldots, x_{ad}(k+i)]^T\).

Step 1: From (13), the altitude tracking error at time \(k + 1\) is presented as
\[
z_1(k+1) = x_1(k) + T_s f_1(k + g_1(k)x_2(k)) - x_{ad}(k+1)
\]
where \(x_{ad}(k+1) = h_d(k+1)\) is the reference command.

The virtual control input \(x_{ad}(k)\) is designed as
\[
x_{ad}(k) = \frac{1}{T_s^2g_1(k)}[-x_1(k) + c_1z_1(k) + x_{ad}(k+1) - T_s f_1(k)]
\]
where \(0 < c_1 < 1\) and \(g_1(k) = V(k)\).

Combining (14), (17) and (18), the following error dynamics can be obtained
\[
z_1(k+1) = c_1z_1(k) + T_s g_1(k)z_2(k)
\]
Step 2: From (14), the tracking error of flight path angle at time \(k + 1\) is presented as
\[
z_2(k+1) = x_2(k) + T_s f_2(k + g_2(k)x_3(k)) - x_{ad}(k+1)
\]
\[
= x_2(k) + T_s f_2(k) + T_s g_2(k)x_3(k) + T_s g_2(k)z_2(k) + T_s \Delta g_2(k)x_3(k) - x_{ad}(k+1)
\]
Define
\[
z_{U2}(k) = \frac{1}{T_s^2g_2(k)} [T_s \Delta f_2(k) + T_s \Delta g_2(k)x_3(k) + T_s g_2(k)z_2(k)]
\]
(21)
\[
z_{U2}(k) = z_{U2}(k - 1) + d_2(\theta_2(k))
\]
where \(f_{27}(k)\) and \(g_{27}(k)\) are the nominal parts of \(f_2(k)\) and \(g_2(k)\), \(x_{ad}(k+1)\) is the future desired control input value as designed in (18), \(\theta_2(k) = [X_1(k), \Delta X_1(k), X_{ad}(k+2)]^T\) and \(d_2(\theta_2(k))\) is the unknown function.

Remark 4: With the functional decomposition, from (7), the states vector \(X_d\) and related information will be introduced for Kriging estimation. The velocity information is used for controller design for the altitude subsystem but it will not be taken into the inputs of Kriging estimation especially in the stability analysis part.

The virtual control input \(x_{ad}(k)\) is designed as
\[
x_{ad}(k) = \frac{1}{T_s^2g_2(k)}[-T_s f_2(k) + (c_2 - 1)z_2(k)]
\]
\[
- z_{U2}(k - 1) - \tilde{d}_2(\theta_2(k))
\]
(22)
where \(0 < c_2 < 1\) and \(\tilde{d}_2(\theta_2(k))\) is the Kriging estimation of \(d_2(\theta_2(k))\).

Combining (15), (20) and (22), the following equation can be obtained
\[
z_1(k+1) = c_2z_1(k) + T_s g_2(k)z_2(k)
\]
\[
+ T_s g_2(k)\tilde{d}_2(\theta_2(k))
\]
(23)
where \(\tilde{d}_2(\theta_2(k)) = d_2(\theta_2(k)) - \tilde{d}_2(\theta_2(k))\).

Step 3: From (15), the tracking error of pitch angle at time \(k + 1\) is presented as
\[
z_3(k+1) = x_3(k) + T_s [f_3(k) + g_3(k)x_4(k)] - x_{ad}(k+1)
\]
(24)
Define
\[
z_{U3}(k) = \frac{1}{T_s} [x_{ad}(k) - x_{ad}(k+1)]
\]
\[
z_{U3}(k) = z_{U3}(k - 1) + d_3(\theta_3(k))
\]
(25)
where \(\theta_3(k) = [X_1(k), \Delta X_1(k), X_{ad}(k+3)]^T\) and \(d_3(\theta_3(k))\) is the unknown function.

The virtual control input \(x_{ad}(k)\) is designed as
\[
x_{ad}(k) = \frac{1}{T_s^2(c_3 - 1)z_3(k) - z_{U3}(k - 1) - \tilde{d}_3(\theta_3(k))}
\]
(26)
where \(0 < c_3 < 1\) and \(\tilde{d}_3(\theta_3(k))\) is the Kriging estimation of \(d_3(\theta_3(k))\).

