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5-COLORING GRAPHS WITH 4 CROSSINGS*

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Abstract. We answer in the negative a question of Oporowski and Zhao [*Discrete Math.*, 309 (2009), pp. 2948–2951] asking whether every graph with crossing number at most 5 and clique number at most 5 is 5-colorable. However, we show that every graph with crossing number at most 4 and clique number at most 5 is 5-colorable. We also show some colorability results on graphs that can be made planar by removing a few edges. In particular, we show that, if a graph with clique number at most 5 has three edges whose removal leaves the graph planar, then it is 5-colorable.

Key words. chromatic number, crossing number, clique number, girth

AMS subject classifications.

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1. Introduction.

The crossing number of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of G in the plane.

The Four Color Theorem states that, if a graph has crossing number zero, then it is 4-colorable. A natural question is to ask whether the chromatic number of a graph is bounded in terms of its crossing number. To answer the question, the concept of crossing cover is crucial. A *crossing cover* of a drawing of a graph is a set of vertices C such that every crossing has an edge incident with a vertex in C. If C is a crossing cover, then $G - C$ is planar, so $\chi(G) \leq 4 + \chi(G\langle C\rangle) \leq 4 + |C|$. Picking one vertex per crossing, we obtain a crossing cover of cardinality at most $\mathrm{cr}(G)$ so $\chi(G) \leq 4 + \mathrm{cr}(G)$.

This upper bound is tight only for $\mathrm{cr}(G) \leq 1$. So a natural question is the following: What is the smallest integer $f(k)$ such that every graph G with crossing number at most k is $f(k)$-colorable? An argument similar to the one above shows that $f(k+1) \leq f(k) + 1$. Settling a conjecture of Albertson [1], Schaefer [14] showed that $f(k) = O\left(k^{1/4}\right)$. This upper bound is tight up to a constant factor since $\chi(K_n) = n$ and $\mathrm{cr}(K_n) \leq \binom{|E(K_n)|}{2} = \binom{\binom{n}{2}}{2} \leq \frac{1}{8}n^4$.

The values of $f(k)$ are known for a number of small values of k. The Four Color Theorem states $f(0) = 4$ and implies easily that $f(1) \leq 5$. Since $\mathrm{cr}(K_5) = 1$, we have $f(1) = 5$. Oporowski and Zhao [13] showed that $f(2) = 5$. Since $\mathrm{cr}(K_6) = 3$, we have $f(3) = 6$. Further, Albertson et al. [2] showed that $f(6) = 6$. Albertson then conjectured that if $\chi(G) = r$, then $\mathrm{cr}(G) \leq \mathrm{cr}(K_r)$. This conjecture was proved by Barát and Tóth [3] for $r \leq 16$.

A graph G is *r-critical* if $\chi(G) = r$ and $\chi(G') < r$ for every proper subgraph G' of G. Oporowski and Zhao [13] proved that K_6 is the unique 6-critical graph with crossing number 3.

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THEOREM 1.1 (Oporowski and Zhao [13]). *If* $\mathrm{cr}(G) \leq 3$ *and* $\omega(G) \leq 5$*, then* $\chi(G) \leq 5$.

Oporowski and Zhao [13] asked whether the conclusion remains true even if $\mathrm{cr}(G) \in \{4, 5\}$.

QUESTION 1.2 (Oporowski and Zhao [13]). *If* $\mathrm{cr}(G) \leq 5$ *and* $\omega(G) \leq 5$*, is* G 5-*colorable?*

We answer in the negative by showing a counterexample. The help of Zdeněk Dvořák was greatly appreciated while obtaining the following theorem.

THEOREM 1.3. *There exists a graph* G *such that* $\mathrm{cr}(G) = 5$*,* $\omega(G) \leq 5$*, and* $\chi(G) = 6$.

On the other hand, we answer Question 1.2 in the affirmative when $\mathrm{cr}(G) = 4$.

THEOREM 1.4. *If* $\mathrm{cr}(G) \leq 4$ *and* $\omega(G) \leq 5$*, then* $\chi(G) \leq 5$.

A key notion in the proof of Theorem 1.4 is the one of dependent crossings. The *cluster* of a crossing is the set of endvertices of its two edges. Two crossings are *dependent* if their clusters intersect.

Settling a conjecture of Albertson [1], Král' and Stacho [12] showed the following.

THEOREM 1.5 (Král' and Stacho [12]). *If a graph* G *has a drawing in the plane in which no two crossings are dependent, then* $\chi(G) \leq 5$.

Loosely speaking, this theorem states that, if the crossings are far apart from each other, then the graph is 5-colorable. On the other hand, if all the crossings are very close, that is, if all their clusters share a common vertex, then the graph is also 5-colorable. In the same vein, we show that, if the crossings are covered by $2k$ edges, then the graph is $(4+k)$-colorable (Theorem 4.1). In particular, if the crossings are covered by three edges, then the graph is 6-colorable. This bound 6 is tight since $\mathrm{cr}(K_6) = 3$, and thus one can remove three edges from K_6 to make it planar. However, by generalizing Theorem 1.1, we show that K_6 is essentially the unique obstruction for such a graph to be 5-colorable.

THEOREM 1.6. *If* $\omega(G) \leq 5$ *and there exists a set* F *of at most three edges such that* $G \setminus F$ *is planar, then* $\chi(G) \leq 5$.

Related open problems are discussed in the final section.

2. Preliminaries.

2.1. Drawings of graphs. A *drawing* $\tilde{G}$ (in the plane or the sphere) of a graph $G = (V, E)$ consists of a bijection D from $V \cup E$ into a set $\tilde{V} \cup \tilde{E}$ such that

(i) $\tilde{V}$ is the image of V and a set of distinct points in the plane;

(ii) for any edge $e = uv$, the element $D(e) = \tilde{e}$ of $\tilde{E}$ is the image of a continuous injective mapping ϕ_e from $[0, 1]$ to the plane which is simple (i.e., does not intersect itself) such that $\phi_e(0) = D(u)$, $\phi_e(1) = D(v)$ and $\phi_e(]0, 1[) \cap \tilde{V} = \emptyset$;

(iii) every point in the plane is in at most two images of edges unless it is in $\tilde{V}$;

(iv) for two distincts edges e_1 and e_2 of E, $\tilde{e_1}$ and $\tilde{e_2}$ intersect in a finite number of points.

We shall often confound the vertex and edge sets of a graph with their image in one of its drawings.

A *crossing* in a drawing of G is a point in the plane minus $\tilde{V}$ that belongs to two edges. Formally, it is a point of $\phi_{e_1}(]0, 1[) \cap \phi_{e_2}(]0, 1[)$ for some edges e_1 and e_2. A *portion* of an edge e is a subarc of $\phi_e[0, 1]$ between two consecutives endpoints or crossings on e. A portion from a to b is called an (a, b)-*portion*.

A graph is *planar* if it has a drawing without crossings. An easy consequence of Euler's Formula is the following well-known proposition.

PROPOSITION 2.1. *If G is planar, then $|E(G)| \leq 3|V(G)| - 6$.*

A drawing of G is *optimal* if it minimizes the number of crossings. Note that two edges may intersect several times, in either endvertices or crossings. However, thanks to the two following lemmas, we will only consider *nice* drawings, i.e., drawings such that two edges intersect at most once.

LEMMA 2.2. *Every graph with crossing number k has a nice drawing with at most k crossings.*

Proof. Let G be a graph with crossing number k. Consider an optimal drawing of G that minimizes the number of crossings between edges with a common vertex. Suppose, by contradiction, that two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ intersect at least twice. Let a and b be two points in the intersection of e_1 and e_2. Without loss of generality we may assume that u_1, u_2, v_1, and v_2 are in the exterior of the closed curve C which is the union of the (a, b)-portion P_1 on u_1v_1 and the (a, b)-portion P_2 on u_2v_2. We may also assume that P_1 contains at least as many crossings as P_2.

Then one can redraw u_1v_1 along the (u_1, a)-portion of e_1, P_2, and the (b, v_1)-portion of e_1 slightly in the exterior of C so that e_1 and e_2 do not cross anymore. Doing so, all the crossings of P_1 including a and b (if they were crossings) disappear while a crossing is created per crossings of P_2 distinct from a and b. Since one of $\{a, b\}$ must be a crossing (there are no parallel arcs), we obtain a drawing with one crossing less, a contradiction. ∎

Similarly, one can show the following lemma.

LEMMA 2.3. *Every graph with a set F of edges whose deletion results in a planar graph has a nice drawing in which each crossing contains at least one edge from F.*

In this paper, we consider only nice drawings. Thus a crossing is uniquely defined by the pair of edges it belongs to. Henceforth, we will often confound a crossing with this set of two edges.

A *face* of a drawing $\tilde{G}$ is a connected component of the space obtained by deleting $\tilde{V} \cup \tilde{E}$ from the plane. We let $F(\tilde{G})$ (or simply F) be the set of faces of $\tilde{G}$. We say that a vertex v or a portion of an edge is incident to $f \in \tilde{F}$ if v is contained in the closure of f. The boundary of f, denoted by $bd(f)$ consists of the vertices and maximum (with respect to inclusion) portions of edges incident to it. An *embedding* of a graph is the set of boundaries of the faces of some drawing of G in the plane.

LEMMA 2.4. *Up to a permutation of the vertices, there is only one embedding of K_6 using exactly 3 crossings (see Figure 1).*

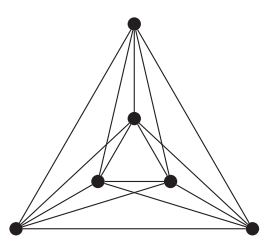

FIG. 1. *Drawing of K_6 with 3 crossings.*

Proof. Let A be an embedding of K_6 using 3 crossings. Let us show that it is unique. First we observe that every edge is crossed at most once. Otherwise, there will be two edges whose removal leaves the graph planar which is a contradiction to Proposition 2.1. As every cluster of a crossing contains four vertices, there must be a

vertex v contained in two of them. Note that v cannot be in all three clusters since $K_6 - v$ (which is isomorphic to K_5) is not planar. Let $e_1 = vx$ and $e_2 = vy$ be the two crossed edges adjacent to v, and let e_3 be one of the edges of the crossing whose cluster does not contain v. The graph $K_6 \setminus \{e_1, e_2, e_3\}$ is a planar triangulation T where $\deg(v) = 3$.

