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The number of zero-frequency modes of an elastic network is an important quantity in determining the stability of the network. We present a constraint-counting method for finding this number in general central-force networks that are under an external tension. The technique involves isolating the backbone and then counting constraints in the same way as for free standing networks. A detailed example of this counting is given for a random two-dimensional network subject to an external tension. The results are shown to agree with the number of zero-frequency modes as determined by a direct matrix diagonalization.

I. INTRODUCTION

The study of the stability of network structures has been of importance since the pioneering but not widely known work of Maxwell.[1,2] This work has recently been extended to describe rigidity percolation in random networks and glasses. Our purpose in this note is to further extend this work to include networks subject to an external tension or pressure.

Maxwell wrote several papers in the mid-nineteenth century on the stability of mechanical systems. The essence of Maxwell's argument is reproduced here.[1–3] Suppose a two-dimensional structure is constructed from b bars and j joints as shown in Fig. 1. The bars are of fixed length and the joints are such that there are no direct constraints on the angles. A single joint is very loose but can be made firm by completing a triangle, which has 3 bars and 3 joints, as occurs in the two-dimensional framework shown in Fig. 1. The smallest three-dimensional rigid unit would be the tetrahedron with 6 bars and 4 joints.

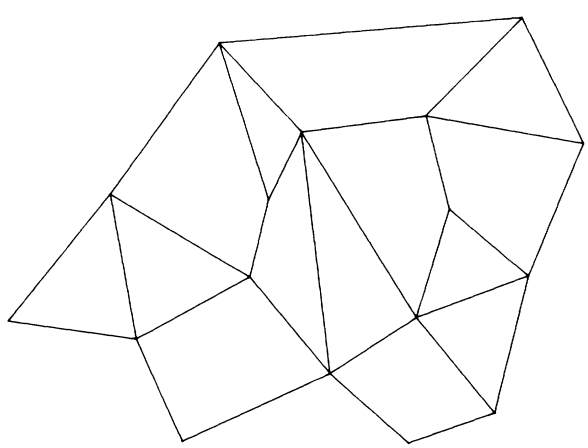

FIG. 1. A typical free-standing network with 30 bonds and 17 joints.

In the network shown in Fig. 1, there are $2j-3$ internal degrees of freedom. Here 2 is the dimensionality and 3 comes from the two rigid translations and one rigid rotation. Maxwell counted the number of constraints as b so that the network is stable if

$$b \geq 2j-3 \tag{1}$$

and unstable if

$$b < 2j-3 \ . \tag{2}$$

Similar arguments in d dimensions lead to a stable network if

$$b \geq dj - n_0 \tag{1'}$$

and an unstable network if

$$b < dj - n_0 \ . \tag{2'}$$

In these expressions d is the dimensionality and

$$n_0 = d(d+1)/2 \tag{3}$$

is the total number of rigid macroscopic translations and rotations. In two dimensions $n_0=3$ as discussed above, and in three dimensions $n_0=6$ which are the three rigid translations and the three rigid rotations. It is important to understand that Eqs. (1) and (2) are not exact because the constraints provided by the bars are not all linearly independent. They do provide an average global condition for mechanical stability and as such should be regarded in the spirit of an effective medium theory—albeit a remarkably good one in networks which are fairly homogeneous.

Another way to envisage this problem is to imagine that the rigid bars are replaced by Hooke springs and to examine the eigenfrequencies of the network. This is a well-defined problem in classical mechanics.[4] The rank R of the dynamical matrix that describes the system is by definition just the number of finite frequency modes.[2] Therefore the number of zero-frequency modes F is the dimensionality of the dynamical matrix dj minus the rank R

$$F = dj - R \ . \tag{4}$$

This is an exact result. Note that F is an intrinsically non-negative quantity. The rank of the dynamical matrix is the number of linearly independent constraints so that in general,

$$R \leq b \ . \tag{5}$$

In many cases it is found that the equality sign holds or almost holds. This means that all the constraints are linearly independent. It is useful to define F' as

$$F' = dj - b \tag{6}$$

then Eqs. (5) and (6) can be rewritten as

$$F \geq F' \ . \tag{7}$$

As the density of bonds per site increases so F' as defined by (6) becomes negative. However the amount of linear dependence among the constraints increases to ensure that F indeed always remains non-negative.

