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H. K. Janssen
B. Schaub
Beate Schmittmann

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H. K. Janssen, B. Schaub, and B. Schmittmann
Institut für Theoretische Physik III, Universität Düsseldorf, Universitätsstrasse 1, D-4000 Düsseldorf 1, West Germany
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The general epidemic process, which is a stochastic multiparticle process belonging to the universality class of dynamic percolation, is studied in a semi-infinite geometry. Critical exponents characterizing the fractal properties are calculated to $O(\epsilon)$ ($\epsilon=6-d$, where d is the spatial dimension) with use of renormalization-group techniques.

I. INTRODUCTION

The understanding of self-similar or fractal structures that characterize a large variety of growth processes in physics, chemistry, and biology is still in its very beginning. One possible route to a deeper understanding of these structures is the observation that self-similarity is also prominent in other phenomena, such as in critical fluctuations.

Critical fluctuations typically arise in macroscopic systems which are close to a second-order phase transition and they can be successfully described using renormalization-group (RG) techniques. The main idea of RG theory consists in the assumption that as the critical point is approached the system can be described by one single length scale. It is this length scale which characterizes the large scale properties of physical quantities such as, for example, correlation functions. At present it seems that the fractal structures observed in certain processes such as diffusion-limited aggregation cannot be explained on the basis of assuming one single length scale only. This may be concluded from the fact that RG transformations applied to such models do not lead to stable fixed points.[1]

It is therefore important to investigate in more detail such fractal structures which can be successfully explained using RG ideas. Of particular interest is the understanding of generic effects of the environment, which will profoundly modify the growth of self-similar clusters.

A simple, yet physically obvious problem is the influence of the global geometry of the environment on the growth process. If the global geometry is changed in a controlled manner, how will typical fractal properties such as the mass of a cluster or the fractal dimension become modified? One way to bring about such a change in the geometry of an environment is the presence of a free surface or a wall. This causes a loss of translational symmetry and generates boundary conditions and therefore one expects significant effects to occur.

The kind of questions we are addressing here for fractal growth processes naturally arise in the context of surface phase transitions, a new research area in condensed matter physics, which has rapidly advanced within recent years. A large variety of physical effects caused by the presence of free surfaces or walls in macroscopic systems which are close to criticality has been found.[2] Early investigations,[3–5] using mean-field theory, revealed an unexpectedly rich structure of possible second- and even first-order[6] phase transitions in such systems. Although these studies did predict qualitatively correct phase diagrams for dimensions $d>2$, the influence of fluctuations on these transitions could only be treated using RG ideas.

Among a variety of techniques based upon these ideas, the powerful ϵ expansion provides one of the most reliable methods to determine critical exponents and scaling functions. It allows the analytic calculation of universal properties by means of a systematic expansion in powers of $\epsilon=d_c-d$ around the upper critical dimension d_c. First attempts to apply these techniques to semi-infinite systems faced two major obstacles, however:[4,5] the loss of translational invariance and the appearance of boundary conditions. An elegant way of resolving these intrinsic difficulties consisted in devising a field-theoretic formulation which (a) works entirely in configuration space and (b) incorporates the boundary conditions in the propagators of the free theory around which perturbative expansions are performed. A recent detailed review of this technique can be found in Ref. 7.

It is one purpose of our paper, utilizing these recently developed techniques, to investigate how typical fractal properties will be affected due to the loss of translational invariance and the appearance of boundary conditions. Another purpose is, as most of the current research on semi-infinite systems has concentrated on *equilibrium* bulk-driven phase transitions, to initiate research on surface phase transitions which are characterized by a nonequilibrium bulk critical point.

In the following we shall study a model of general epidemic processes (GEP's) which belong to the universality class of dynamic percolation in a semi-infinite geometry. The GEP is a stochastic multiparticle process which describes the temporal evolution of a local density of infected individuals $\Phi(x,t)$, where $x=(x_1,\ldots,x_d)$. It is characterized by the following features.

(i) There is an absorbing stationary state at $\Phi(x,t)=0$, corresponding to the situation where the epidemic has become extinct.

(ii) The disease spreads (diffusively) in the available environment.

(iii) Individuals can become immune to the disease. Thus, the net infection rate depends on the number of infected individuals *and* on the number of immune individuals, introducing a memory term into the process.

(iv) Microscopic degrees of freedom are subsumed in the form of a Langevin noise term, which, however, must respect the absorbing state. Hence, its correlations have

to vanish for $\Phi(x,t)=0$.

