Stability of networks under tension and pressure

Anthony Roy Day, John Carroll University
H. Yan
M. F. Thorpe

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H. Yan, A. R. Day, and M. F. Thorpe

Department of Physics and Astronomy and Center for Fundamental Materials Research, Michigan State University, East Lansing, Michigan 48824

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The number of zero-frequency modes of an elastic network is an important quantity in determining the stability of the network. We present a constraint-counting method for finding this number in general central-force networks that are under an external tension. The technique involves isolating the backbone and then counting constraints in the same way as for free standing networks. A detailed example of this counting is given for a random two-dimensional network subject to an external tension. The results are shown to agree with the number of zero-frequency modes as determined by a direct matrix diagonalization.

I. INTRODUCTION

The study of the stability of network structures has been of importance since the pioneering but not widely known work of Maxwell.

Suppose a two-dimensional structure is constructed from $b$ bars and $j$ joints as shown in Fig. 1. The bars are of fixed length and the joints are such that there are no direct constraints on the angles. A single joint is very loose but can be made firm by completing a triangle, which has 3 bars and 3 joints, as occurs in the two-dimensional framework shown in Fig. 1. The smallest three-dimensional rigid unit would be the tetrahedron with 6 bars and 4 joints.

In the network shown in Fig. 1, there are $2j-3$ internal degrees of freedom. Here 2 is the dimensionality and 3 comes from the two rigid translations and one rigid rotation. Maxwell counted the number of constraints as $b$ so that the network is stable if

$$b \geq 2j-3$$

and unstable if

$$b < 2j-3 .$$

Similar arguments in $d$ dimensions lead to a stable network if

$$b \geq dj-n_0$$

and an unstable network if

$$b < dj-n_0 .$$

In these expressions $d$ is the dimensionality and

$$n_0 = d (d+1)/2$$

is the total number of rigid macroscopic translations and rotations. In two dimensions $n_0 = 3$ as discussed above, and in three dimensions $n_0 = 6$ which are the three rigid translations and the three rigid rotations. It is important to understand that Eqs. (1) and (2) are not exact because the constraints provided by the bars are not all linearly independent. They do provide an average global condition for mechanical stability and as such should be regarded in the spirit of an effective medium theory—albeit a remarkably good one in networks which are fairly homogeneous.

Another way to envisage this problem is to imagine that the rigid bars are replaced by Hooke springs and to examine the eigenfrequencies of the network. This is a well-defined problem in classical mechanics. The rank $R$ of the dynamical matrix that describes the system is by definition just the number of finite frequency modes. Therefore the number of zero-frequency modes $P$ is the dimensionality of the dynamical matrix $dj$ minus the rank $R$.

FIG. 1. A typical free-standing network with 30 bonds and 17 joints.
\[ F = dj - R \]  
(4)

This is an exact result. Note that \( F \) is an intrinsically non-negative quantity. The rank of the dynamical matrix is the number of linearly independent constraints so that in general,

\[ R \leq b \]  
(5)

In many cases it is found that the equality sign holds or almost holds. This means that all the constraints are linearly independent. It is useful to define \( F' \) as

\[ F' = dj - b \]  
(6)

then Eqs. (5) and (6) can be rewritten as

\[ F \geq F' \]  
(7)

As the density of bonds per site increases so \( F' \) as defined by (6) becomes negative. However the amount of linear dependence among the constraints increases to ensure that \( F \) indeed always remains non-negative.

The above analysis is not restricted to small systems and can be applied to large networks in the thermodynamic limit where the number of elements approaches infinity. The kind of linear dependence mentioned above is to be expected as a phase transition is approached. Extensive previous work has shown that the equality sign can be taken in the floppy phase, right up to the rigidity transition if the network is sufficiently homogeneous.\(^6,3\text{--}8\)

Therefore we take

\[ F \approx F' = dj - b \text{ for } F' > n_0 \]
\[ \approx 0 \text{ for } F' < n_0 \]  
(8)

or

\[ F \approx \max[F', n_0] = \max[dj - b, n_0] \]  
(9)

For example in the network of Fig. 1 we have \( j = 17 \) and \( b = 30 \) so that \( F' = 4 \) of which \( n_0 = 3 \) are the macroscopic translations and rotations leaving a single internal zero-frequency mode. By direct diagonalization of the dynamical matrix it can be shown that here \( F = F' = 4 \). The single internal zero-frequency mode means that the network is deformable. There is no overcounting of constraints in this case and Eq. (6) holds as an equality.