We denote a, b, c the neighbors of v in T. They must induce a triangle. Without loss of generality, ab and bc are the edges crossed by e_1 and e_2, respectively.

As T is a triangulation abx and bcy form triangles. Moreover, xby is also a triangle as x and y are consecutive neighbors around b. The last two edges, which are not discussed yet, are xc and ya. They must cross inside $bxyc$ (one of them is e_3). Hence A is unique. ∎

LEMMA 2.5. *A drawing of K_5 with all vertices incident to the same face requires 5 crossings.*

Proof. Let us number the vertices of K_5 v_1, v_2, v_3, v_4, v_5 in the clockwise order around the boundary of the face f incident to them. Then free to redraw the edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, and v_5v_1, we may assume that the boundary is the cycle $v_1v_2v_3v_4v_5$ and that f is its interior. Now both v_1v_3 and v_2v_4 are in the exterior of C and thus must cross. Similarly, $\{v_2v_4, v_3v_5\}$, $\{v_3v_5, v_4v_1\}$, $\{v_4v_1, v_5v_2\}$, and $\{v_5v_2, v_1v_3\}$ are crossings. ∎

LEMMA 2.6. *A drawing of $K_{2,3}$ such that vertices of each part are in a common face requires at least one crossing.*

Proof. Let $(\{u_1, u_2\}, \{v_1, v_2, v_3\})$ be the bipartition of $K_{2,3}$. Suppose by contradiction that $K_{2,3}$ has a drawing such that each part of the bipartition is in a common face. Then adding a vertex u_3 is the face incident to the vertices v_1, v_2, and v_3, and connecting u_3 to those vertices by new edges yields a drawing of $K_{3,3}$ with no crossing which contradicts the fact that $K_{3,3}$ is not planar. ∎

2.2. Properties of 6-critical graphs. A *stable crossing cover* of a drawing of a graph is a set of vertices that is both stable and a crossing cover.

LEMMA 2.7. *Every graph with a stable crossing cover is 5-colorable.*

Proof. Let G be a graph having a stable crossing cover W. Use the Four Color Theorem on $G - W$ and extend the coloring to G by using a fifth color on W. ∎

Let G be a graph and u, v be vertices of G. The operation of *identification* of u and v in G results in a graph denoted by $G/\{u, v\}$, which is obtained from $G - \{u, v\}$ by adding a new vertex w and the set of edges $\{wz \mid uz \text{ or } vz \text{ is an edge of } G\}$.

LEMMA 2.8. *Let G be a graph and v be a degree 5 vertex of G. Let u and w be two nonadjacent neighbors of v. If $(G - v)/\{u, w\}$ is 5-colorable, then so is G.*

Proof. A proper 5-coloring of $(G - v)/\{u, w\}$ corresponds to a proper 5-coloring of $G - v$ such that u and w are colored by the same color. So it can be extended to a proper 5-coloring of G by assigning a color to v. ∎

Let G be a graph and a drawing of it in the plane. A cycle is *separating* if it has a vertex in its interior and a vertex in its exterior. A cycle C is *noncrossed* if all its edges are noncrossed. It is *regular* if any cluster of a crossing containing an edge of C contains at least three vertices of C.

LEMMA 2.9. *In every drawing in the plane of a 6-critical graph, there is no separating regular triangle.*

Proof. Let G be a 6-critical graph drawn in the plane. Suppose, by way of contradiction, that there is a regular triangle C. Let G_1 be the graph induced by the vertices in C and inside C, and let G_2 be a graph induced by the vertices in C and outside C. Since C is separating, both G_1 and G_2 have fewer vertices than G. Hence, by

6-criticality of G, they are 5-colorings of those graphs. In addition, in these colorings of G_1 and G_2, the colors of the vertices of C are distinct. So, free to permute the colors, one can assume that the two 5-colorings of G_1 and G_2 agree on C. Hence their union yields a 5-coloring of G. ∎

LEMMA 2.10. *Let G be a 6-critical graph distinct from K_6. In every nice drawing of G, there is no separating triangle such that*

- *at most one of its edges is crossed, and*
- *there is at most one crossing in its interior.*

Proof. Suppose, by way of contradiction, that such a cycle $C = x_1x_2x_3$ exists. Then by Lemma 2.9, one of its edges, say x_2x_3, is crossed. Let uv be the edge crossing it with u inside C and v outside. By Lemma 2.9, C is not regular, so $u \neq x_1$. Moreover, $u \notin \{x_2, x_3\}$ since the drawing is nice.

Let G_1 be the graph induced by C and the vertices outside C. Then G_1 admits a 5-coloring c_1 since G is 6-critical.

Let G_2 be the graph obtained from the graph induced by C and the vertices inside C by adding the edges ux_1, ux_2, and ux_3 if they do not exist. Observe that G_2 has a planar drawing with at most 2 crossings. Indeed the edge ux_1 may be drawn along uv and then a path in the outside of C and the edges ux_2 and ux_3 may be drawn along the edges of the crossing $\{x_2x_3, uv\}$. Thus G_2 admits a 5-coloring c_2.

In both colorings, the colors of the vertices of C are distinct. So, free to permute the colors, we may assume that c_1 and c_2 agree on C. One can also choose for u a color of $\{1, \ldots, 5\} \setminus \{c_2(x_1), c_2(x_2), c_2(x_3)\}$ so that $c_2(u) \neq c_1(v)$. Then the union of c_1 and c_2 is a 5-coloring of G. ∎

LEMMA 2.11. *Let G be a 6-critical graph. In every drawing of G in the plane, there is no noncrossed 4-cycle C such that*

- *C has a chord in its exterior,*
- *C and its interior is a plane graph, and*
- *the interior of C contains at least one vertex.*

Proof. Suppose, by way of contradiction, that there is a 4-cycle $C = tuvw$ satisfying the properties above with vt a chord in its exterior. Consider the graph G_1, which is obtained from G by removing the vertices inside C. Since G is 6-critical, G_1 admits a 5-coloring c_1 in $\{1, 2, 3, 4, 5\}$. Without loss of generality, we may assume that $c_1(v) = 5$. Hence $\{c_1(t), c_1(u), c_1(w)\} \subset \{1, 2, 3, 4\}$.

Now consider the graph G_2 which is obtained from G by removing the vertices outside C. If $c_1(u) = c_1(w)$, let H be the graph obtained from $G_2 - v$ by identifying u and w. If $c_1(u) \neq c_1(w)$, let H be the graph obtained from $G_2 - v$ by adding the edge uw if it does not already exist. In both cases H is a planar graph. Hence H admits a 4-coloring c_2 in $\{1, 2, 3, 4\}$. Moreover, by construction of H, $c_2(u) = c_2(w)$ if and only if $c_1(u) = c_1(w)$. Hence free to permute the colors, we may assume that c_1 and c_2 agree on $\{t, u, w\}$.

Hence the union of c_1 and c_2 is a 5-coloring of G. ∎

2.3. 6-critical graphs embeddable on the torus or the Klein bottle. In the proof of Theorem 1.6, we use the list of all 6-critical graphs embeddable on the torus, which was obtained by Thomassen [17], and the list of all 6-critical graphs embeddable on the Klein bottle, which was obtained independently by Chenette et al. [4] and Kawarabayashi et al. [11].

THEOREM 2.12 (Thomassen [17]). *There are four nonisomorphic 6-critical graphs embeddable on the torus. Three of them are depicted in Figure 2 and the last one is a 6-regular graph on 11 vertices.*

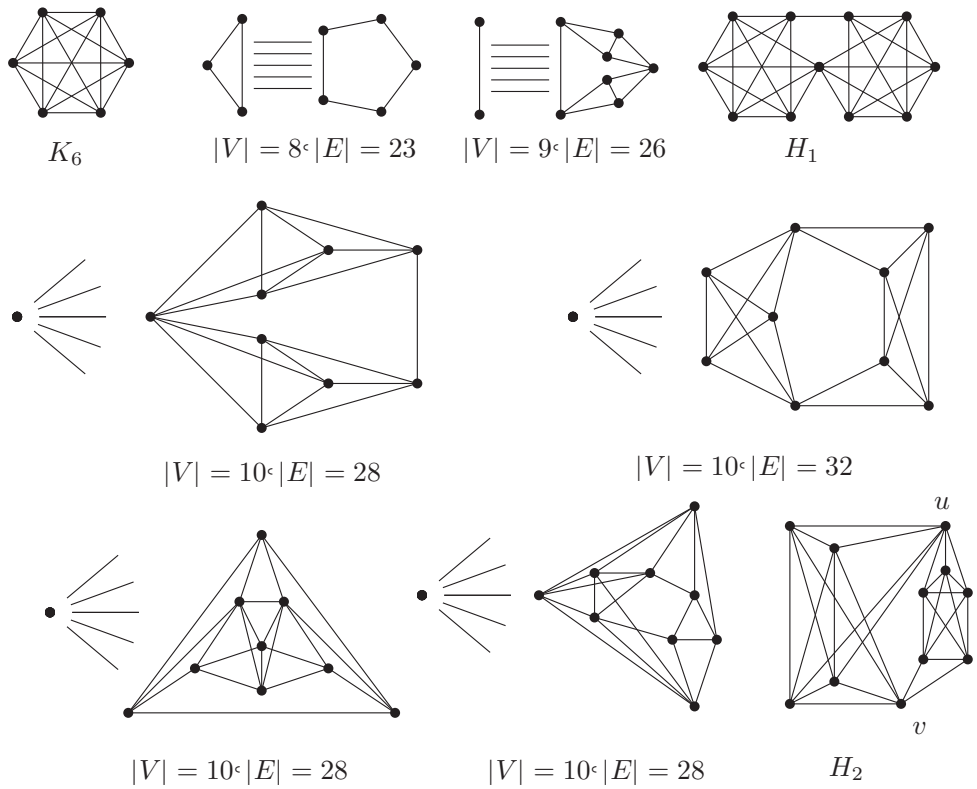

FIG. 2. *The list of* 6*-critical graphs embeddable on the Klein bottle. The first three of them are embeddable on torus as well.*

THEOREM 2.13 (Chenette et al. [4]; Kawarabayashi et al. [11]). *There are nine nonisomorphic 6-critical graphs embeddable on the Klein bottle. They are depicted in Figure* 2.

