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Second Born approximation and Coulomb scattering

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We examine the problem of calculating higher order contributions to the Coulomb scattering amplitude. To make contact with the well known result, it is necessary to modify the conventional definition of the scattering amplitude.
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I. INTRODUCTION

The calculation of the elastic potential scattering amplitude for a particle of mass m is a staple of quantum mechanics courses. The result of the traditional scattering theory calculation yields the exact result[1]

$$\frac{d\sigma}{d\Omega_f}=\left|-\frac{m}{2\pi}T(E_i)\right|^2, \tag{1}$$

where

$$T(E_i)=\langle\phi_f|\hat{V}|\psi_i^{(+)}\rangle \tag{2}$$

is the transition amplitude. Here $|\phi_f\rangle$ is a plane wave state that satisfies the free Schrödinger equation

$$\hat{H}_0|\phi_f\rangle=E_i|\phi_f\rangle, \tag{3}$$

and $|\psi_i^{(+)}\rangle$ is a state with outgoing boundary conditions that satisfies the full Schrödinger equation

$$(\hat{H}_0+\hat{V})|\psi_i^{(+)}\rangle=E_i|\psi_i^{(+)}\rangle. \tag{4}$$

The relation

$$|\psi_i^{(+)}\rangle=|\phi_i\rangle+\frac{1}{E_i-\hat{H}_0+i\epsilon}\hat{V}|\psi_i^{(+)}\rangle, \tag{5}$$

which connects the free and interacting states, allows a perturbative expression for the scattering amplitude known as the Born series in powers of the potential:

$$f_{\rm B}(p_0,\theta)=-\frac{m}{2\pi}\left[\langle\phi_f|\hat{V}|\phi_i\rangle+\sum_n\frac{\langle\phi_f|\hat{V}|n\rangle\langle n|\hat{V}|\phi_i\rangle}{E_i-E_n+i\epsilon}+\ \ldots\ \right]. \tag{6}$$

The leading term in this expansion leads to the first Born approximation for the cross section:

$$\frac{d\sigma_{\rm B,1}}{d\Omega_f}=\left|-\frac{m}{2\pi}\langle\phi_f|\hat{V}|\phi_i\rangle\right|^2. \tag{7}$$

For the Coulomb interaction

$$V_C(r)=\frac{\alpha}{r}, \tag{8}$$

the first Born approximation to the scattering amplitude is

$$f_{\rm B,1}^C(p_0,\theta)=-\frac{m}{2\pi}\int d^3r\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{\alpha}{r}=-\frac{2m\alpha}{\boldsymbol{q}^2}, \tag{9}$$

where $\boldsymbol{q}=\boldsymbol{p}_i-\boldsymbol{p}_f$ is the momentum transfer. The corresponding Born cross section,

$$\frac{d\sigma_{\rm B,1}^{\rm C}}{d\Omega_f}=|f_{\rm B,1}^C(p_0,\theta)|^2=\frac{4m^2\alpha^2}{q^4}=\frac{m^2\alpha^2}{4p_0^4\sin^4(\theta/2)}, \tag{10}$$

agrees not only with its well-known classical value,[2] but also with the exact quantum mechanical solution. The latter agreement occurs because the exact scattering amplitude, obtained from the analytic solution to the Schrödinger equation, differs from its Born value by a simple phase factor[3]

$$f_{\rm exact}^C(p_0,\theta)=-\frac{2m\alpha}{q^2}\exp\left[-2i\arg\Gamma(1-i\xi)-i\xi\log\frac{\boldsymbol{q}^2}{4p_0^2}\right], \tag{11}$$

where $\xi=m\alpha/p_0$ and $E_i=p_0^2/2m$.

We have demonstrated the identity of the exact solution and the Born approximation for lowest order Coulomb scattering

$$f_{{\rm exact},\mathcal{O}(\alpha)}^C(p_0,\theta)=f_{\rm B,1}^C(p_0,\theta)=-\frac{m}{2\pi}\langle\phi_f|\hat{V}|\phi_i\rangle. \tag{12}$$

If we expand in powers of the fine structure constant α, we might expect that the $\mathcal{O}(\alpha^2)$ contribution to the exact scattering amplitude

$$f_{{\rm exact},\mathcal{O}(\alpha^2)}^C(p_0,\theta)=-\frac{2m\alpha}{q^2}\left[-2i\xi\gamma-i\xi\log\frac{\boldsymbol{q}^2}{4p_0^2}\right], \tag{13}$$

where $\gamma=0.5772\ldots$ is Euler's constant, should agree precisely with the second order term in the perturbative expansion of the exact scattering amplitude

$$f_{{\rm exact},\mathcal{O}(\alpha^2)}^C(p_0,\theta)\overset{?}{=}f_{\rm B,2}^C(p_0,\theta)=-\frac{m}{2\pi}\sum_n\frac{\langle\phi_f|\hat{V}|n\rangle\langle n|\hat{V}|\phi_i\rangle}{E_i-E_n+i\epsilon}. \tag{14}$$

In the next section we evaluate the second Born term and find that Eq. (14) is not satisfied.