Combining (16), (24) and (26), the following equation can be obtained
\[
z_1(k+1) = c_3z_1(k) + T_s z_4(k) + T_s \tilde{d}_3(\theta_3(k))
\]
(27)
where \(\tilde{d}_3(\theta_3(k)) = d_3(\theta_3(k)) - \tilde{d}_3(\theta_3(k))\).
Remark 5: The formulation of Step 3 is actually the same to Step 2. The difference is due to the fact that $f_3 = 0$ and $g_3 = 1$.

Step 4: From (16), the tracking error of pitch rate at time $k+1$ is presented as

$$
 z_t(k+1) = x_t(k) + T_z f_t(k) + g_z u_z(k) - x_u(k+1) = x_t(k) + T_z f_t(k) + g_z u_z(k) + T_u \Delta f_u(k) + T_z \Delta g_z u_z(k) - x_u(k+1)
$$

(28)

Assumption 4: $g_z(k)$ can be approximated by a time-varying function $g_z(k)$.

Define

$$
 z_{u4}(k) = \frac{1}{T_z g_z(k)} [ T_z \Delta f_z(k) + T_z \Delta g_z u_z(k) + x_u(k) - x_u(k+1) ]
$$

(29)

$$
 z_{u4}(k) = z_{u4}(k-1) + \bar{d}_z(\theta_z(k))
$$

where $f_z(k)$ and $g_z$ are the nominal parts of $f_z(k)$ and $g_z(k)$, $x_u(k+1)$ is the future desired control input value as designed in (26), $\theta_z(k) = [X_z(k-1), \Delta X_z(k), X_z(k+4)]^T$ and $\bar{d}_z(\theta_z(k))$ is the unknown function. The actual control input of elevator deflection is designed as follows

$$
 u_z(k) = \frac{1}{T_z g_z(k)} [ T_z \Delta f_z(k) + (c_4 - 1)z_z(k) ]
$$

(30)

where $0 < c_4 < 1$ and $\bar{d}_z(\theta_z(k))$ is the Kriging estimation of $d_z(\theta_z(k))$.

Combining (28) and (30), the following equation can be obtained

$$
 z_t(k+1) = c_4 z_t(k) + T_z g_z u_z(k) \bar{d}_z(\theta_z(k))
$$

(31)

where $\bar{d}_z(\theta_z(k)) = d_z(\theta_z(k)) - \bar{d}_z(\theta_z(k))$.

Remark 6: The value of the uncertainty at time $k$ in (21), (25) and (29) can be calculated from the dynamics (9) with (18), (22) and (26) at time $k+1$. Especially, we have the following calculation

$$
 \Delta f_z(k+1) + \Delta g_z u_z(k) x_z(k+1) = x_z(k+1) - x_z(k) - T_z f_z(k) - T_z g_z u_z(k)
$$

(32)

Theorem 1: Considering system (9) under Assumptions 1–4 with the controller (18), (22), (26), (30) where $d_z(\theta_z(k))$, $i = 2, 3, 4$, is the Kriging estimation with (12). Provided the following conditions hold:

(i) The Gaussian functions of $d_z$, $i = 2, 3, 4$, are independent.
(ii) $E[d^2_z(\theta_z(k))] \leq a_j [\theta_z(k) - \theta_z(k)]^2 + e_j, j = 2, 3, 4$, $x_u(k+1) - x_u(k) \leq \Delta X_{\text{max}}, i = 1, 2, 3, 4$, $\Delta X_{\text{max}} > 0$ is constant and positive.
(iii) There exist positive numbers $a_z, b_z, \rho_z$ and $\eta_z$ such that $0 < \eta_z < 1$. The related definitions can be found in the following proof.