LEMMA 2.14. *If three edges are deleted from a* 6*-critical graph embeddable on the torus other than* K_6*, then the resulting graph is nonplanar.*

Proof. We know the complete list of graphs which must be checked due to Theorem 2.12. For all of them except K_6, we have $|E| > 3|V| - 3$. Thus the graphs are not planar after removing three edges according to Proposition 2.1. ∎

LEMMA 2.15. *If three edges are deleted from a* 6*-critical graph embeddable on the Klein bottle other than* K_6*, then the resulting graph is nonplanar.*

Proof. We know the complete list of graphs which must be checked due to Theorem 2.13. For all of those graphs except K_6, H_1, and H_2, we have $|E| > 3|V|-3$. Thus those graphs are not planar after removing three edges according to Proposition 2.1.

Now we need to deal with the last two graphs H_1 and H_2; see Figure 2. Let us first examine H_1. It contains two edge disjoint copies of K_6 without one edge. Each of these copies needs at least two edges to be removed by Proposition 2.1, so H_1 needs at least four edges to be removed.

Let us now examine H_2. Let F be a set of edges such that $H_2 \setminus F$ is planar. Let us denote by u and v the two vertices of the only 2-cut of H_2; see Figure 2. Observe that $H_2 - \{u, v\}$ is a disjoint union of K_5 and K_4. Since K_5 is not planar, one edge

e of this K_5 is in F. But there is still a (u, v)-path P in $K_5 \setminus e$. Then the union of the graph induced by u, v, the vertices of the K_4, and the path P is a subdivision of K_6. Thus, by Proposition 2.1 for K_6, at least three of its edges must be in F. Thus $|F| \geq 4$. ∎

3. 6-critical graph of crossing number 5. We prove Theorem 1.3 by exhibiting a drawing of a 6-critical graph G using 5 crossings which is not K_6.

THEOREM 3.1. *The graph G depicted in Figure 3 is 6-critical.*

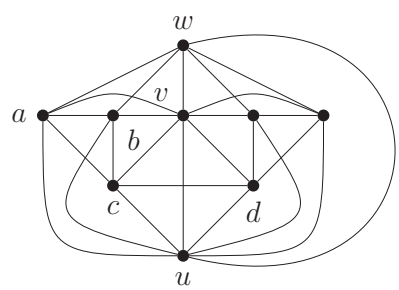

FIG. 3. *A 6-critical graph of crossing number 5.*

Proof. We show by contradiction that G is not 5-colorable. We refer the reader to Figure 3 for names of vertices. Assume that ϱ is a 5-coloring of G. As vertices u, v, and w form a triangle, they must get distinct colors. Without loss of generality, assume that $\varrho(u) = 1, \varrho(v) = 2$, and $\varrho(w) = 3$. The vertices a and b are adjacent to each other and to all the vertices of the triangle uvw; hence $\{\varrho(a), \varrho(b)\} = \{4, 5\}$. Thus $\varrho(c) = 3$ as c is adjacent to a, b, u, and v. By symmetry we obtain that $\varrho(d)$ is also 3, which is a contradiction since cd is an edge.

It can be easily checked that every proper subgraph of G is 5-colorable. So G is 6-critical. ∎

4. Coloring graphs whose crossings are covered by few edges.

THEOREM 4.1. *If a graph can be made planar by deleting a set of at most $2k$ edges, then G is $(4 + k)$-colorable.*

Proof. We proceed by induction on k. The result holds when $k = 0$ by the Four Color Theorem.

Suppose that the result is true for k. Let $G = (V, E)$ be a graph with a set F of at most $2k + 2$ edges such that $G \setminus F$ is planar. Without loss of generality, we may assume that F is minimal, i.e., for any proper subset $F' \subset F$, the graph $G \setminus F'$ is not planar.

Consider a planar drawing of $G \setminus F$. It yields a drawing of G such that each crossing contains an edge of F.

Suppose that $|F| \leq 2k + 1$. Let $e = uv$ be an edge of F. By the induction hypothesis, $G - v$ is $(4+k)$-colorable because $F \setminus e$ is a set of $2k$ edges whose removal leaves $G - v$ planar. Hence $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$.

So we may assume that $|F| = 2k + 2$.

If two edges e and f of F have a common vertex v, then $G - v$ is $(4 + k)$-colorable because $F \setminus \{e, f\}$ is a set of $2k$ edges whose removal leaves $G - v$ planar. So $\chi(G) \leq \chi(G - v) + 1 \leq 4 + k + 1$. So we may assume that the edges of F are pairwise nonadjacent.

Let $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$ be two edges in F. Then the endvertices of these two edges induce a K_4. Suppose for contradiction that u_1 and v_1 are not adjacent. Then $G - \{u_1, v_1\}$ is $(4+k)$-colorable because $F \setminus \{e, f\}$ is a set of $2k$ edges whose removal leaves $G - \{u_1, v_1\}$ planar, and u_1 and v_1 can get the same color. So $\chi(G) \leq \chi(G - \{u_1, v_1\}) + 1 \leq 4 + k + 1$. Hence $X = \{u_1, u_2, v_1, v_2\}$ induces a K_4.

We further distinguish two possible cases:

$k = 0$: Let the edges of F be $e = \{u_1, u_2\}$ and $f = \{v_1, v_2\}$, and let $X = \{u_1, u_2, v_1, v_2\}$. Let C be the 4-cycle induced by X in the plane graph $G \setminus \{e, f\}$. Note that C is a separating cycle; otherwise $G \setminus e$ would be planar. We cut G along C and obtain two smaller graphs G_1 and G_2, where both of them contain X. We 5-color them by induction. A coloring of G can be then obtained from the 5-colorings of G_1 and G_2 by permuting colors on X so that these two colorings agree on $V(C)$.

$k \geq 1$: Note that union of all endvertices of edges from F induce a complete graph $K_{2|F|}$. A $K_{2|F|}$ must be planar after removing at most $|F|$ edges. Hence the following Euler's formula holds:

$$\begin{aligned} |E| &\leq 3|V| - 6 + 2k + 2 \\ \binom{4k+4}{2} &\leq 3(4k+4) + 2k - 4 \\ 8k^2 - 2 &\leq 0. \end{aligned}$$

Hence this case is not possible. ∎

Since $\mathrm{cr}(K_5) = 1$ and $\mathrm{cr}(K_6) = 3$, Theorem 4.1 is tight when $k \leq 2$. But K_6 is the only obstacle for pushing the result further as shown by the following theorem which is a reformulation of Theorem 1.6 in terms of critical graphs.

THEOREM 4.2. *If at most three edges are deleted from a 6-critical graph other than K_6, then the resulting graph is nonplanar.*

Proof. Let G be a 6-critical graph distinct from K_6 and F a set of at most three edges. Assume for a contradiction that $G \setminus F$ is planar.

Let us consider a nice drawing of G. By Lemma 2.7, G has no stable crossing cover.

If $|F| \leq 2$, then Theorem 4.1 contradicts the fact that G is not 5-colorable. Hence we assume that $F = \{e_1, e_2, e_3\}$. Set $e_i = u_i v_i$ for $i \in \{1, 2, 3\}$.

CLAIM 1. *The three edges of F are pairwise vertex-disjoint.*

Proof. If there is a vertex v shared by all three edges, then $\{v\}$ is a stable crossing cover, a contradiction. Hence a vertex u is shared by at most two edges of F. Let s be the number of degree 2 vertices in the graph induced by F.

We now derive a contradiction for each value of $s > 0$. So $s = 0$, which proves the claim.

s = 1: Without loss of generality, $u = u_1 = u_2$. None of $\{u, u_3\}$ and $\{u, v_3\}$ is a stable crossing cover so uu_3 and uv_3 are edges. We redraw the edge e_3 along the path u_3uv_3 such that it crosses only edges incident to u. See Figure 4(a). Then u is a stable crossing cover, a contradiction.

s = 2: Without loss of generality, $u = u_1 = u_2$ and $v = v_2 = v_3$. Then F induces a path. None of $\{v_1, v\}$ and $\{u, u_3\}$ is a stable crossing cover, so v_1v and uu_3 are edges. We add a handle between vertices u and v. Then we draw edges of F using the handle; see Figure 4(b). Hence G can be embedded on the torus, which is a contradiction to Lemma 2.14.

s = 3: Without loss of generality, $u = u_1 = u_2$ is one of the shared vertices. Let v and w be the other two. Note that F induces a triangle. By Proposition 2.1, we have $|E(G)| \leq 3|V(G)| - 3$. Hence there must be at least six vertices of degree 5 as the minimum degree of G is 5.

Let x be a degree 5 vertex different from u, v, and w. By minimality of G, there exists a 5-coloring ϱ of $G - x$. Free to permute the colors, we may assume that $\varrho(u) = 1$, $\varrho(v) = 2$, and $\varrho(w) = 3$. Moreover, the neighbors of x are all colored differently. We denote by y and z the neighbors of x, which are colored 4 and 5, respectively. We assume that G is embedded in the plane such that all crossings are covered by F. There are two consecutive neighbors of x in the clockwise order such that they have colors in $\{1, 2, 3\}$. We denote these vertices by a and b. Without loss of generality let the clockwise order around x be z, y, a, b and $\varrho(a) = 1$ and $\varrho(b) = 2$. See Figure 4(c).