The GEP has been introduced in Refs. 8 and 9 and studied in Refs. 10–13. It was shown to exhibit a second-order phase transition near the absorbing state. Its critical properties have been analyzed within an ϵ expansion about the upper critical dimension $d_c=6$.[12] In Ref. 12, it was also found that the GEP belongs to the universality class of dynamic percolation.

Our paper is organized as follows. In Sec. II we present the generating functional for our model and discuss, in particular, the inclusion of a wall which leads to a semi-infinite geometry. In Sec. III we perform a RG analysis for the model to one-loop order and in Sec. IV we derive a renormalization-group equation (RGE) from which the surface critical exponents can be calculated. Section V contains our conclusions. Details of the renormalization procedure to be discussed in Sec. III are provided in the Appendix.

II. MODEL

Let us consider a Langevin equation for a local density $\Phi(x,t)$ which incorporates the characteristics (i)–(iv) in a simple yet self-contained form (in the RG spirit),

$$\partial_t\Phi(x,t)=\lambda\Delta\Phi(x,t)-\lambda\tau\Phi(x,t)\ ,\qquad -\tfrac{1}{2}\lambda^2 w\Phi(x,t)\int_{-\infty}^{t}dt'\,\Phi(x,t')+\xi(x,t)\ . \tag{2.1}$$

The first term on the right-hand side models the diffusive spreading (ii) of the disease. The next two terms represent the net infection according to (iii). $\xi(x,t)$ is a Langevin force, with correlations subject to (iv):

$$\langle\xi(x,t)\xi(x',t')\rangle=\lambda w'\Phi(x,t)\delta(x-x')\delta(t-t')\ , \tag{2.2}$$

and λ, τ, and w are constant couplings. By a simple rescaling we can ensure $w'=w$ which is in fact preserved under the renormalization group. The characteristic feature of the GEP which distinguishes it from other evolution processes resides in the non-Markovian nature of the nonlinear coupling associated with w. The term

$$\int_{-\infty}^{t}dt'\,\Phi(x,t')$$

sums up the total density of individuals who catch the disease at any time between its outbreak and the time t. Thus it is a measure for the local density of immune individuals.

In the context of ecology, $\Phi(x,t)$ may be interpreted as the local density of, say, a forest fire, where

$$\int_{-\infty}^{t}dt'\,\Phi(x,t')$$

is proportional to the amount of burned fuel (ash) at time t.

Note that the parameter τ in (2.1) measures the distance to the critical point and can be interpreted in terms of percolation probabilities as $\tau\sim p_c-p$. To study critical properties of the GEP within the framework of renormalized field theory it is convenient to recast the Langevin equation (2.1) in conjunction with (2.2) as a dynamic functional,[14–16]

$$J[\tilde\Phi,\Phi]=\int d^dx\int dt\,\tilde\Phi(x,t)\left[\partial_t+\lambda\left[\tau-\Delta-\tfrac{1}{2}w\tilde\Phi(x,t)+\lambda w\int_{-\infty}^{t}dt'\,\Phi(x,t')\right]\right]\Phi(x,t)\ . \tag{2.3}$$

Within this formalism all correlation and response functions can be expressed as functional averages with weight $\exp(-J)$. From dimensional analysis one finds the upper critical dimension $d_c=6$.

Now we turn to a discussion of the model (2.3) in a semi-infinite geometry

$$\{x=(x_\parallel,z)|x_\parallel\in R^{d-1},\ \ 0\le z<\infty\}$$

bounded by a plane at $z=0$. The presence of this plane or wall can be modeled by introducing a new additional interaction term

$$J_s[\tilde\Phi,\Phi]=\int d^{d-1}x\int dt\,\lambda\tau_s\tilde\Phi_s\Phi_s\ , \tag{2.4}$$

where $\tilde\Phi_s=\tilde\Phi(x_\parallel,z=0,t)$, $\Phi_s=\Phi(x_\parallel,z=0,t)$, and $\tau_s\sim p_{sc}-p_s$, with p_s being a surface percolation probability. Depending on the value of p_{sc} the surface may or may not percolate before the bulk becomes critical.