In Fig. 2(a), we show an example of a stable network. Here \( j = 7 \), \( b = 11 \), and \( F = F' = 3 \) which are just the rigid macroscopic translations and rotations. It is clear that this network is stable as it can be completely decomposed into triangles. By the same argument the network in Fig. 2(b) is also stable (hence \( F = 3 \)). Indeed adding an extra bond to a stable network like that in Fig. 2(a) cannot make it unstable. For the network of Fig. 2(b), we have \( j = 7 \), \( b = 12 \), and \( F' = 2 \) so that \( F > F' \). It is found that the approximation (9) works extremely well except for a small critical region around the cusp in the function (4) where \( F \approx F' \approx n_0 \). The reason why the critical region is so small is unclear but has been well documented.\(^9\text{--}13\)

We take Eq. (9) to be the fundamental result for free-standing networks that has been demonstrated to hold in many examples of rigidity percolation.\(^2,9\text{--}13\) In Sec. II we show that this formula can also be applied to networks that are subject to an external tension or pressure if the quantities \( j \) and \( b \) are appropriately generalized and reinterpreted.

**II. NETWORKS UNDER TENSION**

The networks described in the previous section were free-standing and therefore not subject to any external tension or pressure. An example is the network of Fig. 1. It is sometimes convenient, especially when studying percolation type problems, to impose periodic boundary conditions on the network. The only modification required in the formalism is that the rigid macroscopic rotations are no longer allowed so that \( n_0 \) in Eq. (3) is replaced by

\[ n_0 = d \]  
(3')

In Fig. 3 we show a network that has been subject to an external tension.\(^14,15\) We will use this particular network to illustrate the general technique used for counting

![Fig. 3](image_url)

**FIG. 3.** A network of Hooke springs under an external tension. The solid bonds are stretched, the dashed bonds are compressed, and the dotted bonds have their natural length. The sites on the backbone are indicated by the solid circles.
constraints. The network is formed from a regular tri-
angular network that has been stretched by an external
tension so that every bond length has been increased
from its natural length \( l_0 \) to a new length \( l \). In this case
\( l_0/l = 0.75 \). Subsequently, a unit supercell containing
20 \( \times \) 22 \( =440 \) sites is defined and only a fraction \( p =0.55 \)
of the bonds are present. These bonds were chosen ran-
domly. As there were originally 440 sites and hence 1320
bonds in the supercell, there are now only 726 bonds
when 45\% have been removed. The network is relaxed
using conjugate gradient techniques for nonlinear sys-
tems.\(^{16}\) We use the potential,
\[
V = \frac{\alpha}{2} \sum_{ij} (l_{ij} - l_0)^2 ,
\]
where \( \alpha \) is the spring constant. It is the relaxed network
that is shown in Fig. 3. The 134 dotted bonds are un-
strained and have their natural length \( l_0 \). The 540 solid
bonds are stretched and have lengths \( l_{ij} > l_0 \) and the 52
dashed bonds are compressed and have lengths \( l_{ij} < l_0 \).
Except for one or two bonds that have a length very close
to \( l_0 \), there is no problem in dividing the bonds into these
three classes if the relaxation procedure is carried far
enough.\(^{17}\) It is perhaps surprising at first to find such a
large number (52) of compressed bonds. However these
arise quite naturally. For example, when the two oppo-
site corners of a quadrilateral are pulled, the bond joining
the other two corners is compressed. Many of the
compressed bonds in Fig. 3 are of this general type; a few
are more complicated. Note that a few of the dotted
bonds are completely disconnected from the rest of the
lattice and therefore do not contribute towards the elastic
properties of the network. This is also true of some but
not all of the dotted bonds that are attached to the main
part of the network. We may regard the solid and the
dashed bonds as an internal boundary and indicate the
sites on this backbone by the solid circles. There are 348
such sites; all have at least two solid or two dashed bonds
attached to them. As there are 440 sites in all, there are
440 – 348 = 92 sites that are not associated with the rigid
backbone. These are indicated by the open circles.