Let A be the connected component of a in the graph induced by the vertices colored 1 and 5. If A does not contain z, we can switch colors on it. Then x can be colored by 1, and we have a contradiction. Note that the color switch is correct even if u is in A because the new color of u will be 5 which is different from 2 and 3. Thus there must be a path between a and z of vertices colored 1 and 5. A similar argument shows that there is a path between b and y of vertices colored 2 and 4. These paths must be disjoint and they are not using edges of F. But they cannot be drawn in the plane without crossings, a contradiction. ∎

CLAIM 2. *For any distinct integers $i, j \in \{1, 2, 3\}$, an endvertex of e_i is adjacent to at most one endvertex of e_j.*

Proof. Suppose not. Then without loss of generality, we may assume that u_2 is adjacent to u_1 and v_1. First we redraw the edge e_1 along the path $u_1u_2v_1$. Then every edge crossed by e_1, which is not e_3, is incident to e_2. Since $\{u_2, u_3\}$ and $\{u_2, v_3\}$ are not stable crossing covers, u_2u_3 and u_2v_3 are edges. We redraw e_3 along the path $u_3u_2v_3$. Then, again, every edge crossed by e_3, which is not e_1, is incident to e_2. Moreover, the edges e_1 and e_3 cross; otherwise $\{u_2\}$ would be a stable crossing cover. See Figure 4(d).

We distinguish several cases according to the number p of neighbors of v_2 among u_1, v_1, u_3, and v_3.

$p = 0$: The vertex v_2 and a pair of two nonadjacent vertices among u_1, v_1, u_3, and v_3 would form a stable crossing cover. Hence $\{u_1, v_1, u_3, v_3\}$ induces a K_4. See Figure 4(e). By Lemma 2.9, there is no vertex inside each of the triangles $u_2u_1u_3$, $u_2u_3v_1$, $u_2v_1v_3$, and $u_2u_1v_3$. Hence all the vertices are inside the 4-cycle $u_1u_3v_1v_3$. It includes the vertex v_2. We redraw e_1 such that it is crossing only e_3 and u_2v_3. Then $\{v_3, v_2\}$ is a stable crossing cover, a contradiction. See Figure 4(f).

$p = 1$: Without loss of generality we may assume that the neighbor of v_2 is u_1. None of $\{v_2, v_1, u_3\}$ and $\{v_2, v_1, v_3\}$ is a stable crossing cover so u_3v_1 and v_1v_3 are edges. By Lemma 2.9, there is no vertex inside each of the triangles $u_2u_3v_1$ and $u_2v_1v_3$. See Figure 4(g). Thus the edge e_3 could be drawn inside these triangles, and the set F can be changed to $F' = \{e_1, e_2, u_2v_1\}$. Two edges of F' share an endvertex which is a contradiction to Claim 1.

$p \in \{2, 3\}$: We further distinguish two subcases. Either two neighbors of v_2 in $\{u_1, v_1, u_3, v_3\}$ are joined by an edge of F or not.

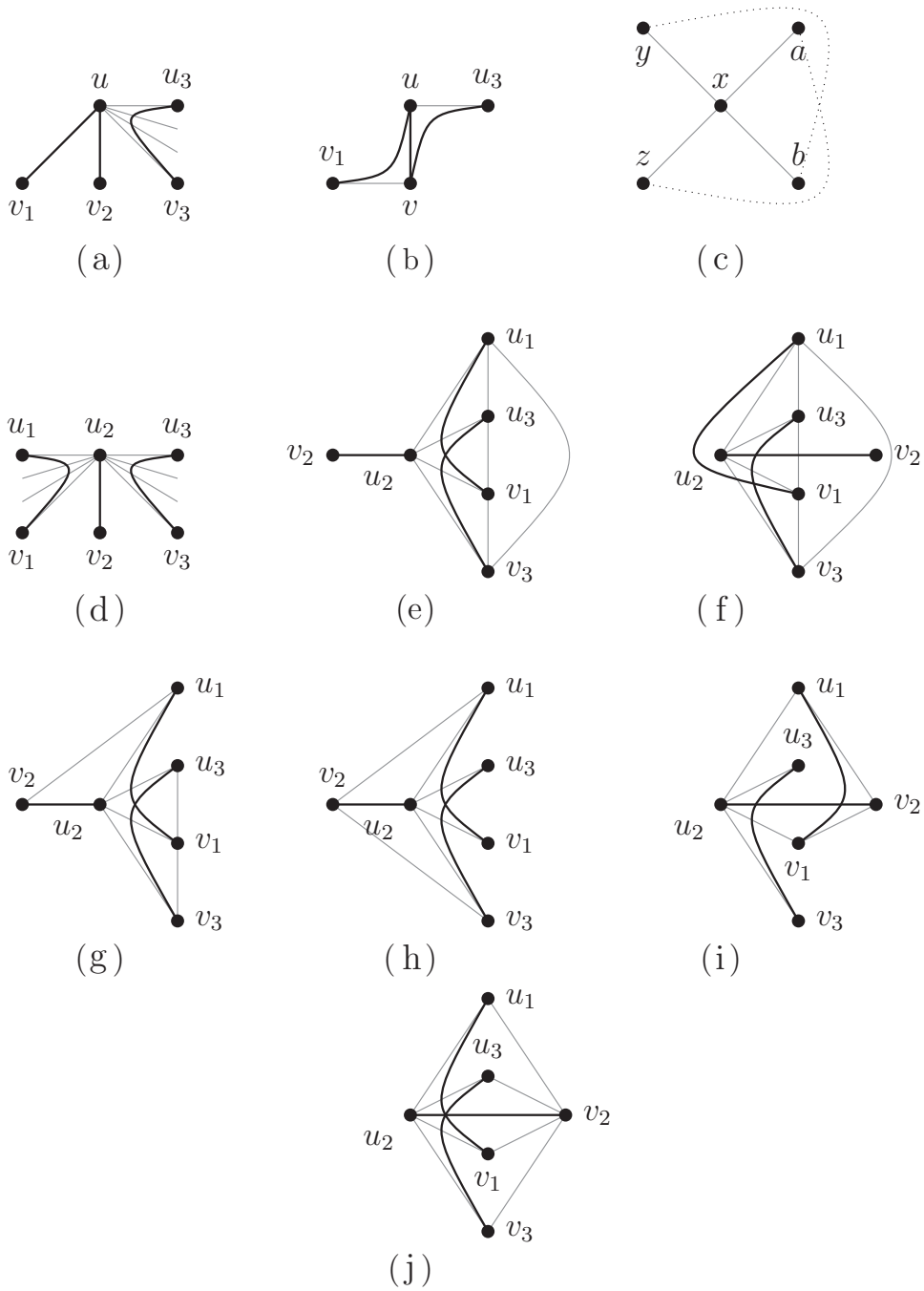

FIG. 4. *The three black edges are covering all the crossings.*

In the second case, without loss of generality, we may assume that the vertices adjacent to v_2 are u_1 and v_3. Now by Lemma 2.11 there is no vertex inside the 4-cycle $v_2u_1u_2v_3$. Hence e_2 can be drawn inside this cycle. See Figure 4(h). Since the removal of $\{e_1, e_3\}$ does not make

G planar, v_1v_3 is inside $v_2u_1u_2v_3$. Hence the set $F' = \{e_1, e_3, u_1v_3\}$ contradicts Claim 1.

In the first case, we may assume, without loss of generality, that v_2 is adjacent to u_1 and v_1. We first redraw e_1 along the path $u_1v_2v_1$. Now all the edges crossing e_1 are incident to v_2. Thus $\{v_2, u_3\}$ or $\{v_2, v_3\}$ form a stable crossing cover. See Figure 4(i).

$p = 4$: See Figure 4(j). We repeatedly use Lemma 2.11 which implies that the 4-cycles $u_2u_3v_2u_1$, $u_2u_3v_2v_1$, $u_2v_1v_2v_3$, and $u_2v_3v_2u_1$ are not separating. This means that the graph contains only six vertices. This is a contradiction because the unique 6-critical graph on six vertices is K_6. ∎

Since $\{u_1, u_2, u_3\}$ is not a stable crossing cover, it must induce at least one edge, say u_1u_2. Then Claim 2 implies that u_1v_2 and v_1u_2 are not edges. Now $\{v_1, u_2, u_3\}$ and $\{v_1, u_2, v_3\}$ are not stable crossing covers. Thus, by symmetry, we may assume that u_2u_3 and v_1v_3 are edges. $\{u_1, v_2, u_3\}$ is not a stable crossing cover so u_1u_3 is an edge; $\{v_1, v_2, u_3\}$ is not a stable crossing cover so v_1v_2 is an edge; $\{u_1, v_2, v_3\}$ is not a stable crossing cover so v_2v_3 is an edge. Hence there are two triangles $u_1u_2u_3$ and $v_1v_2v_3$, which are not separating by Lemma 2.9.

Without loss of generality, two possibilities occur. Either the edges of F do not cross each other or one pair of them is crossing. If they do not cross (Figure 5(a)), G can be embedded on the torus by adding a handle into the triangles and drawing the edges of F on the handle, which contradicts Lemma 2.14.

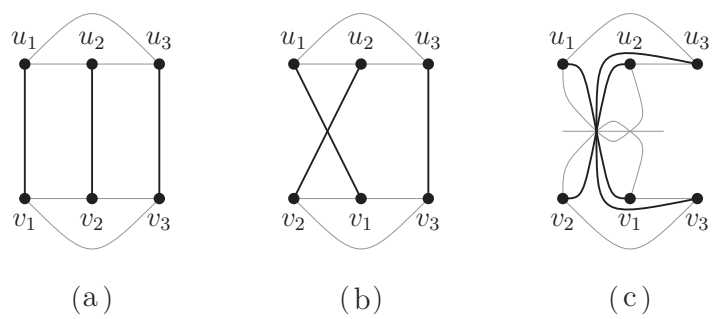

FIG. 5. *The last case of Theorem* 1.6.