The surface interaction term (2.4) is in accordance with the symmetry transformation

$$\tilde\Phi(x,t)\to\lambda\int_{-\infty}^{-t}dt\,\Phi(x,t')\ ,\qquad \Phi(x,t)\to\lambda^{-1}\partial_t\tilde\Phi(x,-t) \tag{2.5}$$

of the bulk system. Further, it respects causality and, most importantly, it is relevant in the RG sense, as $[\tau_s]\sim\mu$ (μ being an inverse length scale). Interaction terms involving a larger number of Φ, $\tilde\Phi$ fields turn out to be irrelevant in the RG sense, because their respective couplings have a negative μ dimension. We also have not included a surface term of the form $\tilde\Phi\partial_z\Phi$ which, although marginal from dimensional arguments, is redundant, since it would correspond (using the boundary condition $\tau_s\Phi|_s=\partial_t\Phi|_s$) to a redefinition of τ_s.[7] Following the pioneering work by Lubensky and Rubin[4,5] it is meanwhile well established that the presence of a term such as (2.4) implies the existence of new phase transitions in a bulk system which is close to criticality. As $[\tau_s]\sim\mu$, the accessible fixed points of the renormalized (R) coupling τ_s^R can only be $\tau_s^R\to+\infty$, $\tau_s^R=0$, and

$\tau_s^R \to -\infty$. The first possibility corresponds to the ordinary (O), the second to the special (S) (a multicritical point), and the third to the extraordinary phase transition.[4,5] All three cases correspond to second-order phase transitions characterized by a diverging bulk correlation length.

III. RENORMALIZATION-GROUP ANALYSIS

It is this fixed point structure of the coupling τ_s^R described above, which enables one to perform perturbative calculations and thus permits a RG treatment of (2.3) and (2.4). For $\tau_s \to (0, \pm\infty)$ the free response propagator G_0 in the presence of a surface can be expressed entirely in terms of the free (bulk) response propagator G_0^B (Refs. 4 and 5) for Eq. (2.3):

$$G_0(x_\parallel, t|z,z') = G_0^B(x_\parallel, t|z,z') \pm G_0^B(x_\parallel, t|z,-z') , \tag{3.1}$$

where

$$G_0^B(x_\parallel, t|z,z') = \frac{\theta(t)}{(4\pi\lambda t)^{d/2}} \exp\left[-\left[\frac{x_\parallel^2}{4\lambda t} + \frac{(z-z')^2}{4\lambda t}\right]\right]$$
$$= \int_q \int_\omega \frac{1}{2\lambda\kappa} \exp[-\kappa|z-z'| + i(\omega t + \mathbf{q}\cdot\mathbf{x}_\parallel)] , \tag{3.2}$$

with $\kappa = (q^2 + i\omega/\lambda)^{1/2}$. Here and in the following we put $\tau = 0$. The upper (+) sign in (3.1) corresponds to the case of Neumann boundary conditions $\partial_z G_0(q,\omega|z,z')|_{z=0} = 0$ which are realized for the special transition ($\tau_s = 0$) and the lower (−) sign corresponds to Dirichlet boundary conditions $G_0(q,\omega|z,z')|_{z=0} = 0$ which are realized for the ordinary transition ($\tau_s \to \infty$).

The strategy for a RG analysis, to be performed below for the special and the ordinary transitions, can be stated in the following way: due to the additional surface interaction (2.4) we expect to find new divergences which are entirely localized in the surface, in addition to the ultraviolet divergences caused by the bulk interactions for $d \to d_c$. This amounts to introducing new renormalization constants[17,18] (Z factors) apart from those necessary to renormalize the bulk divergences. Both bulk and surface renormalizations can be determined by considering the connected response function $G(q,\omega|z_1,z_2) = \langle \Phi(q,\omega|z_1)\tilde{\Phi}(q,\omega|z_2)\rangle_{\text{conn}}$ which reads, utilizing (2.3),

$$G(q,\omega|z_1,z_2) = G_0(q,\omega|z_1,z_2) + \int_0^\infty dz \int_0^\infty dz'\, G_0(q,\omega|z_1,z) I(q,\omega|z,z') G_0(q,\omega|z',z_2) , \tag{3.3}$$

where

$$I(q,\omega|z,z') = -w^2\lambda^3 \int d^{d-1}x\, e^{-i\mathbf{q}\cdot\mathbf{x}_\parallel} \int_0^\infty ds\, e^{-i\omega s} G_0(x_\parallel, s|z,z') \int_0^s d\bar{s}\, G_0(x_\parallel, \bar{s}|z,z') \tag{3.4}$$

to the order of one loop. Evaluating (3.4) with (3.1) we obtain, using the method of dimensional regularization,

$$I(q,\omega|z,z') = I_B(q,\omega|z,z') + I_s(q,\omega|z,z') . \tag{3.5}$$

In (3.5) the first term arises from the usual bulk part of the propagator (see the Appendix) and the second term contains the surface contributions. I_s is of the general form (see the Appendix for details)

$$I_s(q,\omega|z,z') = A[\delta'(z)\delta(z') + \delta(z)\delta'(z')] + \text{finite terms} , \tag{3.6}$$