We make modifications to the arguments given in the
introduction so that we can handle the present case. From Eq. (4) we may write
\[
F = d \left( \text{number of sites not in backbone} \right) - \left( \text{number of constraints not in backbone} \right)
+ d \left( \text{number of sites in backbone} \right) - \left( \text{number of constraints in backbone} \right) .
\]

However because by definition the backbone is rigid, we know that if there is a backbone,
\[
d \left( \text{number of sites in backbone} \right) - \left( \text{number of constraints in backbone} \right) = n_0 ,
\]
and hence we have,
\[
F = d \left( \text{number of sites not in backbone} \right) - \left( \text{number of constraints not in backbone} \right) + n_0 .
\]

If there is no backbone the formulas in the preceding sec-
tion are appropriate rather than (13). The only difference
is the absence of the \( n_0 \) term which is negligibly small in
large systems anyway. The sites in the backbone are the
ones indicated by solid circles in Fig. 3 and the sites not
in the backbone are the ones indicated by the open cir-
cles. The constraints associated with the backbone are
defined as the rank of the dynamical matrix formed when
only those interactions associated with the backbone are
included (i.e., the solid and dashed bonds). The con-
straints not associated with the backbone are defined as
the rank of the dynamical matrix formed when only those
interactions not associated with the backbone are includ-
ed (i.e., the dotted bonds) — with the condition that the
sites on the backbone are frozen. In the same way as be-
fore, we can approximate \( f \) in (13) by \( F' \), where
\[
F' = d \left( \text{number of sites not in backbone} \right) - \left( \text{number of bonds not in backbone} \right) + n_0 .
\]

The number of macroscopic modes is given by Eq. (3')
which for the lattice in Fig. 3 leads to \( n_0 = 2 \) because of
the two-dimensional periodic boundary conditions.

We can apply these ideas to the network shown in Fig.
3. There are 92 sites not in the backbone (open circles)
and 134 (dotted) bonds not in the backbone. Thus from
Eq. (14) we find that \( F' = 2 \times 92 - 134 + 2 = 52 \).

Finally we make a direct determination of the number
of zero-frequency modes \( F \). We examine small displace-
ments about the equilibrium positions of a network\(^{15}\) like
that in Fig. 3. We put \( l_{ij} = l_{ij}^0 + u_{ij} \) in Eq. (10) and expand
up to second order in \( u \). The linear term in \( u \) in the
potential vanishes because the network has been relaxed
which also determines the equilibrium separations \( l_{ij}^0 \).
That this is a local equilibrium is sufficient. Apart from
the constant term, the potential becomes,
\[
V = \frac{\alpha}{2} \sum_{ij} \frac{l_0}{l_{ij}^0} (u_{ij}^2 \dot{r}_{ij})^2 + \left[ 1 - \frac{l_0}{l_{ij}^0} \right] u_{ij}^2 ,
\]
where the unit vector \( \dot{r}_{ij} = l_{ij}^0 / l_{ij}^0 \). A dynamical matrix
can be formed from this potential, using the numerically
determined \( l_{ij}^0 \), and the eigenfrequencies determined by
direct diagonalization. From this the number of zero-
frequency modes can be determined. For the network of
Fig. 3, we find that the number of zero-frequency modes
is 52. This is precisely the same result as obtained via \( F' \)
using constraint counting. This is consistent with the in-
equality (7), and shows that \( F' \) provides a good approxi-
mation to \( F \).
The above analysis only requires that the network be locally stable. The displacements $u$ in the potential (11) are infinitesimally small and the dynamical matrix is positive semidefinite. Our experience with these kinds of networks under tension leads us to believe that in fact the potential minimum found by relaxation is always the absolute minimum in which the positions of the solid circles that comprise the backbone are uniquely determined while some of the other sites are free to move without changing the total energy.\textsuperscript{15}