If they cross (Figure 5(b)), it is possible to draw G on the Klein bottle; see Figure 5(c), which contradicts Lemma 2.15. ∎

5. 5-coloring graphs with 4 crossings. In this section we prove the following Theorem 5.1, which is a reformulation of Theorem 1.4 in terms of critical graphs.

THEOREM 5.1. *The unique* 6*-critical graph with crossing number at most* 4 *is* K_6.

Proof. Suppose, by way of contradiction, that $G = (V, E)$ is a 6-critical graph with crossing number at most 4 distinct from K_6. Moreover, one may assume that G is such a critical graph with the minimum number of vertices and with the maximum number of edges on $|V(G)|$ vertices.

Moreover, assume that we have a nice optimal drawing of G. By Theorem 1.6, there are 4 crossings and every edge is crossed at most once.

Since G is 6-critical, every vertex has degree of at least 5. By Proposition 2.1, we have $|E| \leq 3|V| - 6 + \mathrm{cr}(G) \leq 3|V| - 2$. Hence there are at least four vertices of degree 5.

Let v be an arbitrary degree 5 vertex and v_i, $1 \leq i \leq 5$ be the neighbors of v in the counterclockwise order around v. By criticality of G, the graph $G - v$ admits a 5-coloring ϕ. Necessarily, all the v_i are colored differently; otherwise ϕ could be extended to v.

For any $i \leq j$, there is a path, denoted by $v_i - v_j$, from v_i to v_j such that all its vertices are colored in $\phi(v_i)$ or $\phi(v_j)$. Otherwise, v_j is not in the connected component A of v_i in the graph induced by the vertices colored $\phi(v_i)$ and $\phi(v_j)$. Hence by exchanging the colors $\phi(v_i)$ and $\phi(v_j)$ on A, we obtain a 5-coloring ϕ' of $G - v$ such that no neighbor of v is colored $\phi(v_i)$. Hence by assigning $\phi(v_i)$ to v we obtain a 5-coloring of G, a contradiction.

Let q be the number of crossed edges incident to v.

Claim 3. $q \neq 0$.

Proof. The union of the $v_i - v_j$, for $i \neq j$, is a subdivision of K_5 in $G - v$. If $q = 0$, then the v_i, $1 \leq i \leq 5$, are in one face after the removal of v. By Lemma 2.5, such a subdivision requires 5 crossings which contradicts the assumption of at most 4 crossings. ∎

Claim 4. $q \neq 1$.

Proof. Suppose to the opposite that $q = 1$. Without loss of generality, we may assume that the crossed edge is vv_1.

The path $v_2 - v_4$ must cross the two paths $v_1 - v_3$ and $v_3 - v_5$. Since every edge is crossed at most once, then v_2v_4 is not an edge.

Let G' be the graph obtained from $G - v$ by identifying v_2 and v_4 into a new vertex v'. By Lemma 2.8, G' is not 5-colorable. Now G' has at most 3 crossings because we removed the crossed edge vv_1 together with v. So, by minimality of G, the graph G' contains a subgraph H isomorphic to K_6. Moreover, H must contain v' since G contains no K_6. Since G' has only 3 crossings we can use Lemma 2.4. Let u_1 and u_2 be vertices of H which form a triangular face together with v', and let u_3, u_4, and u_5 be the vertices forming the other triangular face. Without loss of generality, we may assume that $u_3u_4u_5$ is inside $v'u_1u_2$ as in Figure 6(a).

Let us now consider the situation in G. Instead of discussing many rotations of K_6, we rather fix K_6 and try to investigate possible placings of v and its neighbors. We denote the neighbors of v which were identified by x and y (i.e., $\{v_2, v_4\} = \{x, y\}$). Let a and b be the two other neighbors of v such that va and vb are not crossed ($\{a, b\} = \{v_3, v_5\}$). Moreover, we assume that, in the counterclockwise order around v, the sequence is x, a, y, b. Note that the vertex v_1 may be inserted anywhere in the sequence.

One of the identified vertices, say x, is adjacent to at least two vertices of $\{u_3, u_4, u_5\}$.

(1) Assume first that x is adjacent to u_3,u_4, and u_5. Then since G has no K_6, it is not adjacent to some vertex in $\{u_1, u_2\}$, say u_2. Thus $yu_2 \in E$.

The vertex a is either inside u_2yvx or is u_2. See Figure 6(b) and (c), respectively. The path $a - b$ (represented by the dotted line in the figure) necessarily uses u_2. Since colors $\phi(a)$ and $\phi(b)$ alternate on $a - b$, this path cannot contain x nor u_3, u_4, or u_5. The paths $a - b$ and avb separate x and y, and there must be paths $v_1 - x$ and $v_1 - y$. Thus at least one of them

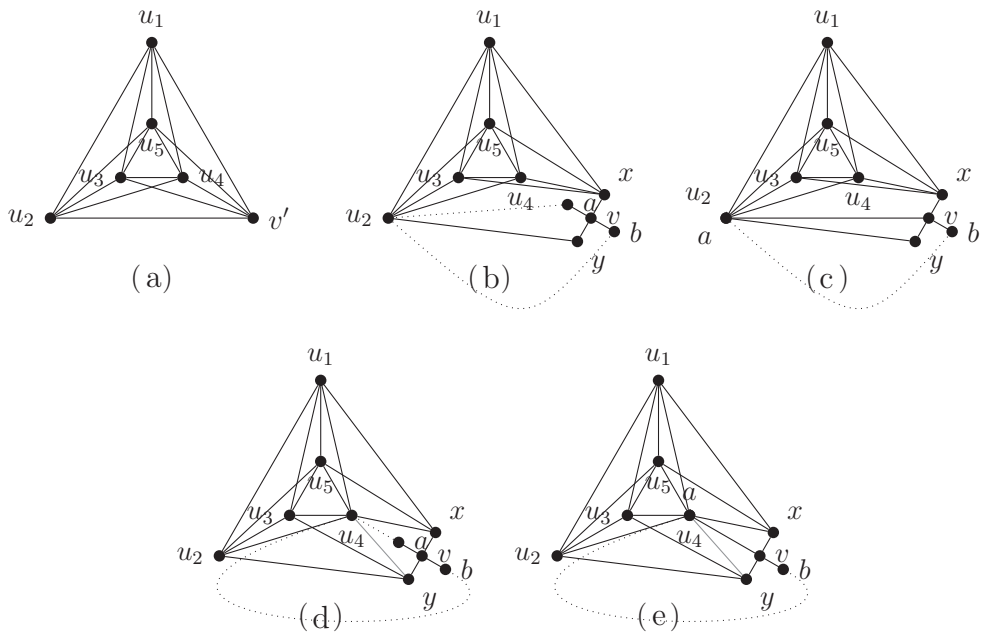

FIG. 6. K_6 *when identifying two neighbours of* v.

must cross the path $a-b$. But none of the 4 crossings is available for that, a contradiction.

(2) Let us now assume that x is adjacent to only two vertices of $\{u_3, u_4, u_5\}$, say u_4 and u_5. Then u_3 is adjacent to y. (Possibly u_4 and y are adjacent too.) The path $a-b$ must go through u_4 and then continue to u_1 or u_2. It cannot go through u_3 or u_5 since the colors on the path alternate. See Figure 6(d) and (e).

The path $x-y$ must cross $a-b$. Hence either $x-y$ goes through u_3y and $a-b$ through u_4u_2, or $x-y$ goes through xu_5 and $a-b$ through u_4u_1. In both cases, one of the paths v_1-x and v_1-y must cross $a-b$. But there are no more crossings available.

This completes the proof of Claim 4. ∎

CLAIM 5. $q \neq 2$.

Proof. Suppose to the opposite that $q = 2$.

We first prove the following assertion that will be used several times.

Assertion. *Let x and y be two neighbors of v. Then x and y are adjacent if one of the following holds:*

- *vx and vy are not crossed;*
- *$\{x, y\}$ is included in the cluster of some crossing.*

Observe that $G-v$ has at most 2 crossings. Suppose that x and y are not adjacent. If vx and vy are not crossed, we can identify x and y along xvy without adding any new crossing. If $\{x, y\}$ is included in the cluster of some crossing, we can identify x and y along the edges of this crossing without adding any new crossing. Hence in both cases $(G-v)/\{x, y\}$ has a planar drawing with at most 2 crossings. Then Lemma 2.8 and Theorem 4.1 yield a contradiction. This proves the Assertion.