where $A = \frac{7}{3}\lambda u/\epsilon$ for S and $A = -\frac{5}{3}\lambda u/\epsilon$ for O, with

$$u = w^2\Gamma(1+\epsilon/2)/(4\pi)^{d/2} ,$$

to one-loop order. The pole terms arising from both the bulk and the surface contribution in connected response functions having external legs on the surface can be absorbed by a new surface renormalization,

$$\Phi_s = Z_0 Z_\Phi^{1/2} \Phi_s^R , \quad \tilde{\Phi}_s = Z_0 Z_{\tilde{\Phi}}^{1/2} \tilde{\Phi}_s^R , \tag{3.7}$$

with $Z_0 = Z_0^S = 1 + \frac{1}{2}(u/\epsilon)$.

For O we consider (due to the Dirichlet boundary conditions) the response function

$$\partial_{z_2} G(q,\omega|z_1,z_2) = \langle \Phi(q,\omega|z_1)\partial_{z_2}\tilde{\Phi}(q,\omega|z_2)\rangle .$$

This corresponds to the general substitution $\Phi(\tilde{\Phi})|_s \to \partial_z\Phi(\tilde{\Phi})|_s =: \Phi_\perp(\tilde{\Phi}_\perp)$ for O. In this case we find $Z_0 = Z_0^O = 1 + u/3\epsilon$.

Apart from Z_0 we will need another Z factor Z_s which renormalizes the new coupling τ_s. For S it is defined by

$$\tau_s = Z_s Z_{\tilde{\Phi}}^{-1} Z_0^{-2} \tau_s^R \tag{3.8}$$

and can be determined from the insertion of the surface operator

$$\lambda\tau_s \int d\bar{t} \int d^{d-1}\bar{x}\, \tilde{\Phi}_s(\bar{x},\bar{t})\Phi_s(\bar{x},\bar{t})$$

into the response function

$$G(x_1,t_1,x_2,t_2) = \langle \Phi(x_1,t_1)\tilde{\Phi}(x_2,t_2)\rangle .$$

We obtain $Z_s^S = 1 + 2(u/\epsilon)$ (for details see Ref. 19, where

we studied a model for directed percolation). For O we define

$$\tau_s^{-1}=Z_s Z_{\tilde{\Phi}}^{-1} Z_0^{-2}(\tau_s^{-1})^R=(\tau_s^{-1})^R \ . \tag{3.9}$$

Thus $Z_s^O=Z_{\tilde{\Phi}}Z_0^2$ to all orders, a result which follows from a Ward identity discussed in Ref. 19. This concludes the necessary renormalizations.

IV. SCALING PROPERTIES

Using the results derived in Sec. III we will now study the scaling properties of a connected renormalized response function $G^{(N,M)}$ (dropping the index R everywhere) composed of $N(\tilde{\Phi},\Phi)$ fields and $M(\tilde{\Phi}_s,\Phi_s)$ fields. Let us first consider the special transition S. Following standard arguments[19] we obtain a RGE which reads

$$[\mu\partial_\mu+\beta_u\partial_u+\kappa\tau\partial_\tau+\kappa_s\tau_s\partial_{\tau_s}+\zeta\lambda\partial_\lambda+\tfrac{1}{2}(N+M)\gamma_\Phi+\tfrac{1}{2}(\tilde{N}+\tilde{M})\gamma_{\tilde{\Phi}}+(M+\tilde{M})\gamma_0]G^{(N,\tilde{N},M,\tilde{M})}=0 \ . \tag{4.1}$$

In (4.1), $\beta_u=\mu\partial_{\mu|0}u$, $\kappa=\mu\partial_{\mu|0}\ln\tau$, $\kappa_s=\mu\partial_{\mu|0}\ln\tau_s$, $\zeta=\mu\partial_{\mu|0}\ln\lambda$, $\gamma_\Phi=\mu\partial_{\mu|0}\ln Z_\Phi$, $\gamma_{\tilde{\Phi}}=\mu\partial_{\mu|0}\ln Z_{\tilde{\Phi}}$, $\gamma_0=\mu\partial_{\mu|0}\ln Z_0$, where $\partial_{\mu|0}$ denotes a derivative at fixed bare parameters. Solving Eq. (4.1) at the fixed point $\beta_u(u^*)=0$ with $u^*=\frac{2}{7}\epsilon$ using the method of characteristics and taking the μ dimensions into account, we secure

