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Gauge-invariant de Gennes model

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A gauge-invariant formulation of the de Gennes model for the nematic—to—smectic-A transition is presented. In this formulation the energy associated with the gauge field $\vec{A}$ reduces to the Frank elastic energy with the application of the constraint $\vec{n}_0 \cdot \vec{A}=0$ where $\vec{n}_0$ is the uniform equilibrium director and $\vec{A}$ is to be identified with deviations $\delta\vec{n}$ of the director from equilibrium. It is shown that thermodynamic quantities and renormalization-group recursion relations are gauge invariant. All gauge dependence appears in the exponent η describing order-parameter correlations. The gauge invariance of a negative dielectric anisotropy smectic-A in an external electric field is also studied.

I. INTRODUCTION

The de Gennes model[1,2] for the nematic—to—smectic-A (N-A) transition of liquid crystals introduces a coupling between the smectic order parameter $\psi(\vec{x})$ and the director $\vec{n}(\vec{x})$ similar to the coupling between the complex order parameter Ψ and the vector field $\vec{A}$ in the Landau-Ginzberg[3] model for the superconducting transition. This similarity has been commented on by many authors[1,2,4–7] but it seemed that, unlike the superconductor, the liquid crystal was not gauge invariant because splay distortions of the director from its uniform equilibrium value $\vec{n}_0$ contribute a term $K_1(\vec{\nabla}\cdot\vec{n})^2$ to the free energy density. Terms of this form are usually associated with gauge fixing.[8] In this case, however, the splay energy merely participates in determining the nature of fluctuations of $\delta\vec{n}(\vec{x})=\vec{n}(\vec{x})-\vec{n}_0$ and does not fix the gauge. This is because the de Gennes model is already written in a specific gauge (called the liquid crystal or LC gauge) where fluctuations $\delta\vec{n}(\vec{x})$ are explicitly forbidden in the direction parallel to $\hat{n}_0$. This could be achieved by the addition of a term $-B(\delta\vec{n}\cdot\hat{n}_0)^2$ with $B=\infty$ in a Hamiltonian with unrestricted variations in $\delta\vec{n}(\vec{x})$ or via the constraint (gauge choice) $\hat{n}_0\cdot\delta\vec{n}=0$. In this paper we will construct a generalization of the de Gennes model that is gauge invariant and that reduces to the original model when the constraint $\hat{n}_0\cdot\delta\vec{n}=0$ is imposed.

Because of this gauge invariance, the thermodynamic quantities should be gauge independent, as in the Landau-Ginzberg model for the superconducting transition. In particular, the critical exponents α and ν should be gauge independent. The correlation functions and hence the critical exponent η are gauge dependent. These cannot be measured in superconductivity or electrodynamics but can be measured, in the LC gauge, in liquid crystals.

We explore the consequences of this in the ϵ expansion. We show that all the gauge dependence is contained in the exponent $\eta=\eta_{SC}+\Delta\eta(\theta)$ where θ is a parameter used to specify the gauge and η_{SC} is the minimum value of η evaluated in the so-called superconducting (SC) gauge where the divergence of the gauge-transformed director field $\vec{A}$ is 0 ($\vec{\nabla}\cdot\vec{A}=0$). $\Delta\eta(\theta)$ diverges in the LC gauge ($\theta=0$) indicating the destruction of long-range order. The other critical exponents are shown to be gauge invariant as are the renormalization-group (RG) recursion relations for the thermodynamic potentials.

We also consider the case of the N-A transition in an external field which suppresses fluctuations of $\delta\vec{n}(\vec{x})$ in all but one direction.[9] Once again the Hamiltonian can be written in a gauge-invariant form: the thermodynamic critical exponents and the RG recursion relations are gauge independent while $\eta=\eta_{SC}+\Delta\eta(\theta)$ is not. However, because of the external field, $\Delta\eta(\theta)$ no longer diverges and long-range order exists. We further note that this transition is in the same universality class as the N-A transition in the absence of a field when the twist elastic constant K_2 is infinite.