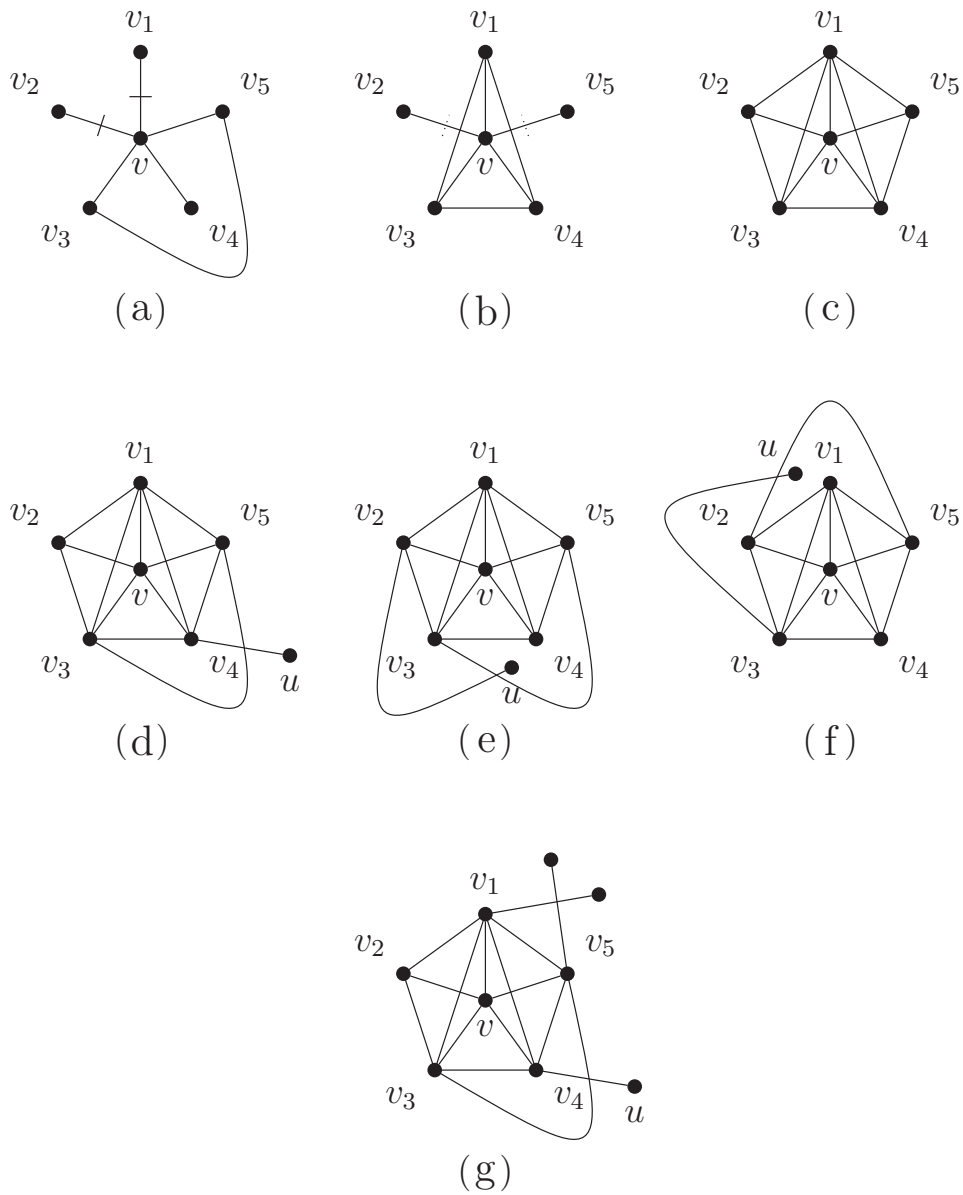

FIG. 7. *Two crossed edges.*

Assume that the crossed edges are consecutive, say vv_1 and vv_2. By the Assertion, v_3v_5 is an edge. See Figure 7(a). If v_3v_5 is not crossed or crosses either vv_1 and vv_2, then the cycle vv_3v_5 is regular, which contradicts Lemma 2.9. If v_3v_5 is crossed by another edge, then the cycle vv_3v_5 contradicts Lemma 2.10. Henceforth, we may assume that the two crossed edges are not consecutive, say vv_2 and vv_5.

By the Assertion, v_1v_3, v_1v_4, and v_3v_4 are edges. If v_1v_3 is not crossed, then the triangle vv_1v_3 is separating because v_2 and v_4 are on the opposite sides. This contradicts Lemma 2.9. If v_1v_3 is crossed, it can be redrawn along the path v_1vv_3

with 1 crossing with vv_2. Symmetrically, we assume that v_1v_4 is crossing vv_5. See Figure 7(b).

By the Assertion, $\{v_1v_2, v_2v_3, v_4v_5, v_5v_1\} \subset E(G)$. See Figure 7(c).

Let $C = \{c_1c_2, c_3c_4\}$ and $D = \{d_1d_2, d_3d_4\}$ be the 2 crossings not having v in their cluster. For convenience and with a slight abuse of notation, we denote by C (resp. D) both the crossing C (resp. D) and its cluster. For $X \in \{C, D\}$, let $a(X) := |X \cap N(v)|$. Without loss of generality, we may assume that $a(C) \leq a(D)$.

A vertex u is a *candidate* if it is not adjacent to v. There is no candidate u common to both C and D; otherwise $\{u, v\}$ would be a stable crossing cover. There are no nonadjacent candidate vertices $c \in C$ and $d \in D$; otherwise $\{v, c, d\}$ would be a stable crossing cover.

Assume that $a(D) = 4$. The vertex v_1 cannot be in D because it is already adjacent to all the other neighbors of v by edges not in D. Thus $D = \{v_2, v_3, v_4, v_5\}$. But then, by the Assertion, v_2v_5 is an edge. So $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence $a(C) \leq a(D) \leq 3$.

Suppose now that $X \in \{C, D\}$ does not induce a K_4. Then two vertices x_1 and x_2 of X are not adjacent. One can add the edge x_1x_2 and draw it along the edges of the crossing such that no new crossing is created. Hence by the choice of G, the obtained graph $G \cup x_1x_2$ contains a K_6. Since K_6 has crossing number 3, one of the crossings containing v in its cluster must be used. So v belongs to the K_6, and hence the K_6 is induced by $\{v\} \cup N(v)$. In such case edges v_2v_4 and v_3v_5 cross and hence form C or D, which is not possible since $a(C) \leq a(D) \leq 3$.

Hence both C and D induce a K_4. Thus the candidates in $C \cup D$ induce a complete graph. So there are at most five of them. Since $C \cap D$ contains no candidate, we have $a(C) + a(D) \geq 3$ and so $2 \leq a(D) \leq 3$.

Assume that $a(D) = 2$ and thus $1 \leq a(C) \leq 2$. Then C (resp. D) contains a set C' (resp. D') of two candidates. All the vertices of C' are adjacent to all the vertices of D'. But since both C and D contain a vertex in $N(v)$, drawing all the edges between these two sets requires one more crossing, a contradiction. Hence $a(D) = 3$.

Thus, an edge of D has its two endvertices in $N(v)$ and so it is v_2v_5, v_2v_4, or v_3v_5. Let u be the unique candidate of D.

Assume first that $v_1 \in D$. Then v_1u is an edge of D. Moreover, C must be on the paths $v_2 - v_4$ and $v_3 - v_5$. Since edges are crossed at most once $D = \{v_1u, v_2v_5\}$. Let w be a candidate vertex in C. Then w is outside the cycle vv_2v_5. But the only neighbor of v_1 outside this cycle is u which is distinct from w because the crossings C and D have no candidate vertex in common. Thus $\{w, v_1\}$ is a stable crossing cover, a contradiction to Lemma 2.7. So $v_1 \notin D$.

By symmetry, we may assume that D is either $\{v_3v_5, v_4u\}$ (Figure 7(d)) or $\{v_3v_5, v_2u\}$ (Figure 7(e)) or $\{v_2v_5, v_3u\}$ (Figure 7(f)). In the second and third cases, Lemma 2.10 is contradicted by the cycle $v_3v_4v_5$ and $v_1v_2v_5$, respectively. Hence $D = \{v_3v_5, v_4u\}$.

The set $\{v_2, v_4\}$ is stable and covers the 3 crossings distinct from C. Hence $\{v_2, v_4\}$ does not intersect C; otherwise it would be a stable crossing cover. So $C \cap N(v) \subset \{v_1, v_3, v_5\}$. The edge v_1v_5 is not crossed; otherwise it could be redrawn along the edges of the crossing $\{vv_5, v_1v_4\}$ to obtain a drawing of G with fewer crossings. Furthermore, v_1v_3 and v_3v_5 are not in C because they are in some other crossing. Hence $a(C) \leq 2$.

Let B be the set of candidates of C. Recall that all vertices of B are adjacent to u. Moreover, every vertex $b \in B$ is adjacent to a vertex of $\{v_2, v_4\}$; otherwise $\{v_2, v_4, b\}$ is a stable crossing cover. But v_4 and u are separated by $v_3v_4v_5$, so all vertices of B

are adjacent to v_2. Now the graph induced by the edges between B and $\{u, v_2\}$ is a complete bipartite graph. Moreover, its induced drawing has no crossing, and the vertices of each part are in a common face. Thus, by Lemma 2.6, $|B| \leq 2$. So $a(C) = 2$.

Recall that $C \cap N(v) \subset \{v_1, v_3, v_5\}$. Suppose that $C \cap N(v) = \{v_1, v_3\}$. The closed curve formed by the path $v_3 v v_1$ and the two "half-edges" connecting v_1 to v_3 in C separates v_2 and u. Then the vertices of B cannot be adjacent to both u and v_2, a contradiction. Similarly, we obtain a contradiction if $C \cap N(v) = \{v_3, v_5\}$. Hence we may assume that $C \cap N(v) = \{v_1, v_5\}$. But then connecting the vertices of B to those of $\{v_2, v_4\}$ would require one more crossing. See Figure 7(g).

This completes the proof of Claim 5. ∎

Claim 6. $q \neq 3$.

Proof. Suppose that $q = 3$.

Let C be the crossing whose cluster does not contain v. It contains no candidate u; otherwise $\{u, v\}$ would be a stable crossing cover. Hence $C \subset N(v)$.

Assume first that the three crossed edges incident to v are consecutive, say the crossed edges are vv_1, vv_2, and vv_5. By the Assertion, $v_3 v_4$ is an edge. See Figure 8(a). Up to symmetry, the cluster of C is one of the following three sets: $\{v_1, v_2, v_3, v_4\}$, $\{v_2, v_3, v_4, v_5\}$, or $\{v_1, v_2, v_4, v_5\}$.