The above analysis is not restricted to small systems and can be applied to large networks in the thermodynamic limit where the number of elements approaches infinity. The kind of linear dependence mentioned above is to be expected as a phase transition is approached. Extensive previous work has shown that the equality sign can be taken in the floppy phase, right up to the rigidity transition if the network is sufficiently homogeneous.[2,5–8] Therefore we take

$$\begin{aligned} F \approx F' = dj - b \quad &\text{for } F' > n_0 \\ \approx 0 \quad &\text{for } F' < n_0 \end{aligned} \tag{8}$$

or

$$F \approx \max[F', n_0] = \max[dj - b, n_0] \ . \tag{9}$$

For example in the network of Fig. 1 we have $j = 17$ and $b = 30$ so that $F' = 4$ of which $n_0 = 3$ are the macroscopic translations and rotations leaving a single internal zero-frequency mode. By direct diagonalization of the dynamical matrix it can be shown that here $F = F' = 4$. The single internal zero-frequency mode means that the network is deformable. There is no overcounting of constraints in this case and Eq. (6) holds as an equality.

In Fig. 2(a), we show an example of a stable network. Here $J = 7$, $b = 11$, and $F = F' = 3$ which are just the rigid macroscopic translations and rotations. It is clear that this network is stable as it can be completely decomposed into triangles. By the same argument the network in Fig. 2(b) is also stable (hence $F = 3$). Indeed adding an extra bond to a stable network like that in Fig. 2(a) cannot make it unstable. For the network of Fig. 2(b), we have $j = 7$, $b = 12$, and $F' = 2$ so that $F > F'$. It is found that the approximation (9) works extremely well except for a small critical region around the cusp in the function (4) where $F \approx F' \approx n_0$. The reason why the critical region is so small is unclear but has been well documented.[9–13] We take Eq. (9) to be the fundamental result for free-standing networks that has been demonstrated to hold in many examples of rigidity percolation.[2,9–13] In Sec. II we show that this formula can also be applied to networks that are subject to an external tension or pressure if the quantities j and b are appropriately generalized and reinterpreted.

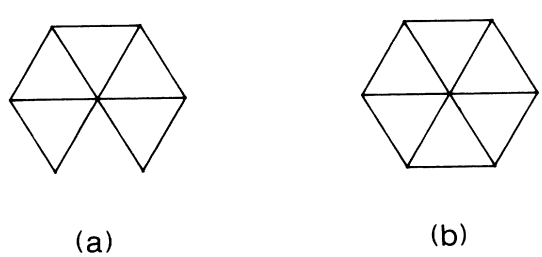

FIG. 2. Two related free-standing networks. Here (a) has 11 bonds and 7 joints while (b) has 12 bonds and 7 joints. In (b) not all the constraints are linearly independent.

II. NETWORKS UNDER TENSION

The networks described in the previous section were free-standing and therefore not subject to any external tension or pressure. An example is the network of Fig. 1. It is sometimes convenient, especially when studying percolation type problems, to impose periodic boundary conditions on the network. The only modification required in the formalism is that the rigid macroscopic rotations are no longer allowed so that n_0 in Eq. (3) is replaced by

$$n_0 = d \ . \tag{3'}$$

In Fig. 3 we show a network that has been subject to an external tension.[14,15] We will use this particular network to illustrate the general technique used for counting