Then all the signals involved are a.s. bounded and the tracking errors $z_z(k), k \geq 2, i = 1, 2, 3, 4$ are exponentially bounded in mean square with

$$
 E[z^2_z(k)] \leq \frac{g_z^2 K_z + g_z^2 T_z k^2}{1 - \gamma_w} \times \left( \sum_{j=1}^4 \left( \frac{z_j(2)}{g_z} \right)^2 + x_j(z_j(1))^2 + x_z(z_j(0))^2 \right)
$$

(33)

**Proof:** Define the positive function

$$
 L(k) = L_1(k) + L_2(k) + L_3(k) + L_4(k)
$$

where $L_1(k) = (z_1(k)/g_z)^2$, $L_2(k) = (z_2(k)/g_z)^2$, $L_3(k) = (z_3(k)/g_z)^2$ and $L_4(k) = (z_4(k)/g_z)^2$. Note that $g_z = 1$.

The fact is known as

$$
 2ab \leq \frac{\varepsilon a^2}{\varepsilon} + \frac{1}{\varepsilon} b^2
$$

(34)

where $\varepsilon$ is constant and positive.

The first item

$$
 L_1(k + 1) = \left( \frac{z_1(k + 1)}{g_z} \right)^2 = \left( \frac{c_1 z_1(k) + T_z g_z(k) z_z(k)}{g_z} \right)^2
$$

$$
 = \frac{c_1^2}{g_z^2} z_z^2(k) + \left( \frac{T_z g_z(k) z_z(k)}{g_z} \right)^2 + 2 \frac{c_1 z_1(k) T_z g_z(k) z_z(k)}{g_z^2}
$$

(35)

where $K_{11} = \left( \frac{c_1^2}{g_z^2} + \varepsilon_1 c_1^2 \right), K_{12} = T_z^2 \left( 1 + \frac{1}{\varepsilon_1} \right)$.

The second item: The conditional expected value of at time $k + 1$ gives

$$
 E[L_z(k + 1) | F(k)]
$$

$$
 = E \left[ \left( \frac{z_z(k + 1)}{g_z} \right)^2 \right] = \left( \frac{c_1 z_z(k)}{g_z} \right)^2
$$

$$
 + \left( \frac{T_z g_z(k) z_z(k)}{g_z^2} \right)^2 + 2 \frac{c_1 z_1(k) T_z g_z(k) z_z(k)}{g_z^2}
$$

$$
 + E \left[ \frac{T_z g_z(k) \bar{d}_z(\theta_z(k)) z_z(k)}{g_z^2} \right] + E \left[ \frac{2 T_z g_z(k) \bar{d}_z(\theta_z(k)) z_z(k)}{g_z^2} \right]
$$

(36)

$$
 E[L_z(k + 1) | F(k)] \leq K_{22} z_z^2(k) + K_{23} z_z^2(k) + E[T_z^2 \bar{d}_z^2(\theta_z(k))]
$$

(37)

where $K_{22} = ((1/g_z^2) + \varepsilon_2)c_1^2$ and $K_{23} = T_z^2(1 + (1/\varepsilon_2))$.
Similarly for the third and fourth items the following conclusions can be acquired.

\[
E[L_3(k+1)F(k)] \leq K_{33}z_2^2(k) + K_{34}z_2^2(k) + E[T_i^2\bar{d}^2_j(\theta_s(k))]
\]

(38)

where \( K_{33} = (1 + \varepsilon_3)c_1^2, K_{34} = T_i^2 \left( 1 + \frac{1}{\varepsilon_3} \right) \)

\[
E[L_4(k+1)F(k)] \leq K_{41}z_2^2(k) + E[T_i^2\bar{d}^2_j(\theta_s(k))]
\]

(39)

where \( K_{44} = (1/\bar{g}_2^2)c_1^2 \).

\[
E[L(k+1)F(k)] \leq \sum_{i=1}^{4} [K_iz_i^2(k) + P_iz_i^2(k - 1) + Mz_i^2(k - 2)] + K_3
\]

(40)