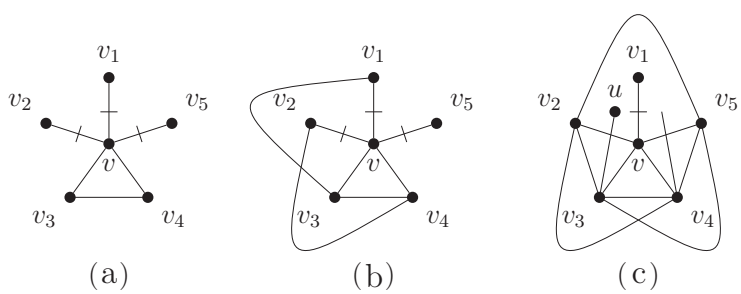

Fig. 8. *Three consecutive crossed edges.*

- $C = \{v_1, v_2, v_3, v_4\}$. Then the edges of C are not $v_1 v_4$ and $v_2 v_3$ because it is impossible to draw them such that each is crossed exactly once. Hence $C = \{v_1 v_3, v_2 v_4\}$. The Jordan curve formed by the path $v_1 v v_4$ and the two "half-edges" connecting v_1 to v_4 in C separates $\{v_2, v_3\}$ and v_5. See Figure 8(b). Moreover, it is crossed only once (on edge $v_1 v$), while 2 crossings are needed, one for each of the disjoint paths $v_2 - v_5$ and $v_3 - v_5$, a contradiction.
- $C = \{v_2, v_3, v_4, v_5\}$. Then the edges of C are not $v_2 v_3$ and $v_4 v_5$ because it is impossible to draw them such that each is crossed exactly once. Hence $C = \{v_2 v_4, v_3 v_5\}$. Hence by the Assertion, $v_2 v_3$, $v_4 v_5$, and $v_2 v_5$ are edges. The triangle $v v_2 v_3$ has only one crossed edge. So, by Lemma 2.10, it is not separating. Thus its interior is empty, and the edge crossing $v v_2$ is incident to v_3. Let u be the second endvertex of this edge. By symmetry, the interior of $v v_4 v_5$ is empty and the edge crossing $v v_5$ is $v_4 t$ for some vertex t.

  If $u = t = v_1$, then by the Assertion, $v_1 v_2$ and $v_1 v_5$ are edges. So $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence, without loss of generality, we may assume that $u \neq v_1$. See Figure 8(c).

  The interiors of the cycles $v v_2 v_3$, $v v_3 v_4$, and $v_2 v_3 v_4$ contain no vertices by Lemma 2.9. Hence v_3 is a degree 5 vertex. Moreover, its two neighbors u

and v are not adjacent, and $(G - v_3)/\{u, v\}$ has at most 2 crossings. Then Theorem 4.1 and Lemma 2.8 yield a contradiction.

- $C = \{v_1, v_2, v_4, v_5\}$. The crossing C is neither $\{v_1v_2, v_4v_5\}$ nor $\{v_1v_5, v_2v_4\}$ since it is impossible to draw so that every edge is crossed exactly once. Hence $C = \{v_1v_4, v_2v_5\}$. By the Assertion, $v_2v_4 \in E(G)$. Then the triangle vv_2v_4 contradicts Lemma 2.10.

Suppose now that the three crossed edges incident to v are not consecutive. Without loss of generality, we assume that these edges are vv_1, vv_3, and vv_4.

By the Assertion, v_2v_5 is an edge. If v_2v_5 is not crossed, then vv_2v_5 is a separating triangle, contradicting Lemma 2.9. So v_2v_5 is crossed. It could not cross vv_3 or vv_4; otherwise vv_2v_5 would be a regular cycle contradicting Lemma 2.9. Moreover, v_2v_5 cannot be in C; otherwise vv_2v_5 would contradict Lemma 2.10. Hence v_2v_5 crosses vv_1.

By the Assertion, v_1v_2 and v_1v_5 are edges. Moreover they are not crossed; otherwise they could be redrawn along the edges of the crossing $\{vv_1, v_2v_5\}$ to obtain a drawing of G with fewer crossings. See Figure 9(a).

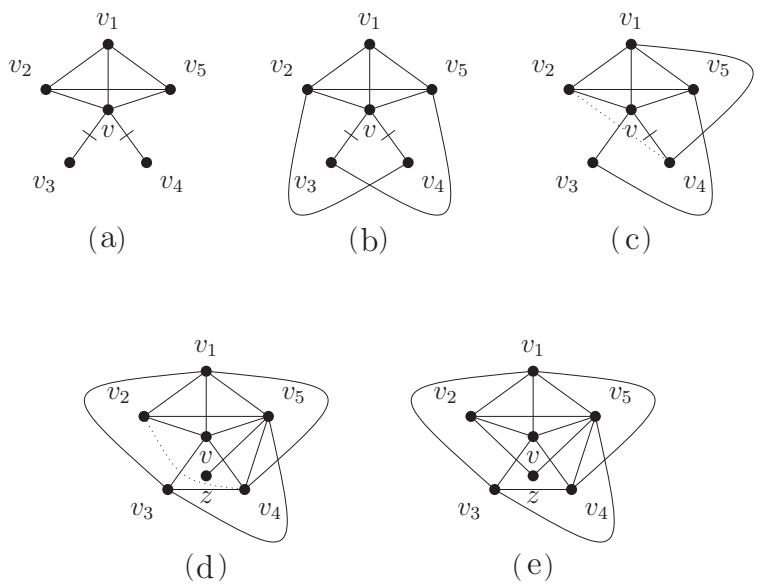

FIG. 9. *Three nonconsecutive crossed edges.*

Consider the paths $v_2 - v_4$ and $v_3 - v_5$. If they cross, it is through C. Since $C \subset N(v)$, the paths $v_2 - v_4$ and $v_3 - v_5$ are actually edges. See Figure 9(b). But one can redraw v_2v_5 along the edges of C to obtain a drawing of G with fewer crossings, a contradiction.

Suppose now that $v_2 - v_4$ and $v_3 - v_5$ do not cross. By symmetry, we may assume that $v_2 - v_4$ crosses vv_3. The paths $v_1 - v_4$ and $v_3 - v_5$ cross. It must be through C so v_1v_4 and v_3v_5 are both edges. See Figure 9(c). By the Assertion, v_1v_3, v_3v_4, and v_4v_5 are edges.

If v_2v_4 is an edge, the Assertion implies that v_2v_3 is an edge. Then $N(v) \cup \{v\}$ induces a K_6, a contradiction. Hence $v_2v_4 \notin E(G)$.

By Lemma 2.10, the cycle vv_4v_5 is not separating, so its interior contains no vertex, and vv_4 is crossed by an edge with v_5 as an endvertex. Let z be the other endvertex of this edge. As an edge is crossed at most once, z is inside vv_3v_4. See Figure 9(d).

Let ab be the edge which is crossing vv_3. The sets $\{v_5, a\}$ and $\{v_5, b\}$ are not stable; otherwise they would be a stable crossing cover. Hence v_5a and v_5b are both edges. Thus $ab = v_2z$. See Figure 9(e). Now v_1z is not an edge, and hence $\{v_1, z\}$ is a stable crossing cover, contradicting Lemma 2.7.

This completes the proof of Claim 6. ∎

CLAIM 7. $q \neq 4$

Proof. By way of contradiction, suppose that $q = 4$. Then $\{v\}$ is a stable crossing cover, a contradiction. ∎

Combining Claims 3, 4, 5, 6, and 7 yields a contradiction. This finishes the proof of Theorem 5.1. ∎

6. Further research.

6.1. Extending our results. Theorem 4.1 states that, if a graph can be made planar by removing at most $2k$ edges, then it is $(4+k)$-colorable. We believe that this is not tight. Thus a natural question is the following.

PROBLEM 6.1. *What is the maximum chromatic number of all graphs that contain a set of at most k edges whose deletion results in a planar graph?*

Let us denote this maximum by $g(k)$. Clearly, $g(1) = g(2) = 5$ by Theorem 4.1 and because K_5 is not planar and $g(3) = 6$ by Theorem 4.1 and because $cr(K_6) = 3$. For a larger value of k, we also believe that the optimal value is given by a complete graph. It is also very likely that the complete graph $K_{g(k)}$ is the unique $g(k)$-critical graph that can be made planar by removing k edges. It is, in particular, the case for $k = 6$ and $k = 7$. Indeed, by Proposition 2.1, at least six edges are needed to make K_7 planar, and there is a set of six edges whose removal leaves K_7 planar. See Figure 10.

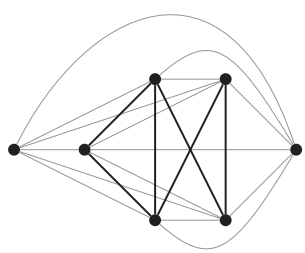

FIG. 10. *The graph K_7 and a set of edges (in bold) whose removal yields a planar graph.*

THEOREM 6.2. *Let G be a graph with clique number at most 6. If there is a set of at most seven edges whose deletion results in a planar graph, then G is 6-colorable.*

Proof. To prove this theorem, we show that K_7 is the unique 7-critical graph G for which there exists a set of at most seven edges whose removal leaves G planar. A famous result of Dirac [5] states that, if G is an r-critical graph and is not K_r, then $2|E(G)| \geq (r-1)|V(G)| + r - 3$. In particular, if $r = 7$, then $|E(G)| \geq 3|V(G)| + 2$. Hence, by Proposition 2.1, we need to remove at least eight edges to make it planar. ∎

One of the first problems to tackle is the following conjecture which extends both Theorem 1.6 and Theorem 1.4.

CONJECTURE 6.3. *Every graph with clique number at most* 5 *and that contains a set of at most* 4 *edges whose deletion results in a planar graph is* 5*-colorable.*

6.2. Critical graphs and colorability. It is easy to derive from Proposition 2.1 that, for $r \geq 8$, there are only finitely many r-critical graphs that can be embedded on a fixed surface. As pointed out by Thomassen in [17], the number of 7-critical graphs that can be embedded on a fixed surface is also finite. Finally, Thomassen [18] completed the results by showing that the number of 6-critical subgraphs is finite for any fixed surface Σ. This implies in particular that the $(r-1)$-colorability problem for graphs embeddable on Σ is decidable in polynomial time for any $r \geq 6$. On the other hand, deciding 3-colorability is NP-complete for planar graphs (see [8]) and thus also for graphs embeddable on any other surface. The complexity of 4-colorability remains open.