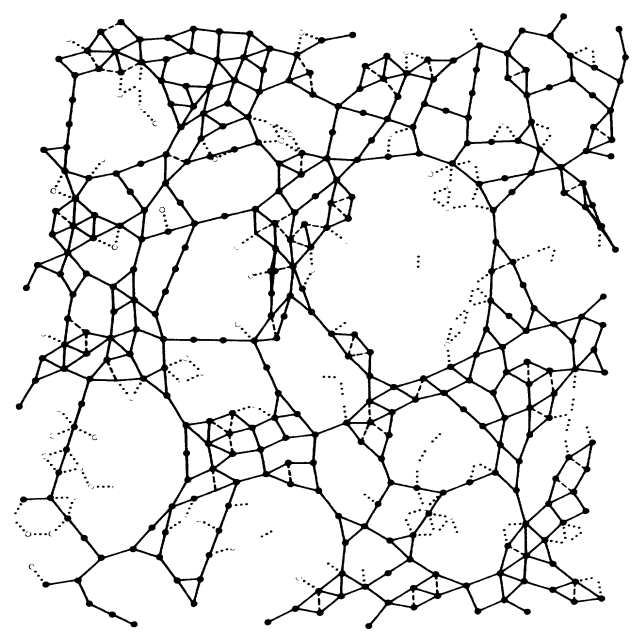

FIG. 3. A network of Hooke springs under an external tension. The solid bonds are stretched, the dashed bonds are compressed, and the dotted bonds have their natural length. The sites on the backbone are indicated by the solid circles.

constraints. The network is formed from a regular triangular network that has been stretched by an external tension so that every bond length has been increased from its natural length l_0 to a new length l. In this case $l_0/l=0.75$. Subsequently, a unit supercell containing $20\times 22=440$ sites is defined and only a fraction $p=0.55$ of the bonds are present. These bonds were chosen randomly. As there were originally 440 sites and hence 1320 bonds in the supercell, there are now only 726 bonds when 45% have been removed. The network is relaxed using conjugate gradient techniques for nonlinear systems.[16] We use the potential,

$$V=(\alpha/2)\sum_{ij}(l_{ij}-l_0)^2 , \tag{10}$$

where α is the spring constant. It is the relaxed network that is shown in Fig. 3. The 134 dotted bonds are unstrained and have their natural length l_0. The 540 solid bonds are stretched and have lengths $l_{ij}>l_0$ and the 52 dashed bonds are compressed and have lengths $l_{ij}<l_0$. Except for one or two bonds that have a length very close to l_0, there is no problem in dividing the bonds into these three classes if the relaxation procedure is carried far enough.[17] It is perhaps surprising at first to find such a large number (52) of compressed bonds. However these arise quite naturally. For example, when the two opposite corners of a quadrilateral are pulled, the bond joining the other two corners is compressed. Many of the compressed bonds in Fig. 3 are of this general type; a few are more complicated. Note that a few of the dotted bonds are completely disconnected from the rest of the lattice and therefore do not contribute towards the elastic properties of the network. This is also true of some but not all of the dotted bonds that are attached to the main part of the network. We may regard the solid and the dashed bonds as an internal boundary and indicate the sites on this backbone by the solid circles. There are 348 such sites; all have at least two solid or two dashed bonds attached to them. As there are 440 sites in all, there are $440-348=92$ sites that are not associated with the rigid backbone. These are indicated by the open circles.

We make modifications to the arguments given in the introduction so that we can handle the present case. From Eq. (4) we may write

$$F=d(\text{number of sites not in backbone})-(\text{number of constraints not in backbone})$$
$$+d(\text{number of sites in backbone})-(\text{number of constraints in backbone}) . \tag{11}$$

However because by definition the backbone is rigid, we know that if there is a backbone,

$$d(\text{number of sites in backbone})-(\text{number of constraints in backbone})=n_0 , \tag{12}$$

and hence we have,

$$F=d(\text{number of sites not in backbone})-(\text{number of constraints not in backbone})+n_0 . \tag{13}$$

If there is no backbone the formulas in the preceding section are appropriate rather than (13). The only difference is the absence of the n_0 term which is negligibly small in large systems anyway. The sites in the backbone are the ones indicated by solid circles in Fig. 3 and the sites not in the backbone are the ones indicated by the open circles. The constraints associated with the backbone are defined as the rank of the dynamical matrix formed when only those interactions associated with the backbone are included (i.e., the solid and dashed bonds). The constraints *not* associated with the backbone are defined as the rank of the dynamical matrix formed when only those interactions *not* associated with the backbone are included (i.e., the dotted bonds)—with the condition that the sites on the backbone are frozen. In the same way as before, we can approximate f in (13) by F', where

$$F'=d(\text{number of sites not in backbone})$$
$$-(\text{number of bonds not in backbone})+n_0 . \tag{14}$$

The number of macroscopic modes is given by Eq. (3′) which for the lattice in Fig. 3 leads to $n_0=2$ because of the two-dimensional periodic boundary conditions.