According to the Appendix, we have

\[
K_1 = K_{11} + \sum_{j=2}^{4} T_i^2a_j(3 + 3\rho_j)
\]

\[
K_2 = K_{12} + K_{22} + \sum_{j=2}^{4} T_i^2a_j(3 + 3\rho_j)
\]

\[
K_3 = K_{23} + K_{33} + \sum_{j=2}^{4} T_i^2a_j(3 + 3\rho_j)
\]

\[
K_4 = K_{34} + K_{44} + T_i^2a_4(3 + 3\rho_4)
\]

\[
K_5 = \sum_{j=1}^{4} \sum_{k=1}^{j} T_i^2 \left[ a_j \left( 6 + \frac{1}{\eta_j} + \frac{4}{\rho_j} \right) \Delta x_{\text{max}}^2 + \varepsilon_j \right]
\]

\[+ T_i^2 \left[ a_2 \left( 6 + \frac{1}{\eta_23} + \frac{4}{\rho_23} \right) \Delta x_{\text{max}}^2 \right]
\]

If \( i = 1, 2, 4 \), we have

\[
P_i = \sum_{j=i}^{4} T_i^2a_j(14 + 2\eta_j + 12\rho_j)
\]

\[
M_i = \sum_{j=i}^{4} T_i^2a_j(5 + 2\eta_j + 3\rho_j)
\]

For \( i = 3 \), we have

\[
P_3 = \sum_{j=3}^{4} T_i^2a_j(14 + 2\eta_j + 12\rho_j)
\]

\[+ T_i^2a_2(14 + 2\eta_23 + 12\rho_23) \]

\[
M_3 = \sum_{j=3}^{4} T_i^2a_3(5 + 2\eta_j + 3\rho_j) + T_i^2a_2(5 + 2\eta_23 + 3\rho_23)
\]

Define the stochastic Lyapunov function candidate

\[
J(k) = L(k) + \sum_{i=1}^{4} [\alpha_i z_i^2(k - 1) + \beta_i z_i^2(k - 2)]
\]

Then

\[
E[J(k + 1)F(k)]
\]

\[
\leq E[L(k + 1)F(k)] + \sum_{i=1}^{4} [\alpha_i z_i^2(k) + \beta_i z_i^2(k - 1)]
\]

\[
\leq \sum_{i=1}^{4} (K_i + \alpha_i)z_i^2(k) + (P_i + \beta_i)z_i^2(k - 1)
\]

\[+ Mz_i^2(k - 2)] + K_5
\]

\[
\leq \gamma_n J(k) + K_5
\]

(42)

where \( \gamma_n = \max\{(K_i + \alpha_i)/\bar{g}_i, (P_i + \beta_i)/\alpha_i, (M_i/\beta_i)\}, i = 1, 2, 3, 4 \).

From Definitions 1 and 2, \( J(k), k \geq 2 \) is a.s. bounded and for every \( k \geq 2 \)

\[
E[J(k)] \leq \gamma_n^{k-2} J(2) + K_5 \sum_{i=0}^{k-3} \gamma_n^i
\]

\[
\leq \gamma_n^{k-2} J(2) + \frac{K_5}{1 - \gamma_n}
\]

(43)

Therefore \( z_i(k), k \geq 2, i = 1, 2, 3, 4 \) satisfies

\[
\sup_{k \geq 2} z_i^2(k) \leq \bar{g}_i^2 E[J(k)] < \infty
\]

(44)

\[
E[z_i^2(k)] \leq \bar{g}_i^2 E[J(k)]
\]

\[
\leq \bar{g}_i^2 \left( \gamma_n^{k-2} J(2) + \frac{K_5}{1 - \gamma_n} \right)
\]

\[
= \frac{\bar{g}_i^2 K_5}{1 - \gamma_n} + \bar{g}_i^2 \gamma_n^{k-2}
\]

\[\times \left( \sum_{j=1}^{4} \left( \frac{z_j(2)}{\bar{g}_i} \right)^2 + \alpha_j(z_j(1))^2 + \beta_j(z_j(0))^2 \right) \]

(45)

This completes the proof.

**Remark 7:** One more item is included in \( P_3, M_3 \), because \( x_3(k) \) is added as the input of Kriging system \( \hat{d}_3(\theta_s(k)) \).