II. GAUGE-INVARIANT MODEL

We start by defining the de Gennes model for the N-A transition. In the nematic phase the orientation of the bar-like molecules is given by the unit vector, $\vec{n}(\vec{x})$. It has small local variations $\delta\vec{n}(\vec{x})$ from its uniform equilibrium value $\hat{n}_0$:

$$\vec{n}(\vec{x})=\hat{n}_0 + \delta\vec{n}(\vec{x}) . \tag{2.1}$$

The energy of small, nonuniform distortions of $\vec{n}(\vec{x})$ is described by the reduced Frank Hamiltonian[10]

$$\beta H_n=\int d^3x \,[\, K_1(\vec{\nabla}\cdot\vec{n})^2 + K_2(\vec{n}\cdot\vec{\nabla}\times\vec{n})^2 + K_3(\vec{n}\times(\vec{\nabla}\times\vec{n}))^2 \,], \tag{2.2}$$

where K_1, K_2, and K_3 are the Frank elastic constants for splay, twist, and bend. We choose the 1 axis to be along the $\hat{n}_0$ direction and for small fluctuations we have

$$\delta\vec{n}(\vec{x})\cdot\hat{n}_0 = 0 . \tag{2.3}$$

Under this constraint and extending to d dimensions, we have

$$\beta H_n = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\,[\,(K_1 q_\perp^2 + K_3 q_\parallel^2)(\hat{e}_\perp)_i(\hat{e}_\perp)_j + (K_2 q_\perp^2 + K_3 q_\parallel^2)(\hat{e}_t)_i(\hat{e}_t)_j\,] \times \delta n_i(\vec{q})\delta n_j(-\vec{q}), \tag{2.4}$$

where $\delta\vec{n}(\vec{q})$ is the Fourier transform of $\delta\vec{n}(\vec{x})$ and $\hat{e}_\perp$ and $\hat{e}_t$ are defined in Fig. 1. The order parameter for the smectic phase is the complex amplitude $\psi(\vec{x})$ of a mass density wave. The reduced Hamiltonian for ψ in the de Gennes model is

$$\beta H_\psi = \int d^d x\,[\,r\,|\psi|^2 + |(\vec{\nabla} - iq_0\delta\vec{n}\,)\,\psi|^2 + \tfrac{1}{2}u\,|\psi|^4\,]\,. \tag{2.5}$$

The full de Gennes Hamiltonian describing the N and A phases is thus

$$\beta H = \beta H_n + \beta H_\psi\,. \tag{2.6}$$

The de Gennes model for liquid crystals closely resembles the Landau-Ginzberg model for the superconducting transition and gauge transformations of the form

$$\psi' = \psi e^{iq_0 L}\,, \tag{2.7a}$$

$$\vec{A} = \delta\vec{n} + \vec{\nabla} L \tag{2.7b}$$

can be introduced. βH_ψ is invariant under this transformation but βH_n as defined in Eq.(2.4) is not manifestly so. However, it is already written in a particular gauge, called the liquid crystal (LC) or physical gauge, where $\hat{n}_0\cdot\delta\vec{n}=0$. To derive a gauge-invariant form of the de Gennes model we first perform a gauge transformation of the original model to the superconducting (SC) gauge defined via

$$\vec{\nabla}\cdot\vec{A} = 0 = \vec{\nabla}\cdot\delta\vec{n} + \nabla^2 L\,. \tag{2.8}$$

This and Eq. (2.7b) define a unique relation between $\vec{A}$ and $\delta\vec{n}$:

$$A_i(\vec{q}) = P_{ij}(\vec{q})\delta n_j(-\vec{q}), \tag{2.9}$$

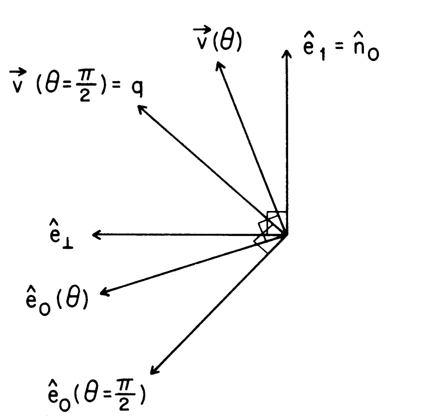

FIG. 1. Bases used in the $\hat{n}_0 - \hat{q}$ plane. $\hat{e}_t$ is orthogonal to this plane.

where $P_{ij}(\vec{q}) = \delta_{ij} - q_i q_j/q^2$ is the projection operator onto directions perpendicular to $\vec{q}$. In other words, the transformations defined in Eqs. (2.7) to (2.9) represent simply a change of variables from ψ and $\delta\vec{n}$ to ψ' and $\vec{A}$. In three dimensions $\vec{A}$ has two independent components just as $\delta\vec{n}$. Defining $\hat{e}_0$ to be the vector normal to $\vec{q}$ in the $\hat{e}_1$-$\hat{e}_\perp$ plane (see Fig. 1) we have $\vec{A} = A_0\hat{e}_0 + A_t\hat{e}_t$ and