We can apply these ideas to the network shown in Fig. 3. There are 92 sites not in the backbone (open circles) and 134 (dotted) bonds not in the backbone. Thus from Eq. (14) we find that $F'=2\times 92-134+2=52$.

Finally we make a direct determination of the number of zero-frequency modes F. We examine small displacements about the equilibrium positions of a network[15] like that in Fig. 3. We put $l_{ij}=l_{ij}^0+\mathbf{u}_{ij}$ in Eq. (10) and expand up to second order in u. The linear term in u in the potential vanishes because the network has been relaxed which also determines the equilibrium separations l_{ij}^0. That this is a local equilibrium is sufficient. Apart from the constant term, the potential becomes,

$$V=\frac{\alpha}{2}\sum_{ij}\left[\frac{l_0}{l_{ij}^0}(\mathbf{u}_{ij}\cdot\hat{\mathbf{r}}_{ij})^2+\left[1-\frac{l_0}{l_{ij}^0}\right]u_{ij}^2\right] , \tag{15}$$

where the unit vector $\hat{\mathbf{r}}_{ij}=\mathbf{l}_{ij}^0/l_{ij}^0$. A dynamical matrix can be formed from this potential, using the numerically determined l_{ij}^0, and the eigenfrequencies determined by direct diagonalization. From this the number of zero-frequency modes can be determined. For the network of Fig. 3, we find that the number of zero-frequency modes is 52. This is precisely the same result as obtained via F' using constraint counting. This is consistent with the inequality (7), and shows that F' provides a good approximation to F.

TABLE I. Showing the number of zero-frequency modes F obtained numerically and also the estimate F' obtained by constraint counting using formula (14) as functions of the fraction p of bonds present in a 40×44 triangular lattice with periodic boundary conditions. The original lattice was under a large tension such that $l_0/l=0.50$. The rigidity transition takes place at $p\approx0.36$.

p	Computed F	F' from constraint counting
1.000 00	2	2
0.975 00	2	2
0.950 00	2	2
0.925 00	2	2
0.900 00	2	2
0.875 00	2	2
0.850 00	4	4
0.825 00	4	4
0.800 00	4	4
0.775 00	5	5
0.750 00	10	10
0.725 00	13	13
0.700 00	18	18
0.675 00	27	27
0.650 00	39	39
0.625 00	60	60
0.600 00	84	84
0.575 00	123	123
0.550 00	162	162
0.525 00	232	232
0.500 00	306	306
0.487 00	357	357
0.475 00	397	397
0.462 50	457	457
0.450 00	525	525
0.437 50	591	591
0.425 00	641	640
0.412 50	737	737
0.400 00	876	876
0.393 75	941	941
0.387 50	1039	1039
0.381 25	1233	1231
0.375 00	1332	1330
0.368 75	1446	1446
0.362 50	1480	1480
0.356 25	1639	1639
0.350 00	1672	1672
0.343 75	1705	1705
0.337 50	1738	1738
0.325 00	1804	1804
0.312 50	1870	1870
0.300 00	1936	1936
0.275 00	2068	2068
0.250 00	2200	2200
0.225 00	2332	2332
0.200 00	2464	2464
0.150 00	2728	2728
0.100 00	2992	2992
0.050 00	3256	3256
0.000 00	3520	3520

The above analysis only requires that the network be locally stable. The displacements u in the potential (11) are infinitesimally small and the dynamical matrix is positive semidefinite. Our experience with these kinds of networks under tension leads us to believe that in fact the potential minimum found by relaxation is always the absolute minimum in which the positions of the solid circles that comprise the backbone are uniquely determined while some of the other sites are free to move without changing the total energy.[15]

We have studied networks similar to that of Fig. 3 but with $40\times44=1760$ sites and hence 3520 degrees of freedom and 5280 bonds. In Table I we show values of F and F' as obtained by the methods described above, for various values of p for a network stretched so that $l_0/l=0.5$. Note that $F=F'$ except for $p=0.375\,00$, 0.381 25, and 0.425 00. These small differences may be real or occur because it is sometimes difficult to know if a bond has its natural length or is stressed. The impressive agreement between F and F' establishes that constraint counting is a reliable method of obtaining the number of zero-frequency modes in stressed networks. Note that as $p\rightarrow1$, both $F=F'=n_0=2$ as would be expected. As $p\rightarrow0$, both $F=F'=3520$ which is the total number of degrees of freedom.

