

**University of Massachusetts Amherst**

---

**From the Selected Works of Barry R Holstein**

---

May 29, 2007

# Parametrization dependence of the energy-momentum tensor and the metric

NEJ Bjerrum-Bohr  
JF Donoghue  
BR Holstein



Available at: [https://works.bepress.com/barry\\_holstein/91/](https://works.bepress.com/barry_holstein/91/)

# Parametrization dependence of the energy-momentum tensor and the metric

N. E. J Bjerrum-Bohr,<sup>1,2</sup> John F. Donoghue,<sup>3</sup> and Barry R. Holstein<sup>3</sup>

<sup>1</sup>*Department of Physics, University of Wales Swansea, Swansea, SA2 8PP, United Kingdom*

<sup>2</sup>*Institute for Advanced Study, Princeton, New Jersey 08540, USA*

<sup>3</sup>*Department of Physics University of Massachusetts Amherst, Massachusetts 01003, USA*

(Received 13 October 2006; published 29 May 2007)

We use results by Kirilin to comment that in general relativity the nonleading terms in the energy-momentum tensor of a particle depends on the parameterization of the gravitational field. The classical observables are parameterization independent after a change in coordinates. The quantum effects that emerge within the same calculation of the metric also depend on the parameterization and a full quantum calculation requires the inclusion of further diagrams. However, within a given parameterization the quantum effects calculated by us in a previous paper are well defined. Flaws of Kirilin's proposed alternate metric definition are described and we explain why the diagrams that we calculated are the appropriate ones.

DOI: [10.1103/PhysRevD.75.108502](https://doi.org/10.1103/PhysRevD.75.108502)

PACS numbers: 04.60.-m

In Ref. [1], we calculated the long-distance one-loop corrections to the energy-momentum tensor of spin zero and spin one-half particles due to graviton loops and used this result to derive the long-distance corrections to the metric. The diagrams are shown in Fig. 1. This procedure reproduces the leading classical nonlinearities in the Schwarzschild and Kerr metrics—in harmonic gauge—and produces novel quantum corrections linear in  $\hbar$ .

As noted in our paper, this calculation is not a full quantum calculation. The metric is not a fully quantum concept and further diagrams are needed in order to produce a proper quantum amplitude. This feature is known in the context of the semiclassical Einstein equations [2,3] but is also manifest in quantum loop calculations. Indeed, in the companion paper [4], we completed the calculation of the classical and quantum corrections to the gravitational scattering amplitude. The full set of diagrams is shown in Fig. 2. This result has also been calculated by Khriplovich and Kirilin [5] and our results agree.

In the preceding comment [6], Kirilin criticizes our choice of diagrams to include in the definition of the metric, and proposes a metric derived from the work on the scattering amplitude [5]. This criticism is based on a calculation which shows that the metric derived from Fig. 1 is not invariant under the reparameterization of the graviton field. This is interesting and appears to be correct—we will comment further below. It implies that when discussing the quantum corrections to the metric one must specify not only the gauge but also the field parameterization. However, we will argue that our definition of the metric is still to be preferred and that the one proposed by Kirilin has a number of flaws.

Kirilin considers the family of parameterizations of the metric field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \frac{a}{4} h_{\mu\lambda} h_{\nu}^{\lambda}, \quad (1)$$

where  $a$  is a free parameter. Changes in  $a$  lead to changes

in the various vertices entering the Feynman diagrams [6], in particular, the triple graviton coupling and also the coupling of two gravitons to the matter field. For example, when the results of [6] are applied to the energy-momentum tensor of a scalar particle

$$\langle p_2 | T_{\mu\nu}(x) | p_1 \rangle = \frac{e^{i(p_2 - p_1) \cdot x}}{\sqrt{4E_2 E_1}} [2P_{\mu} P_{\nu} F_1(q^2) + (q_{\mu} q_{\nu} - \eta_{\mu\nu} q^2) F_2(q^2)], \quad (2)$$

with  $P_{\mu} = (p_1 + p_2)_{\mu}/2$  and  $q_{\mu} = (p_1 - p_2)_{\mu}$ , the consideration of the diagrams of Fig. 1(a) and 1(b), show that the matrix element of  $T_{\mu\nu}$  depends on the parameterization of the gravitational field. Specifically

$$F_1(q^2) = 1 + \frac{Gq^2}{\pi} \left( -\frac{3}{4} \log \frac{-q^2}{m^2} + \frac{1}{16} \frac{\pi^2 m}{\sqrt{-q^2}} \right) + \dots$$

$$F_2(q^2) = -\frac{1}{2} + \frac{Gm^2}{\pi} \left( \left( -2 - \left( \frac{11}{3} a + a^2 \right) \right) \log \frac{-q^2}{m^2} + \frac{7 + 4a}{8} \frac{\pi^2 m}{\sqrt{-q^2}} \right) + \dots \quad (3)$$