$$\delta n_t = A_t\,, \qquad \delta n_\perp = \frac{q}{q_1}A_0\,. \tag{2.10}$$

Thus, the de Gennes model in this gauge is

$$\beta H = \beta H_A + \int d^d x\,[r\,|\psi'|^2 + |(\vec{\nabla} - iq_0\vec{A})\psi'|^2 + \tfrac{1}{2}u\,|\psi'|^4], \tag{2.11}$$

where

$$\beta H_A = \int \frac{d^3 q}{(2\pi)^3}[\,K_\perp q^2(\hat{e}_0)_i(\hat{e}_0)_j + K_t q^2(\hat{e}_t)_i(\hat{e}_t)_j\,] \times A_i(\vec{q})A_j(-\vec{q}) \tag{2.12}$$

and

$$K_\perp = K_1\frac{q_\perp^2}{q_\parallel^2} + k_3\,,$$

$$K_t = K_2 q_\perp^2 + K_3 q_\parallel^2\,.$$

Using $\hat{e}_0 = \hat{e}_t\times\hat{q}$ and $\hat{e}_t = \hat{q}\times\hat{e}_0$ where $\hat{q} = \vec{q}/|\vec{q}\,|$, we obtain immediately

$$\beta H_A = \int \frac{d^3 q}{(2\pi)^3}\,[\,K_\perp(\hat{e}_t)_i(\hat{e}_t)_j + K_t(\hat{e}_0)_i(\hat{e}_0)_j\,] \times(\vec{q}\times\vec{A})_i(\vec{q}\times\vec{A})_j \equiv \int d^3 x \bar{K}_{ij}(\vec{\nabla}\times\vec{A})_i(\vec{\nabla}\times\vec{A})_j \tag{2.13}$$

and so βH_A depends only on $\vec{\nabla}\times\vec{A}$. If we now lift the constraint that $\nabla\cdot\vec{A}=0$ and treat $\vec{A}$ as an unconstrained three-component vector, βH is gauge invariant just as are electrodynamics and the Landau-Ginzberg model for superconductivity. The original de Gennes model is regained by imposing the constraint $\hat{e}_1\cdot\vec{A}=0$.

Though we have cast the original de Gennes model in a gauge-invariant form, it is worth emphasizing a difference between it and the more familiar Landau-Ginzberg model for a superconductor. The elastic constant $K_\perp$ is highly anisotropic and highly singular. In particular, it diverges as $q_\parallel\to 0$ for nonzero K_1 and $q_\perp$. Thus, fluctuations in $A_0(q_\parallel=0,\ \vec{q}_\perp)$ are effectively excluded from the model. In practice, this presents no problems in field theoretic calculations in infinite systems but it can lead to some problems in finite-size lattice systems.[11]

III. RG RECURSION RELATIONS IN AN ARBITRARY GAUGE

A. Gauge transformation

In this section we will discuss the $\epsilon=4-d$ expansion for the continuum of gauges, introduced by Dunn and Lubensky[6], parametrized by a single variable $0\leq\theta\leq\pi/2$. $\theta=0$ is what we have called the LC gauge and $\theta=\pi/2$ is the SC gauge. We set $L=L(\theta)$ and introduce the gauge condition

$$\vec{v}^{\,\theta}(\vec{q})\cdot\vec{A}^{\,\theta}(\vec{q})=0, \tag{3.1}$$

where

$$\vec{v}^{\,\theta}(\vec{q})=\hat{e}_1\cos\theta + i\hat{q}\sin\theta . \tag{3.2}$$

From these conditions and Eq.(2.7b) we can derive a transformation matrix

$$P^{\theta}_{ij}(\vec{q}) = (\delta_{ij}-f(\theta,\vec{q})\hat{q}_i\hat{q}_j) , \tag{3.3}$$

where

$$f(\theta,\vec{q})=\frac{\sin\theta}{\sin\theta-i\hat{q}_1\cos\theta} \tag{3.4}$$

such that

$$A^{\theta}_i(\vec{q})=P^{\theta}_{ij}(\vec{q})A^{\theta=0}_j(\vec{q}) . \tag{3.5}$$

The propagators $D^{\theta}_{ij}(\vec{q})=\langle A^{\theta}_i(\vec{q})A^{\theta}_j(-\vec{q})\rangle$ satisfy

$$D^{\theta}_{ij}(\vec{q})=P^{\theta}_{ik}(\vec{q})P^{\theta}_{lj}(-\vec{q})D_{kl}(\vec{q}), \tag{3.6}$$

where from Eq.(2.4) we have

$$D_{ij}(\vec{q}) = (\hat{e}_\perp)_i(\hat{e}_\perp)_j(K_1q_\perp^2 + K_3q_\parallel^2)^{-1} + (\hat{e}_t)_i(\hat{e}_t)_j(K_2q_\perp^2 + K_3q_\parallel^2)^{-1} . \tag{3.7}$$

We note that the perpendicular part of this propagator diverges when $q_\parallel=0$ and $K_1=0$.