We have studied networks similar to that of Fig. 3 but with $40 \times 44 = 1760$ sites and hence 3520 degrees of freedom and 5280 bonds. In Table I we show values of $F$ and $F'$ as obtained by the methods described above, for various values of $p$ for a network stretched so that $l_0/l = 0.5$. Note that $F = F'$ except for $p = 0.375, 0.381, 25$, and $0.425, 0.45$. These small differences may be real or occur because it is sometimes difficult to know if a bond has its natural length or is stressed. The impressive agreement between $F$ and $F'$ establishes that constraint counting is a reliable method of obtaining the number of zero-frequency modes in stressed networks. Note that as $p \to 1$, both $F = F' = n_0 = 2$ as would be expected. As $p \to 0$, both $F = F' = 3520$ which is the total number of degrees of freedom.

In Fig. 4 we plot similar results to those of the network in Table I, but with $l_0/l = 0.01$ and 0.85. We use

$$f = F/dN,$$

where the dimension $d = 2$. It is known that the rigidity percolation transition moves from $p = 0.3473$ at $l_0/l = 0$ to $p \approx 0.65$ when $l_0/l = 1.14$.\textsuperscript{15} In Fig. 4, the transitions are at $p^* = 0.36$ for $l_0/l = 0.01$ and $p^* = 0.46$ for $l_0/l = 0.85$. The results are for single samples and are not ensemble averaged. The jump in $f$ at $p^*$ is caused by the tension being relieved completely when a single bond is cut. For larger samples this jump would get smaller. It does not signify a first-order transition. Similar behav-

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**Table I.** Showing the number of zero-frequency modes $F$ obtained numerically and also the estimate $F'$ obtained by constraint counting using formula (14) as functions of the fraction $p$ of bonds present in a $40 \times 44$ triangular lattice with periodic boundary conditions. The original lattice was under a large tension such that $l_0/l = 0.50$. The rigidity transition takes place at $p = 0.36$.

<table>
<thead>
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<th>$p$</th>
<th>Computed $F$</th>
<th>$F'$ from constraint counting</th>
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<td>2</td>
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<td>0.975 00</td>
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<tr>
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<td>3520</td>
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</table>
ior is exhibited by the infinite cluster in conventional connectivity percolation.\textsuperscript{18}

For small $(1-p)$ we can make an expansion for $f$. The leading term is given when $z - 1$ of the $z$ bonds around a site are removed.\textsuperscript{7} The one remaining bond associated with this site has a single zero-frequency mode associated with it. Hence

\[ f = z (1-p)^{-1/d} \quad (17) \]

with the number of nearest neighbors $z = 6$ for the triangular net and the dimension $d = 2$. This is shown by the solid line in Fig. 4.

We have made some similar preliminary studies of networks subject to an external pressure. In that case the network tends to collapse in on itself and there can be many locally stable minima. There are many more bonds that are compressed than are stretched. Nevertheless the procedure described in this section works just as well. The backbone is associated with the compressed and stretched bonds. Backbone sites are identified as sites connected to at least two dashed bonds or two solid bonds and the determination of $F'$ proceeds as before using Eq. (14). Again $F'$ forms a good approximation to $F$ and obeys the inequality (7).

III. CONCLUSIONS

We have shown how the number of zero-frequency modes can be well estimated by a counting procedure that involves identifying the backbone via a relaxation technique. This somewhat limits the utility of the method when compared with free-standing networks where no such relaxation is required prior to the counting procedure. However this procedure is still superior to a relaxation followed by some kind of numerical matrix manipulation to find the total number of zero-frequency modes. There is no way to avoid the relaxation and counting is the simplest subsequent procedure. We also find this approach useful conceptually.

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