Here we have displayed only the nonanalytic terms that give long range modifications. As shown in Ref. [1], the square-root terms lead to classical corrections, while the logarithms lead to quantum corrections. We notice that the

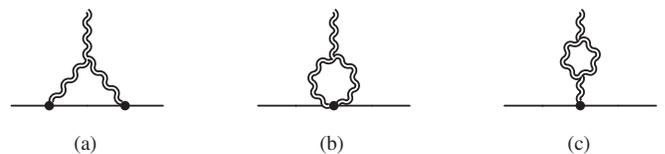


FIG. 1. Feynman diagrams used to calculate the long-distance corrections to the energy-momentum tensor and the metric of a scalar particle. Here the doubly wiggly lines represent gravitons.

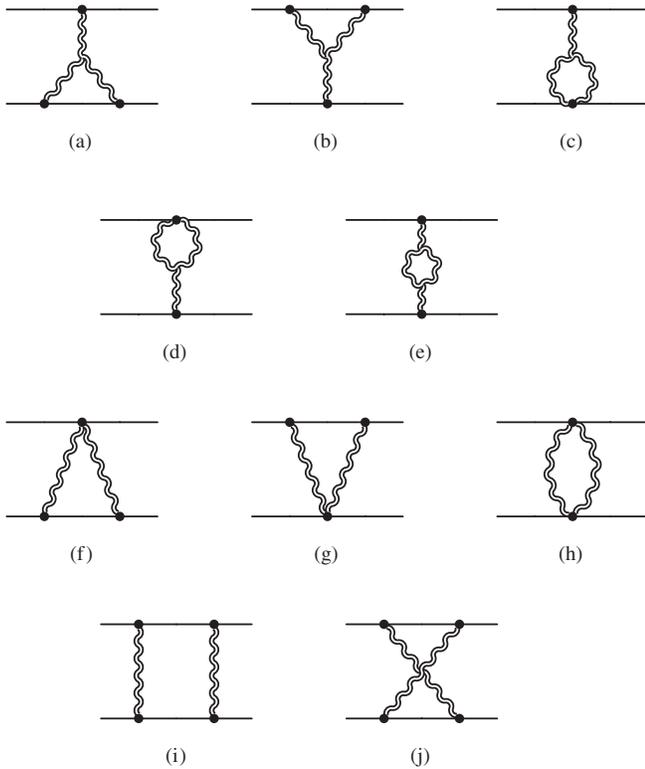


FIG. 2. The diagrams used to calculate the long-distance corrections to the scattering amplitude.

form factor  $F_1$  is unaffected, while  $F_2$  is modified in *both* the classical and quantum components.

It is interesting that the energy-momentum tensor depends on the gravitational field parameterization. In non-gravitational theories, the energy and momentum are well-defined quantities and such a matrix element suffers no ambiguities. However, in general relativity, Noether's theorem does not produce a well-defined energy because the action is invariant under general coordinate transformations. This feature is manifest in the one-loop corrections to the particle's energy, and also in the shift of the gravitational vertices. For example, the  $a$ -dependent modification of the classical component arises from the energy and momentum contained in the classical gravitational field surrounding a particle, which in the loop expansion is described by Fig. 1(a) [7]. However, the energy-momentum tensor for the gravitational field is a pseudo-tensor<sup>1</sup> that depends on the field parameterization (i.e. on  $a$ ). The variation in that tensor means that the amount of energy and momentum that is carried in the classical field also varies with the parameterization. The variation in the quantum component comes from this effect, plus an additional change which results from a shift in the two graviton ( $\phi\phi hh$ ) coupling, which is involved in Figs. 1(b).

<sup>1</sup>It is equivalent to the triple graviton coupling when one of the gravitons is taken as an external gravitational field.

While the classical parameterization dependence can be shown to not influence observables—causing an modification that can be removed by a change in coordinates—the quantum parameterization dependence cannot be removed by a change in coordinates. The consequence of this residual dependence is that, when performing the calculation of a full quantum amplitude, one must take care to use the same field parameterization for the entire calculation. This aspect is not unique to gravity. For example, in the internucleon potential, a reparameterization of the pion field modifies the central nuclear potential [8], although the sum of pionic exchanges and the central potential is invariant. In both cases, a consistent calculation of intermediate results must involve a full specification of the field variables, which in the gravitational case means both the gauge and the field definition.