### 6.2 Adaptive Kriging control for the velocity subsystem

Define the velocity error

\[
z_V(k) = V(k) - V_d(k)
\]

(46)

From (46), we have

\[
z_V(k + 1) = V(k + 1) - V_d(k + 1)
\]

\[
= V(k) + T_i[f_{VN}(k) + g_{VN}(k)u_V(k)] - V_d(k + 1)
\]

\[+ T_i[\Delta f_V(k) + \Delta g_V(k)u_V(k)]
\]

(47)

The uncertainty is defined as

\[
z_{UV}(k) = \frac{1}{T_i} f_{VN}(k) T_i[\Delta f_V(k) + \Delta g_V(k)u_V(k)]
\]

\[
z_{UV}(k) = z_{UV}(k - 1) + d_V(\theta_V(k))
\]

(48)

where \( f_{VN}(k) \) and \( g_{VN}(k) \) are the nominal parts of \( f_V(k) \) and \( g_V(k) \), \( V_d(k + 1) \) is the future reference value,
\[ \theta_V(k) = [X_T^4(k-1), \Delta X_T^4(k), V(k), \Delta V(k)]^T \] and \( d_V(\theta_V(k)) \) is the unknown function.

The throttle setting is designed as
\[
u_V(k) = \frac{1}{T_{gV\gamma}(k)} [-V(k) - T_{fV}(k) + c_V z_V(k) \\
+ V_d(k + 1) - z_{UV}(k - 1) - \tilde{d}_V(\theta_V(k))] \tag{49}
\]
where \( \tilde{d}_V(\theta_V(k)) \) is the Kriging estimation of \( d_V(\theta_V(k)) \).

Then (47) can be derived as
\[
z_V(k + 1) = c_V z_V(k) + T_{gV\gamma}(k) \tilde{d}_V(\theta_V(k)) \tag{50}
\]
where \( \tilde{d}_V(\theta_V(k)) = d_V(\theta_V(k)) - \tilde{d}_V(\theta_V(k)) \).

**Theorem 2:** Considering system (8) with the controller (49), where \( \tilde{d}_V(\theta_V(k)) \) is the Kriging estimation with (12), the velocity is a.s. bounded. The proof is quite similar to Theorem 1 and thus omitted here.

---

**Fig. 1** System tracking: altitude and velocity tracking (I)

---

**Fig. 2** System states and tracking error (I)
Simulations

In this section, the simulation test of the proposed adaptive Kriging controller is compared with the discrete neural controller [18]. Reference commands are generated by the filter

\[ h_i = \frac{\omega_{n1}\omega_{n2}^2}{(s + \omega_{n1})(s^2 + 2\epsilon_c\omega_{n2}s + \omega_{n2}^2)} \quad (51) \]

where \( \omega_{n1} = 0.7, \omega_{n2} = 0.3, \epsilon_c = 0.7, \omega_{v1} = 0.5, \omega_{v2} = 0.2, \epsilon_{vc} = 0.7. \) The perturbation is set to be 3% for the parameter set \((m, \mu, Iyy, S)\).

The parameters for the Kriging controller are selected as \( T_s = 0.05 \text{ s}, c_1 = 0.95, c_2 = 0.9, c_1 = 0.9, c_4 = 0.9, c_v = 0.9, h_0 = 0.1, C^0 = 1, \tau_0 = 1, M = 10. \)

\[ V_d = \frac{\omega_{v1}\omega_{v2}^2}{(s + \omega_{v1})(s^2 + 2\epsilon_{vc}\omega_{v2}s + \omega_{v2}^2)} \quad (52) \]

Fig. 3  Control inputs: elevator deflection and throttle setting (I)

Fig. 4  Kriging tracking (I)
7.1 Step tracking of altitude and velocity

Fig. 1 depicts the response performance that the altitude controller tracks the step change with magnitude 500 ft while the velocity steps from 15 060 to 15 160 ft/s. For both methods, the performance of the altitude and velocity is quite similar. However from the altitude tracking error in Fig. 1, we can see the Kriging method got less stable error. The system states response of flight path angle, pitch angle and pith rate with the tracking error from virtual control inputs are demonstrated in Fig. 2. It shows the rapid tracking of the Kriging controller. Furthermore, we find that compared with neural control, the Kriging controller achieved smoother adjustment (Fig. 3). From Fig. 4, the estimation error for the defined uncertainty in (21), (25), (29) and (48) is presented. It shows that the estimation method with Kriging system is efficient.