II. SECOND BORN APPROXIMATION

A. Paradise lost

In Sec. I we showed the identity of the leading order contribution to the exact Coulomb scattering amplitude and the lowest order Born approximation [see Eq. (12)]. To check our speculation, Eq. (14), we need to evaluate the second Born contribution to Coulomb scattering

$$f^C_{\mathrm{B},2}(p_0,\theta) = -\frac{m}{2\pi}\sum_n \frac{\langle\phi_f|\hat{V}|n\rangle\langle n|\hat{V}|\phi_i\rangle}{E_i - E_n + i\epsilon} \tag{15a}$$

$$= -\frac{m}{2\pi}\int \frac{d^3\ell}{(2\pi)^3} \times \frac{-4\pi\alpha}{|\boldsymbol{p}_f - \ell|^2}\frac{1}{\frac{p_0^2}{2m} - \frac{\ell^2}{2m} + i\epsilon}\frac{-4\pi\alpha}{|\ell - \boldsymbol{p}_i|^2}. \tag{15b}$$

The integration can be performed by combining the second and third pieces via the Feynman parameter x:

$$\frac{1}{\frac{p_0^2}{2m} - \frac{\ell^2}{2m}}\frac{1}{|\ell - \boldsymbol{p}_i|^2} = -2m\int_0^1 dx \times \frac{1}{[(\ell - \boldsymbol{p}_i x)^2 + p_i^2 x(1-x) - p_0^2(1-x)]^2}. \tag{16}$$

Then, by changing variables to $\boldsymbol{s} = \ell - \boldsymbol{p}_f$ we find

$$f^C_{\mathrm{B},2}(p_0,\theta) = \frac{2\alpha^2 m^2}{\pi^2}\int_0^1 dx\int_0^\infty ds\int d\Omega \frac{1}{(s^2 - 2\boldsymbol{s}\cdot\boldsymbol{u} + K)^2}, \tag{17}$$

where $\boldsymbol{u} = \boldsymbol{p}_f - \boldsymbol{p}_i(1-x)$ and $K = \boldsymbol{q}^2 x - (p_0^2 - p_f^2)(1-x)$. The integration over solid angle and s may now be performed directly, leaving only a one-dimensional integration[4]

$$f^C_{\mathrm{B},2}(p_0,\theta) = i8\pi m^2 p_0\alpha^2 \times\int_0^1 dy \frac{1}{4yp_0^2\boldsymbol{q}^2 + (1-y)^2(p_0^2 - p_f^2)(p_0^2 - p_i^2)}. \tag{18}$$

(It is important here that we keep $p_0^2 \neq p_f^2, p_i^2$ because otherwise the result diverges.) The remaining integration can now be done directly, yielding, in the limit $p_0^2 - p_f^2 \sim p_0^2 - p_i^2 \ll |\boldsymbol{q}|^2$

$$f^C_{\mathrm{B},2}(p_0,\theta) = i\frac{2m\alpha\xi}{q^2}\log\frac{4p_0^2 q^2}{(p_0^2 - p_f^2)(p_0^2 - p_i^2)}. \tag{19}$$

The problem is now clear: if the $\mathcal{O}(\alpha^2)$ exact and second Born expressions are subtracted, the result is nonzero!

$$f^C_{\mathrm{exact},\mathcal{O}(\alpha^2)}(p_0,\theta) - f^C_{\mathrm{B},2}(p_0,\theta) = -\frac{2m\alpha}{q^2}\left[-i2\xi\gamma - i\xi\log\frac{(p_0^2 - p_i^2)(p_0^2 - p_f^2)}{16p_0^4}\right] \neq 0. \tag{20}$$