$$G^{(N,\tilde{N},M,\tilde{M})}(\{x,t\},\tau,\tau_s,\lambda,\mu)=l^{(1/2)(N+M)(d+2+\eta)}l^{(1/2)(\tilde{N}+\tilde{M})(d-2+\tilde{\eta})}l^{(M+\tilde{M})\eta_0}$$
$$\times G^{(N,\tilde{N},M,\tilde{M})}(\{lx,l^z t\},l^{-1/\nu}\tau,l^{-1/\nu_s}\tau_s,\lambda,\mu) \ , \tag{4.2}$$

where $\eta=\gamma_\Phi^*=-\frac{8}{21}\epsilon$, $\tilde{\eta}=\gamma_{\tilde{\Phi}}^*=-\frac{1}{21}\epsilon$, $\eta_0=\gamma_0^*=-\frac{1}{7}\epsilon$ (for S), $z=2+\zeta^*=2-(\epsilon/6)$, $1/\nu=2-\kappa^*=2-\frac{5}{21}\epsilon$, and $1/\nu_s=1-\kappa_s^*=1-\frac{5}{21}\epsilon$ (for S). In terms of the parameter τ (setting $\tau l^{-1/\nu}=1$) we have in (4.2) $\tau_s\tau^{-\Phi}$ where $\Phi=\nu/\nu_s=\frac{1}{2}(1-\frac{5}{42}\epsilon)$ is a crossover exponent which describes the behavior of the surface transition temperature as a function of τ_s near the multicritical point S.[2,7]

We can extract static correlations from (4.1) and (4.2) by setting the time arguments of $\tilde{\Phi}(\tilde{\Phi}_s)$ to zero and performing an extra time integration for all fields $\Phi(\Phi_s)$. Then we find

$$G_{\text{stat}}^{(1,0,0,0)}(x)=|\tau|^\beta f\left[\frac{x}{\xi}\right] , \tag{4.3}$$

where $\beta=\frac{1}{2}\nu(d-2+\tilde{\eta})=1-(\epsilon/7)$, $\xi=\xi_0|\tau|^{-\nu}$ is the correlation length, and f is some scaling function. Putting $N=\tilde{N}=0$, $M=1$, $\tilde{M}=0$ we obtain a new exponent $\beta_s=\beta+\nu\eta_0=\frac{1}{2}\nu(d-2+\eta_\parallel)=1-\frac{3}{14}\epsilon$ with $\eta_\parallel=\tilde{\eta}+2\eta_0=-\epsilon/3$. Furthermore, it follows from a short-distance expansion[17] that $f(z/\xi)\sim(z/\xi)^\Psi$ for $z\to 0$ with an exponent $\Psi=\eta_0=-\epsilon/7$ (for S).

Now we consider the mean size of a cluster. In the bulk this is given by

$$\int d^dx\, G_{\text{stat}}^{(1,1,0,0)}(x)\sim|\tau|^{-\gamma} , \tag{4.4}$$

with $\gamma=\nu(2-\tilde{\eta})=1+(\epsilon/7)$.[12]

For $N=\tilde{M}=1$, $\tilde{N}=M=0$ we determine a surface exponent $\gamma_s=\nu(2-\tilde{\eta}-\eta_0)=:\nu(2-\eta_\perp)=1+\frac{3}{14}\epsilon$ with $\eta_\perp=\tilde{\eta}+\eta_0=-\frac{4}{21}\epsilon$ (for S). If $N=\tilde{N}=0$, $M=\tilde{M}=1$ we find an exponent $\gamma_{ss}=\nu(1-\tilde{\eta}-2\eta_0)=\frac{1}{2}+\frac{19}{84}\epsilon$.

Further cluster properties of interest can be obtained from the static moments of the form[12]

$$\left\langle\left[\int d^dx\int_{-\infty}^{+\infty}dt\,\Phi(x,t)\right]^k\tilde{\Phi}(0,0)\right\rangle\sim G_{k,1}(\{q,\omega\}=0) \ . \tag{4.5}$$

For the bulk, we find

$$G_{k,1}(\{q,\omega\}=0)\sim\tau^{\nu(d-2+\tilde{\eta})/2-k\nu d_F} , \tag{4.6}$$

where the fractal dimension d_F of a cluster is given by $d_F=\frac{1}{2}(d+2-\tilde{\eta})=4-\frac{10}{21}\epsilon$. A quantity of interest in dynamic percolation phenomena is the spreading dimension[1] d_e defined by

$$d_e=\frac{d_F}{z} , \tag{4.7}$$

with z being the dynamic critical exponent as given above. For the bulk we have $d_e=2-(\epsilon/14)$ to $O(\epsilon)$.