7.2 Step tracking of altitude

In this test, the altitude controller tracks step command with magnitude 500 ft whereas the velocity is maintained in the neighbourhood of 15 060 ft/s. In Fig. 5, it demonstrates the Kriging controller is with smaller velocity tracking error. Referred to Fig. 6, the same conclusion can be made that Kriging controller is with smoother adjustment. Compared with Fig. 3, the elevator deflection value is almost the same whereas throttle setting is smaller. It indicates that the altitude subsystem is mainly controller by elevator deflection.

8 Conclusions

Taking advantage of Kriging system, the adaptive discrete controller by the back-stepping scheme is proposed for HFV control. Considering the characteristics of HFV, the nominal non-linearity during each step is eliminated to ensure the performance. The system uncertainty is modelled as the realisations of Gaussian random functions and estimated by the recursive Kriging algorithm. The stochastic stability analysis for back-stepping design is presented. Simulation result shows the Kriging controller is with smoother adjustments and less stable error than the neural controller.

9 Acknowledgments

This work was supported by Sino Swiss Science and Technology Cooperation and the National Science Foundation of China (Grants Nos: 90716021, 61?134004, 60736023). The authors would like to thank Hao Wu and the anonymous reviewers for constructive comments that helped to improve the quality and presentation of this paper.
10 References


11 Appendix

For $j = 3, 4$

\[ E[D_j^2(\theta j)] \leq a_j \left[ \sum_{i=1}^{j} \left[ \frac{(1 + \eta_j)(z_i(k - 1) - z_i(k - 2))^2 + \left(1 + \frac{1}{\rho_j}\right) \Delta x_{\text{max}}^2}{\max} \right] + \epsilon_j \right] \]

\[ \leq a_j \left[ \sum_{i=1}^{j} \left( 3 + 3 \rho_j \right) z_i^2(k) + \left( 14 + 4 \eta_j + 12 \rho_j \right) z_i^2(k - 1) + \left( 5 + 2 \eta_j + 3 \rho_j \right) z_i^2(k - 2) 
+ \left( 6 + \frac{1}{\eta_j} + \frac{4}{\rho_j} \right) \Delta x_{\text{max}}^2 \right] + \epsilon_j \]

\[ E[D_j^2(\theta j)] \leq a_j \left[ \sum_{i=1}^{j} \left( 3 + 3 \rho_j \right) z_i^2(k) + \left( 14 + 4 \eta_j + 12 \rho_j \right) z_i^2(k - 1) + \left( 5 + 2 \eta_j + 3 \rho_j \right) z_i^2(k - 2) 
+ \left( 6 + \frac{1}{\eta_j} + \frac{4}{\rho_j} \right) \Delta x_{\text{max}}^2 \right] + \epsilon_j \]

\[ E[D_j^2(\theta j)] \leq a_j \left[ \sum_{i=1}^{j} \left( 3 + 3 \rho_j \right) z_i^2(k) + \left( 14 + 4 \eta_j + 12 \rho_j \right) z_i^2(k - 1) + \left( 5 + 2 \eta_j + 3 \rho_j \right) z_i^2(k - 2) + \left( 6 + \frac{1}{\eta_j} + \frac{4}{\rho_j} \right) \Delta x_{\text{max}}^2 \right] + \epsilon_j \]
\[-2x_d(k+1) + x_d(k+2)^2 \leq (1 + \rho_{ji})(z_i(k) - 2z_i(k-1) + z_i(k-2))^2 + 4\left(1 + \frac{1}{\rho_{ji}}\right)\Delta x_{max}\]  

\[(55)\]

\[(x_{1d}(k+i) - x_{1d}(k+i-1))^2 \leq \Delta x_{max}^2\]  

\[(56)\]

Then we have (see (57))

For \(j = 2\), as \(x_1(k)\) is involved as the input for \(ZU_2\), we have (see (58))