B. Paradise regained

The resolution of the "mystery" presented in Eq. (20) is associated with the definition of the scattering amplitude. For ordinary scattering the full momentum space transition amplitude, which represents the change from initial momentum $\boldsymbol{p}_i$ to the final momentum $\boldsymbol{p}_f$, is given by[5]

$$-iK_F(\boldsymbol{p}_f,\boldsymbol{p}_i) = D_F^{(0)}(\boldsymbol{p}_f,p_0)\Big[-iV(\boldsymbol{p}_f - \boldsymbol{p}_i) + (-i)^2 \times\int\frac{d^3\ell}{(2\pi)^3}V(\boldsymbol{p}_f - \ell)D_F^{(0)}(\ell,p_0)V(\ell - \boldsymbol{p}_i) + \ldots\Big]D_F(\boldsymbol{p}_i,p_0), \tag{21}$$

where

$$D_F^{(0)}(\boldsymbol{p},p_0) = \frac{i}{p_0^2/2m - p^2/2m + i\epsilon} \tag{22}$$

is the free propagator. The scattering amplitude is then given by

$$f^{\mathrm{Born}}(p_0,\theta) = -\frac{m}{2\pi}\frac{K_F(\boldsymbol{p}_f,\boldsymbol{p}_i)}{D_F^{(0)}(\boldsymbol{p}_f,p_0)D_F^{(0)}(\boldsymbol{p}_i,p_0)}. \tag{23}$$

The free propagator has the Fourier transform

$$\left\langle \boldsymbol{r}_f\left|\frac{i}{E_i - \hat{H}_0 + i\epsilon}\right|\boldsymbol{r}_i\right\rangle = -\frac{m}{2\pi r}e^{ip_0 r}, \tag{24}$$

which is associated with the asymptotic form of the outgoing scattered wave

$$\psi(r) \xrightarrow{r\to\infty} \frac{1}{r}e^{ip_0 r}. \tag{25}$$

The problem is that in the Coulomb case the outgoing wave is not spherical because of the long range nature of the force. Instead the asymptotic form of the Coulomb wavefunction is[3]

$$\psi_C(r) \xrightarrow{r\to\infty} \frac{1}{r}e^{i(p_0 r - \xi\log 2p_0 r)}, \tag{26}$$

whose Fourier transform yields the propagator[6]

$$D_F^C(\boldsymbol{p},p_0) = D_F^{(0)}(\boldsymbol{p},p_0)\Gamma(1 - i\xi)\exp\left[-i\xi\log\frac{p_0^2 - \boldsymbol{p}^2}{4p_0^2}\right]. \tag{27}$$

Equation (27), which generates the proper asymptotic dependence of the wavefunction, should be used to define the Coulomb scattering amplitude[7]

$$f^C_{\mathrm{exact}}(p_0,\theta) = -\frac{m}{2\pi}\frac{K_F(\boldsymbol{p}_f,\boldsymbol{p}_i)}{D_F^C(\boldsymbol{p}_f,p_0)D_F^C(\boldsymbol{p}_i,p_0)} \tag{28a}$$

$$= -\frac{m}{2\pi}\frac{K_F(\boldsymbol{p}_f,\boldsymbol{p}_i)}{D_F^{(0)}(\boldsymbol{p}_f,p_0)D_F^{(0)}(\boldsymbol{p}_i,p_0)} \times \frac{1}{[\Gamma(1-i\xi)\exp(i\xi\log[(p_0^2 - p_f^2)/4p_0^2])][\Gamma(1-i\xi)\exp(i\xi\log[(p_0^2 - p_i^2)/4p_0^2])]}. \tag{28b}$$

The Coulomb scattering amplitude thus differs from that given by the usual scattering series via

$$f^C_{\rm exact}(p_0,\theta)=\frac{f^{\rm Born}_C(p_0,\theta)}{[\Gamma(1-i\xi)\exp{(i\xi\log{[(p_0^2-p_f^2)/4p_0^2]})}]\Gamma(1-i\xi)\exp{(i\xi\log{[(p_0^2-p_i^2)/4p_0^2]})}}. \tag{29}$$

To $\mathcal{O}(\alpha^2)$ we find

$$f^C_{{\rm exact},\mathcal{O}(\alpha^2)}(p_0,\theta)-f^C_{{\rm B},2}(p_0,\theta) = \frac{2m\alpha^2}{q^2}\left[-2i\xi\gamma-i\xi\log\frac{(p_0^2-p_i^2)(p_0^2-p_f^2)}{16p_0^4}\right], \tag{30}$$

which is in precise agreement with Eq. (20) found earlier—the apparent paradox has been resolved.

III. CONCLUSIONS

If the conventional scattering series is used to calculate the Coulomb scattering amplitude, then although the exact and Born series results agree at lowest order, discrepancies between the two expressions arise for higher order Born terms. We calculated the second Born term explicitly and showed that this disagreement can be resolved by using the incoming and outgoing propagators, which take into account the proper asymptotic behavior of the Coulomb wavefunction. The agreement at lowest order is due to the fact that the difference between the free and Coulomb propagators in Eq. (27) begins at $\mathcal{O}(\alpha)$. This discussion could stimulate interest in the context of an advanced quantum mechanics course.

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