If we take $N=0$, $\tilde{N}=1$, $M=k$, $\tilde{M}=0$ we find, using

$$G_{k,1}^s(\{q,\omega\}=0)$$
$$\sim\left\langle\left[\int d^{d-1}x\int_{-\infty}^{+\infty}dt\,\Phi_s(x,t)\right]^k\tilde{\Phi}(0,0)\right\rangle , \tag{4.8}$$

$d_F^s=\frac{1}{2}(d-\tilde{\eta}-2\eta_0)=3-(\epsilon/3)$ and $d_e^s=\frac{3}{2}-(\epsilon/24)$.

Now we turn to a discussion of the ordinary transition O where $\tau_s\to\infty$. For this case we consider $N(\Phi)$, $\tilde{N}(\tilde{\Phi})$ fields and $M(\Phi_\perp)$, $\tilde{M}(\tilde{\Phi}_\perp)$ fields. Omitting the term $\tau_s\partial_{\tau_s}$ in (4.1) we obtain a solution in analogy to (4.2) [for $N+\tilde{N}\neq 0$, $M+\tilde{M}\neq 2$, otherwise (4.1) becomes inhomogeneous[20]] where now $\eta_0=\gamma_0^*=-\frac{2}{21}\epsilon$. Corrections due to τ_s^{-1} lead to a dependence of the form $\tau_s^{-1}\tau^\nu$. The exponents follow as $\eta_\parallel=2+\tilde{\eta}+2\eta_0=2-\frac{5}{21}\epsilon$, $\Psi=1-\frac{2}{21}\epsilon$, $\gamma_s=\nu(1-\tilde{\eta}-\eta_0)=\frac{1}{2}+\frac{11}{84}\epsilon$, $\eta_\perp=1+\tilde{\eta}+\eta_0=1-(\epsilon/7)$, and $\gamma_{ss}=\nu(-1-\tilde{\eta}-2\eta_0)=-\frac{1}{2}+\frac{5}{84}\epsilon$. For the fractal dimension we find $d_F^s=\frac{1}{2}(d-2-\tilde{\eta}-2\eta_0)=2-\frac{8}{21}\epsilon$ and the spreading dimension is given by $d_e^s=1-\frac{3}{28}\epsilon$.

V. CONCLUSIONS

We have presented a study of the GEP in a semi-infinite geometry. The reason for our investigation has been twofold. Fractal structures which have emerged in a large variety of growth processes in all natural sciences

are still an object of intensive research. In particular, the influence of the global environment geometry on a growth process is an important issue. The modifications of the global geometry are expected to affect typical fractal properties such as the mass of a cluster, the fractal dimension, the spreading dimension, etc. One simple, yet physically important way to change the global symmetry is by introduction of a free surface or wall in an environment. This kind of problem has the additional advantage that the mathematical techniques for its solution have already been developed in the context of surface phase transitions. As most of the research in this direction has concentrated on equilibrium bulk-driven phase transitions, we find it necessary to initiate research on systems which are characterized by a nonequilibrium (bulk) phase transition. Nonequilibrium phase transitions occur in various evolution processes ranging from reaction-diffusion systems in chemistry to population dynamics in biology. The study of such phenomena which is ultimately aimed at the understanding of self-organization in nonequilibrium systems is of particular importance for the areas of epidemiology and ecology. And it is just in these fields where all processes taking place under realistic conditions are subjected to boundaries, i.e., walls, surfaces, and finite environments. Thus it should become increasingly clear that the investigation of surface effects for processes such as the one studied here is of principal interest and further research in this direction should be conducted.

In conclusion, we should mention that the *static exponents* which we have calculated in Sec. IV can also be determined starting from the Potts model, which for $n \to 0$ (n being the number of components) describes the percolation problem.[21] This has been done in Ref. 22 for the ordinary transition and the results obtained agree with ours. Meanwhile also a study for the special transition has been performed.[23] The exponents reported there are identical to those presented in Sec. IV.

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APPENDIX

In this appendix we provide some details of the renormalization procedure discussed in Sec. III. We start from a modified version of Eq. (3.4),

$$I(\omega|z,z') = -w^2\lambda^3 \int d^{d-1}y \int_0^\infty ds\, e^{-i\omega s} G_0(y,s|z,z') \int_0^s dx\, G_0(y,x|z,z') \,, \tag{A1}$$

where we have put $\mathbf{q}=0$ for computational simplicity. The Green's functions in (A1) are given by

$$G_0(y,s|z,z') = G_0^{(d-1)}(y,s)(4\pi\lambda s)^{-1/2}\left[\exp\left[-\frac{(z-z')^2}{4\lambda s}\right] \pm \exp\left[-\frac{(z+z')^2}{4\lambda s}\right]\right] , \tag{A2}$$

etc. according to (3.1) and (3.2). $G_0^{(d-1)}$ in (A2) denotes the (bulk) response propagator for the $(d-1)$ parallel directions.