We will now use these propagators to evaluate the RG recursion relations as a function of θ. We will show how all the gauge dependence is contained in the critical exponent η : The RG recursion relations for the thermodynamic potentials (and hence all the other critical exponents) are gauge invariant.

B. Recursion relations

Near the critical point, we expect the propagators to obey the following homogeneity relations:[5]

$$G(\vec{q},t,K_1) = e^{l(2-\eta)}G(e^{l(1+\mu)}q_\parallel,e^{l}q_\perp,e^{l/\nu_\perp}t,e^{-\tau l}K_1) , \tag{3.8}$$

$$D_{ij}(\vec{q},t,K_1) = e^{l(2-\eta_A)}D_{ij}(e^{l(1+\mu)}q_\parallel,e^{l}q_\perp,e^{l/\nu_\perp}t,e^{-\tau l}K_1) \tag{3.9}$$

where $q_\parallel$ and $q_\perp$ are components of the wave vector $\vec{q}$ parallel and perpendicular to $\hat{n}_0$.

We perform the renormalization-group calculation in the normal manner,[12] allowing for anisotropic rescaling. First we integrate out all fluctuations with wave vector $\vec{q}$ in the momentum shell between $|\vec{q}|=1$ and the ellipsoid $q_\perp^2+q_\parallel^2e^{l(1+\mu)}=1$. We then rescale the lengths anisotropically, $q_\perp\to q_\perp e^{-l}$, $q_\parallel\to q_\parallel e^{-l(1+\mu)}$, to regain the unit Brillouin zone. $\psi(\vec{q})$ and $\vec{A}(\vec{q})$ rescale so that $G(\vec{q})$ and $D_{ij}(\vec{q})$ obey the homogeneity relations, and η and μ are chosen such that

$$\frac{\partial^2G^{-1}(k)}{\partial k_\perp^2}=1 \tag{3.10a}$$

and

$$\frac{\partial^2G^{-1}(k)}{\partial k_\parallel^2}=1 . \tag{3.10b}$$

We choose $\eta_A=\epsilon-\mu$ so that q_0 remains constant under rescaling.

If l is an infinitesimal this procedure generates differential recursion relations for the potentials. The diagrams that contribute are shown in Fig. 2. Note that diagrams 6–8 do not contribute in the SC gauge. In other gauges, they are necessary to ensure gauge invariance. These are evaluated an arbitrary gauge θ in the Appendix and yield the following recursion relations:

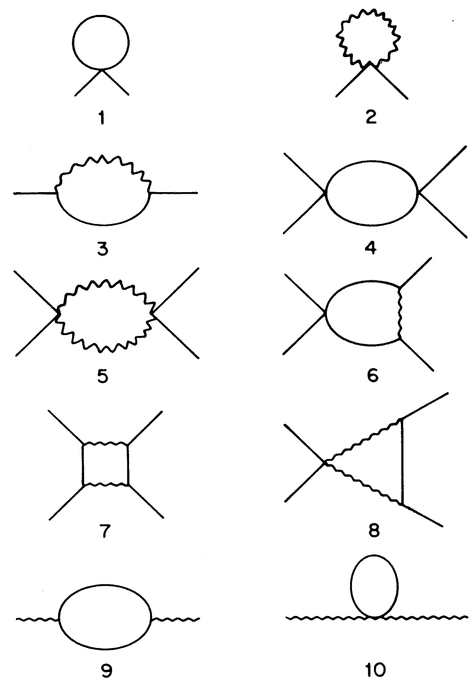

FIG. 2. Diagrams contributing to the renormalization of the potentials to first order in ϵ . A wiggly line represents the propagator $D_{ij}(\vec{q})$ and a solid line represents the propagator $G(\vec{q})$. The three-point vertex is $(2k+q)_i$. Diagrams 1, 2, and 3 renormalize r , diagrams 4 through 8 renormalize u, and diagrams 9 and 10 renormalize K_2 and K_3. Diagram 3 fixes η and μ.