In Fig. 4 we plot similar results to those of the network in Table I, but with $l_0/l=0.01$ and 0.85. We use

$$f=F/dN\ , \tag{16}$$

where the dimension $d=2$. It is known that the rigidity percolation transition moves from $p=0.3473$ at $l_0/l=0$ to $p\approx0.65$ when $l_0/l=1$.[14,15] In Fig. 4, the transitions are at $p^*=0.36$ for $l_0/l=0.01$ and $p^*=0.46$ for $l_0/l=0.85$. The results are for single samples and are not ensemble averaged. The jump in f at p^* is caused by the tension being relieved completely when a single bond is cut. For larger samples this jump would get smaller. It does *not* signify a first-order transition. Similar behav-

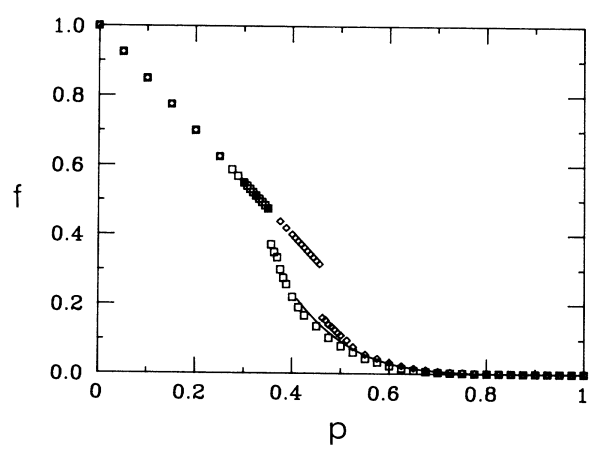

FIG. 4. The fraction f of zero-frequency modes computed for $l_0/l=0.01$ (squares) and 0.85 (diamonds) for $40\times44=1760$ site triangular networks. The squares are for $l_0/l=0.01$ and the small symbols are for $l_0/l=0.85$ The solid line is the exact result (17) for small $(1-p)$. The rigidity transition occurs around $p\approx0.36$ for l_0/l and at $p\approx0.46$ for $l_0/l=0.85$.

ior is exhibited by the infinite cluster in conventional connectivity percolation.[18]

For small $(1-p)$ we can make an expansion for f. The leading term is given when $z-1$ of the z bonds around a site are removed.[7] The one remaining bond associated with this site has a single zero-frequency mode associated with it. Hence

$$f = z(1-p)^{z-1}/d \tag{17}$$

with the number of nearest neighbors $z=6$ for the triangular net and the dimension $d=2$. This is shown by the solid line in Fig. 4.

We have made some similar preliminary studies of networks subject to an external pressure. In that case the network tends to collapse in on itself and there can be many locally stable minima. There are many more bonds that are compressed than are stretched. Nevertheless the procedure described in this section works just as well. The backbone is associated with the compressed and stretched bonds. Backbone sites are identified as sites connected to at least two dashed bonds or two solid bonds and the determination of F' proceeds as before using Eq. (14). Again F' forms a good approximation to F and obeys the inequality (7).

III. CONCLUSIONS

We have shown how the number of zero-frequency modes can be well estimated by a counting procedure that involves identifying the backbone via a relaxation technique. This somewhat limits the utility of the method when compared with free-standing networks where no such relaxation is required prior to the counting procedure. However this procedure is still superior to a relaxation followed by some kind of numerical matrix manipulation to find the total number of zero-frequency modes. There is no way to avoid the relaxation and counting is the simplest subsequent procedure. We also find this approach useful conceptually.

ACKNOWLEDGMENTS

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