Upon inserting the Green's functions in (A1), using the property

$$\int d^{d-1}y\, G_0^{(d-1)}(y,s) G_0^{(d-1)}(y,x) = G_0^{(d-1)}(x+s) \tag{A3}$$

and performing a variable transformation $x=\alpha s$ we obtain

$$\begin{aligned} I(\omega|z,z') = -w^2\lambda^3 \int_0^\infty ds\, e^{-i\omega s} \int_0^1 d\alpha\, G_0^{(d-1)}(s(1+\alpha))(4\pi\lambda\alpha^{1/2})^{-1} \\ \times \left\{ \exp\left[-\frac{(z-z')^2}{4\lambda s}\left[\frac{\alpha+1}{\alpha}\right]\right] \pm \exp\left[-\frac{1}{4\lambda s}[(z-z')^2+\alpha^{-1}(z+z')^2]\right] \right. \\ \left. \pm \exp\left[-\frac{1}{4\lambda s}[(z+z')^2+\alpha^{-1}(z-z')^2]\right] + \exp\left[-\frac{(z+z')^2}{4\lambda s}\left[\frac{\alpha+1}{\alpha}\right]\right] \right\} , \end{aligned} \tag{A4}$$

with

$$G_0^{(d-1)}(s(1+\alpha)) = (4\pi\lambda)^{(1-d)/2}[s(1+\alpha)]^{(1-d)/2} \,. \tag{A5}$$

Consider the first term in (A4) which stems from the bulk part of the propagators and thus exhibits translational invariance in $(z-z')$:

$$I_B(\omega|z,z')=-w^2\lambda\int_0^1 d\alpha\,\alpha^{-1/2}(1+\alpha)^{(1-d)/2}(4\pi)^{-d/2}\pi^{-1/2}$$
$$\times\left[\left[\frac{\alpha+1}{\alpha}\right]^{1/2}|z-z'|\left[\frac{i\omega}{\lambda}\right]^{1/2}\right]^{(3-d)/2}\left[\frac{2i\omega}{\lambda}\right]^{(d-3)/2}$$
$$\times K_{(d-3)/2}\left[\left[\frac{i\omega}{\lambda}\right]^{1/2}|z-z'|\left[\frac{\alpha+1}{\alpha}\right]^{1/2}\right] \tag{A6}$$

utilizing formula 8.432(8) of Ref. 24. The very fact that $I_B(\omega|z-z')$ contains pole terms can be seen in the following way. For $|z-z'|=\bar{z}$ we find, using

$$K_\nu(\bar{z})=\frac{\pi}{2\sin(\nu\pi)}\left[\left[\frac{\bar{z}}{2}\right]^{-\nu}\sum_{k=0}^{\infty}\frac{(\bar{z}/2)^{2k}}{k!\Gamma(k+1-\nu)}-\left[\frac{\bar{z}}{2}\right]^{\nu}\sum_{k=0}^{\infty}\frac{(\bar{z}/2)^{2k}}{k!\Gamma(k+1+\nu)}\right], \tag{A7}$$

$$I_B(\omega|\bar{z})=-w^2\lambda\int_0^1 d\alpha\,\alpha^{-1/2}(1+\alpha)^{(1-d)/2}(4\pi)^{-d/2}\pi^{-1/2}\left[\frac{2i\omega}{\lambda}\right]^{(d-3)/2}\Gamma\left[\frac{d-3}{2}\right]2^{(1-d)/2}$$
$$\times\left[\left[\frac{a\bar{z}}{2}\right]^{3-d}+\frac{2}{5-d}\left[\frac{a\bar{z}}{2}\right]^{5-d}+\cdots\right], \tag{A8}$$

where $a=(i\omega/\lambda)^{1/2}[(\alpha+1)/\alpha]^{1/2}$. From (A8) we may extract the distribution content with the help of exponential test functions

$$\int_0^\infty dz\int_0^\infty dz'\,|z-z'|^{-3+\epsilon}e^{-\beta z-\beta' z'}=\frac{1}{2\epsilon}\frac{\beta^2+\beta'^2}{\beta+\beta'}+O(\epsilon^0), \tag{A9}$$

$$\int_0^\infty dz\int_0^\infty dz'\,|z-z|^{-1+\epsilon}e^{-\beta z-\beta' z'}=\frac{1}{\epsilon}\frac{2}{\beta+\beta'}+O(\epsilon^0), \tag{A10}$$