Kirilin [6] proposes instead to use a different definition of the metric, based on the full set of diagrams shown in Fig. 2. We do not feel that this alternative definition is acceptable. Flaws of this set of diagrams include:

- (1) These diagrams do not reproduce the classical corrections to the metric. Instead they reproduce the classical terms in the post-Newtonian potential. These are *not* the same. As discussed above, our diagrams capture the correct classical metric corrections. There are additional classical corrections are found in diagrams 2(f)–2(j), shifting the result to the post-Newtonian potential rather than the metric. Kirilin's set of diagrams does not give the correct classical metric.
- (2) The diagrams of Fig. 2 form a gauge-invariant set. In contrast, the metric depends very explicitly on the gauge for the gravitational field. In Ref. [5], the authors correct for this deficiency by hand, but the correct gauge dependence is not a feature of either the classical or quantum components of these diagrams.
- (3) The diagrams proposed by Kirilin depend on the existence of a second nonrelativistic particle in the diagrams. For example, it includes the vertex correction to the gravitational vertex of the *other* particle, Fig. 2(b). The classical corrections depend on the mass of the other particle, and there exist terms in the interaction that depend on the spin of the other particle. There is also no indication that the same result would be obtained if the other particle were an ultrarelativistic particle such as a photon. Of course, one can deal with these features in an ad-hoc way by taking one mass much larger than the other and excising the spin dependence by hand. However, the basic feature of a metric is that it is the property of a single object independent of the existence of a second body. These diagrams do not have that property.

- (4) The mechanism that is required in order to obtain the invariance of the classical observables, described above and in Ref. [6], does not work with the classical part of the diagrams of Fig. 2 but does if one uses those of Fig. 1.
- (5) In the case of electromagnetic corrections to the metric, there is no ambiguity due to reparameterization of the gravitational field, because gravity in this case is purely classical. Here, both the diagrams corresponding to Fig. 1 and 2 have been calculated [9,10],<sup>2</sup> and it has been demonstrated that the diagrams of Fig. 1 give the correct metric. Kirilin's procedure in this case would be incorrect. The above comments 1–4 also apply to the photonic calculation.

Our definition of the metric includes the full set of one-loop diagrams for the gravitational field around a single body. It reproduces correctly the classical terms in the

metric, and displays the correct gauge dependence. It parallels the well-defined corrections due to photonic loops. Because of the parameterization dependence of the energy-momentum matrix element, the metric also displays a parameterization dependence. However, the classical dependence within these diagrams is exactly what is required in order that classical observables be parameterization independent. Kirilin's alternate definition is shown to be inappropriate for the classical components. While the metric is not a full quantum calculation, within a given parameterization our definition can be completed to give the calculation of the full quantum amplitude. In summary, we stand by our choice of diagrams describing the classical and quantum corrections to the metric.

The research of J. F. D. and B. R. H. has been supported in part by the U.S. National Science Foundation under Grant No. NSF 05-53304 and that of N. E. J. B. B. by the PPARC of the UK and the Department of Energy under Grant No. DE-FG02-90ER40542.

<sup>2</sup>The vacuum polarization diagram is not relevant in this case.

- 
- [1] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, *Phys. Rev. D* **68**, 084005 (2003); **71**, 069904(E) (2005).
- [2] G. A. Vilkovisky, *Nucl. Phys. B* **234**, 125 (1984); B. S. DeWitt, *Quantum Field Theory and Quantum Statistics* (Hilger, Bristol, 1987); G. A. Vilkovisky, in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. DeWitt* (Adam Hilger Ltd., Bristol, England, 1984), p. 169.
- [3] D. A. R. Dalvit and F. D. Mazzitelli, *Phys. Rev. D* **56**, 7779 (1997).
- [4] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, *Phys. Rev. D* **67**, 084033 (2003); **71**, 069903(E) (2005).
- [5] I. B. Khriplovich and G. G. Kirilin, *Zh. Eksp. Teor. Fiz.* **95**, 1139 (2002) [*J. Exp. Theor. Phys.* **95**, 981 (2002)]. *Zh. Eksp. Teor. Fiz.* **125**, 1219 (2004) [*J. Exp. Theor. Phys.* **98**, 1063 (2004)].
- [6] G. G. Kirilin, arXiv:gr-qc/0601020.
- [7] B. R. Holstein and J. F. Donoghue, *Phys. Rev. Lett.* **93**, 201602 (2004).
- [8] J. L. Friar, *Phys. Rev. C* **60**, 034002 (1999).
- [9] J. F. Donoghue, B. R. Holstein, B. Garbrecht, and T. Konstandin, *Phys. Lett. B* **529**, 132 (2002).
- [10] N. E. J. Bjerrum-Bohr, *Phys. Rev. D* **66**, 084023 (2002).