PROBLEM 6.4. *For any fixed surface, does there exist a polynomial time algorithm for deciding if a graph embeddable on this surface is* 4*-colorable?*

The answer to Problem 6.4 is only known for the sphere by the Four Color Theorem. An affirmative answer cannot be obtained in the same way as for $r-1 \geq 5$ because there are infinitely many 5-critical graphs as implied by a result of Fisk [7].

If $\mathrm{cr}(G) = k$, then G is embeddable in $\mathbb{S}_k$ and in $\mathbb{N}_k$ as well, where $\mathbb{S}_k$ is an orientable surface of genus k and $\mathbb{N}_k$ is a nonorientable surface of genus k. Hence for any k and $r \geq 6$, the number of r-critical graphs of crossing number k is finite, and so the $(r-1)$-colorability problem for graphs of crossing number k is decidable in polynomial time. However, the design of such a polynomial time algorithm requires the knowledge of the list of 6-critical graphs.

PROBLEM 6.5. *Let* $k \geq 0$. *What is the list of* 6*-critical graphs with crossing number at most* k*?*

When $k \leq 3$, the list is empty and if $k = 4$, then the list is $\{K_6\}$. If $k = 5$, then the list contains K_6 and the graph depicted in Figure 3. But are there any others?

Similarly to graphs embeddable on a fixed surface, the complexity of 4-colorability problem for graphs with crossing number k is not known.

PROBLEM 6.6. *Let* $k \geq 1$. *Does there exist a polynomial time algorithm for deciding if a given graph with crossing number* k *is* 4*-colorable?*

This would be true if there were a finite number of 5-critical graphs with crossing number at most k. However, as observed by Zdeněk Dvořák, this is not the case. Indeed, every graph obtained from an odd cycle by adding two new vertices adjacent to all the vertices of the cycle and adjacent to each other is 5-critical.

6.3. Choosability. A *list assignment* of a graph G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. An *L-coloring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G. If G admits an L-coloring, then it is *L-colorable*. A graph G is *k-choosable* if it is L-colorable for every list assignment L such that $|L(v)| \geq k$ for all $v \in V(G)$. The *choose number* of G, denoted by $\mathrm{ch}(G)$, is the minimum k such that G is k-choosable.

Similarly to the chromatic number, one may seek for bounds on the choose number of a graph with few crossings or with independent crossings.

Thomassen [16] showed that every planar graph is 5-choosable. In fact, he proved a stronger result.

DEFINITION 6.7. An *inner triangulation* is a plane graph such that every inner face of G is bounded by a triangle and its outer face by a cycle $F = (v_1 v_2 \ldots v_k v_1)$.

A list assignment L of an inner triangulation G is *suitable* if

- $|L(v_1)| = 1$ and $|L(v_2)| = 2$,
- for every $v \in V(F) \setminus \{v_1, v_2\}$, $|L(v)| \geq 3$, and
- for every $v \in V(G) \setminus V(F)$, $|L(v)| \geq 5$.

THEOREM 6.8 (Thomassen [16]). *If L is a suitable list assignment of an inner triangulation G, then G is L-colorable.*

THEOREM 6.9. *Every graph with crossing number* 1 *is* 5*-choosable.*

Proof. Let G be a graph with crossing number 1. Consider a plane embedding of G with 1 crossing $C = \{x_1y_1, x_2y_2\}$. Without loss of generality, we may assume that C is in the outer face. Free to add edges, we may assume that the outer face is bounded by the 4-cycle $x_1x_2y_1y_2x_1$ and that $G \setminus C$ is an inner triangulation.

Let L be a 5-list assignment of G. Set $c_1 \in L(x_1)$ and $c_2 \in L(x_2) \setminus \{c_1\}$. Let L' be the list assignment defined by $L'(x_1) = \{c_1\}$, $L'(x_2) = \{c_1, c_2\}$, $L'(y_i) = L(y_i) \setminus \{c_1, c_2\}$ for $i = 1, 2$ and $L'(v) = L(v)$ for every $v \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$. Then L' is a suitable list assignment of $G \setminus C$. Hence $G \setminus C$ admits a proper L'-coloring, which is an L-coloring of G by the definition of L'. ∎

PROBLEM 6.10. *Is every graph with crossing number* 2 5*-choosable?*

6.4. Graphs with small clique number or large girth. The celebrated Grötzsch Theorem [9] asserts that triangle-free (i.e., with clique number at most 2) planar graphs are 3-colorable. (See also [19] for a short and elegant proof.) Together with Theorem 1.4, this suggests that the upper bounds on the chromatic number of graphs which are almost planar (they have few crossings or pairwise nondependent crossings or contain few edges whose deletion makes them planar) may be lowered when considering graphs with small clique number. We now prove a result analogous to Theorem 1.5 for K_4-free graphs.

THEOREM 6.11. *If a K_4-free graph has a drawing in the plane in which no two crossings are dependent, then it is* 4*-colorable.*

Proof. Consider a drawing of a K_4-free graph G in which no two crossings are dependent. Let $C_i = \{u_iv_i, x_iy_i\}$, $i \in I$, be the crossings. Since G is K_4-free, without loss of generality, we may assume that u_ix_i is not an edge for all $i \in I$. Let G' be the graph obtained from G by identifying u_i with x_i for every $i \in I$ into a vertex z_i. The graph G' is planar. Thus, by the Four Color Theorem, G' admits a proper 4-coloring c'. Let us define c by $c(x_i) = c(x_i) = c'(z_i)$ for every $i \in I$ and $c(v) = c'(v)$ for every vertex $v \in V(G) \cap V(G')$. Since, for every $i \in I$, x_i and u_i are not adjacent, c is a proper 4-coloring of G. ∎

Note that Theorem 6.11 is tight because there exist K_4-free planar graphs which are not 3-colorable. But can it be improved for triangle-free graphs, or is there a triangle-free graph which has a drawing in the plane in which no two crossings are dependent and which is not 3-colorable?

For triangle-free graphs, one can show an analogue to Theorem 1.6.

THEOREM 6.12. *If a triangle-free graph contains a set of (at most)* 4 *edges whose deletion results in a planar graph, then it is* 4*-choosable.*

Proof. By induction on the number n of vertices of the triangle-free graph G, the result holds trivially when $n \leq 4$. A triangle-free planar graph on n vertices has at most $2n - 4$ edges. Hence G has at most $2n$ edges. Thus either G has a vertex v of degree at most 3 or it is 4-regular.

In the first case, by the induction hypothesis $\mathrm{ch}(G - v) = 4$. Let L be a 4-list assignment of $V(G)$. The graph $G - v$ admits an L-coloring c that can be extended to G by assigning to v a color in its list not assigned to any of its neighbors. So G is 4-choosable.

In the second case, since G is triangle-free it contains no K_5, and thus by Brooks's Theorem for list-coloring, $\mathrm{ch}(G) \leq 4$. $\square$

For C_3 and K_4 and, more generally, for any graph or any family of graphs $\mathcal{F}$, one can ask the following questions.

PROBLEM 6.13. What is the smallest integer $f_{\mathcal{F}}(k)$ (resp. $g_{\mathcal{F}}(k)$) such that every $\mathcal{F}$-free graph G and crossing number at most k is $f_{\mathcal{F}}(k)$-colorable (resp. $g_{\mathcal{F}}(k)$-choosable)?

In particular, for $\mathcal{C}_g = \{C_i | i = 3, \ldots, g-1\}$ the family of cycles of length less than g, the $\mathcal{C}_g$-free graphs are graphs with girth at least g. Set $f_g(k) = f_{\mathcal{C}_g}(k)$. Trivially, $f_g(k) \leq f_h(k)$ if $g \geq h$. In particular, for any $g \geq 3$, $f_g(k) \leq O\left(k^{1/4}\right)$ since $f_3(k) = f(k) = O\left(k^{1/4}\right)$. Erdös [6] showed that there are graphs with arbitrarily large girth and chromatic number. Hence for any fixed g, $f_g(k)$ tends to infinity when k tends to infinity. The Grötzsch graph is triangle-free and has crossing number at most 5 and chromatic number 4, so $f_3(5) \geq 4$. Thomas and Walls [15] proved that every graph of girth at least 5 which admits an embedding in the Klein bottle is 3-colorable. Since every graph with crossing number at most 2 is embeddable in the Klein bottle, it follows that every graph of girth at least 5 and crossing number at most 2 is 3-colorable.

Jensen and Royle [10] showed a K_4-free graph with crossing number at most 6 and chromatic number 5, so $f_{K_4}(6) \geq 5$.

One can prove an analogue to Theorem 1.5 for graphs of large girth.

PROPOSITION 6.14.

Let G be a graph having a drawing in the plane in which no two crossings are dependent.

(i) *If G has girth at least 5, then G is 4-choosable.*

(ii) *If G has girth at least 10, then G is 3-choosable.*

Proof. Let us prove that G has a vertex of degree at most 3 (resp. at most 2) if G has girth at least 5 (resp. 10). Then an easy induction would give the result.

Let n be the number of vertices of G. Since no two crossings are dependent, then G has at most $n/4$ crossings. Hence there is a set F of at most $n/4$ edges such that $G \setminus F$ is planar. Moreover, $G \setminus F$ has girth at least 5 (resp. 10), so $G \setminus F$ has fewer than $\frac{10}{6}n$ (resp. $\frac{3}{4}n$) edges. Hence G has fewer than $\frac{23}{12}n$ (resp. n) edges. Hence G has a vertex of degree at most 3 (resp. 2). $\square$

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