where we have put $d=6-\epsilon$. This corresponds to

$$\frac{1}{|z-z'|^{3-\epsilon}}=\frac{1}{2\epsilon}[\delta''(z|z')+\delta''(z'|z)]+O(\epsilon^0), \tag{A11}$$

$$\frac{1}{|z-z'|^{1-\epsilon}}=\frac{2}{\epsilon}\delta(z-z')+O(\epsilon^0), \tag{A12}$$

where we have introduced the definition

$$\int_0^\infty dz\,\delta''(z|z')f(z):=f''(z'). \tag{A13}$$

Performing the remaining α integrations we find

$$I_B(\omega|z-z')=-\tfrac{1}{6}(4\pi)^{-d/2}w^2\lambda\left[\frac{i\omega}{\lambda}\right]^{-\epsilon/2}\frac{1}{\epsilon}\left[\frac{1}{2}[\delta''(z|z')+\delta''(z'|z)]-\frac{9}{2}\left[\frac{i\omega}{\lambda}\right]\delta(z-z')\right]+O(\epsilon^0). \tag{A14}$$

Now we turn to a discussion of the remaining three terms in (A4) which contribute to the surface term $I_s(\omega|z,z')$. We are interested in evaluating the pole terms for $z,z'\to 0$ and thus the main point consists in determining the distribution content of I_s. In principle we could proceed as in the case of the bulk part by using for each term the (modified) Bessel function representation similar to (A6).

It is simpler, however, to determine the distribution character of I_s in the following way: as $z,z'\to 0$ separately, $I_s(\omega|z,z')$ has to be of the form

$$I_s(\omega|z,z')=A\,[\delta^k(z)\delta(z')+\delta(z)\delta^k(z')], \tag{A15}$$

where k denotes some derivative acting on the argument of $\delta(z)$ or $\delta(z')$ and A is proportional to the pole term. Now k can be found from simple dimensional considerations (by simply taking the μ dimensions into account) and one finds $k=1$. Then A can be calculated by considering moments such as

$$\int_0^\infty dz\int_0^\infty dz'(z+z')\exp\left[-\frac{1}{4\lambda s}\left[\frac{\alpha+1}{\alpha}\right](z+z')^2\right]=2\pi^{1/2}(\lambda s)^{3/2}\left[\frac{\alpha}{\alpha+1}\right]^{3/2}, \tag{A16}$$

where we have taken the last term in (A4) as an example. In this way, we obtain, adding up all three terms,

$$I_s(\omega) =: \tfrac{1}{2}\int_0^\infty ds\, e^{-i\omega s}\int_0^\infty dz\int_0^\infty dz'(z+z')I_s(z,z'|s)$$
$$= -\tfrac{1}{2}w^2\lambda^3\int_0^\infty ds\, e^{-i\omega s}\int_0^1 d\alpha\, G_0^{(d-1)}(s(1+\alpha))(4\pi\lambda\alpha^{1/2})^{-1}$$
$$\times\{2\pi^{1/2}(\lambda s)^{3/2}[\alpha(1+\alpha)^{-1}]^{3/2}\pm 2\pi^{1/2}(\lambda s)^{3/2}[\alpha(1+\alpha)]^{1/2}\}\ . \tag{A17}$$

Performing the α and the s integration we secure

$$I_s(\omega) = -\lambda w^2 G_\epsilon \frac{1}{\epsilon}\left[\frac{\lambda}{i\omega}\right]^{\epsilon/2}\times\begin{cases}\frac{7}{12}\\ -\frac{5}{12}\end{cases}, \tag{A18}$$

where $G_\epsilon = \Gamma(1+\epsilon/2)/(4\pi)^{d/2}$ and where $\frac{7}{12}$ refers to S and $-\frac{5}{12}$ refers to O. Extracting the pole term we find to one-loop order

$$I_s(\omega|z,z') = \lambda w^2\frac{G_\epsilon}{\epsilon}\times\begin{Bmatrix}\frac{7}{3}\\ -\frac{5}{3}\end{Bmatrix}\times[\delta'(z)\delta(z')+\delta(z)\delta'(z')]\ , \tag{A19}$$

where we have used

$$\int_0^\infty dx\, f(x)\delta^{(n)}(x) = \frac{(-1)^n}{2}f^{(n)}(0)\ . \tag{A20}$$

The pole terms arising from both the bulk and the surface contribution in connected response functions having external legs on the surface can be determined with the help of (A6) and (A14). This concludes our discussion.

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