$$\frac{dr}{dl}=(2-\eta_{SC})r+\tfrac{1}{2}(n+2)C_4\frac{u}{1+r}$$
$$+C_4q_0^2\left[\frac{1}{(\sqrt{K_1}+\sqrt{K_3})^2}+\frac{4}{\sqrt{K_2}}\frac{1}{\sqrt{K_2}+\sqrt{K_3}}\right],$$
$$\frac{du}{dl}=(\epsilon-\mu-2\eta_{SC})u-\tfrac{1}{2}(n+8)C_4u^2$$
$$-2C_4q_0^4\left[\frac{1}{\sqrt{K_3}(\sqrt{K_1}+\sqrt{K_3})^3}+\frac{2}{\sqrt{K_3}K_2^{3/2}}\right],$$
$$\frac{dK_1}{dl}=-(\epsilon-\mu)K_1\ ,$$
$$\frac{dK_2}{dl}=-(\epsilon-\mu)K_2+\tfrac{1}{6}nC_4q_0^2\ ,$$
$$\frac{dK_3}{dl}=-(\epsilon+\mu)K_3+\tfrac{1}{6}C_4q_0^2\ , \qquad (3.11)$$
$$\eta_{SC}(l)=-4C_4q_0^2\left[\frac{1}{12}\frac{3\sqrt{K_1}+\sqrt{K_3}}{(\sqrt{K_1}+\sqrt{K_3})^3}+\frac{4}{3}\frac{1}{\sqrt{K_2}(\sqrt{K_2}+\sqrt{K_3})}\right],$$
$$\mu(l)=-2C_4q_0^2\left[\frac{2}{3}\frac{\sqrt{K_3}}{(\sqrt{K_1}+\sqrt{K_3})^3}-\frac{4}{3}\frac{1}{\sqrt{K_2}(\sqrt{K_2}+\sqrt{K_3})}\right],$$
$$\Delta\eta(\theta,l)=q_0^2\int\frac{d\Omega_4}{(2\pi)^4}[1-f(\theta,\hat{q})f^*(\theta,\hat{q})]\times\frac{\sin^2\gamma}{K_3\cos^2\gamma+K_1\sin^2\gamma}\ ,$$
$$\eta_A(l)=\epsilon-\mu(l)\ ,$$

where $C_4=1/8\pi^2$ and $d\Omega_4$ is the differential of solid angle in four dimensions.

The recursion relations for the gauge-independent quantities are the same as those obtained by Lubensky and Chen[5] in the SC gauge with $\theta=\pi/2$. We refer the reader to Ref. 5 for a detailed analysis of the fixed points of these equations. The gauge-dependent exponent $\Delta\eta(\theta)$ is zero at $\theta=\pi/2$. When $\theta=0$, $1-|f|^2=1$ and $\Delta\eta(0)$ diverges as K_1^{-1} as $K_1\to 0$. $K_1=0$ at the critical point with no anisotropy in the correlation-length exponents. Thus, at this critical point, $G(\vec{q})$ does not exhibit simple power-law behavior in $|\vec{q}|$. A detailed analysis of the behavior of G in the LC gauge is given in Ref. 7.

IV. *N-A* TRANSITION IN A FIELD

Recently Halsey and Nelson[9] discussed the *N-A* transition in a negative dielectric anisotropy system with the external field $\vec{E}$ perpendicular to $\hat{n}_0$. In this section we will discuss the gauge-dependent properties of this transition. As in the previous case, we find that all gauge dependence is in $\Delta\eta(\theta)$. However, in this case $\Delta\eta(\theta)$ is not divergent even in the LC gauge.

A. Model

The dielectric tensor $-\epsilon_a n_i n_j + \epsilon_\perp \delta_{ij}$ is anisotropic in liquid crystals, providing a coupling between an external electric field $\vec{E}$ and the director described by the Hamiltonian

$$\beta H_{\text{ext}} = \tfrac{1}{2}\epsilon_a \int d^3x(\vec{E}\cdot\vec{n})^2 \qquad (4.1)$$

which suppresses fluctuations of $\delta\vec{n}$ along the direction parallel to $\vec{E}$. To generalize this Hamiltonian to d dimensions, we follow Halsey and Nelson[5] and specify that the effect of $\vec{E}$ is to suppress fluctuations of $\delta\vec{n}$ in $d-2$ directions perpendicular to n_0 leaving a single easy direction in all dimensions. Thus choosing $\vec{n}_0$ along the 1 axis and the easy direction of $\delta\vec{n}$ to be along the 2 axis, we obtain

$$\beta H_{\text{ext}} = \tfrac{1}{2}\epsilon_a E^2 \sum_{i=3}^{d} \int d^dx(\delta n_i)^2\ . \qquad (4.2)$$

Other generalizations of Eq. (4.1) are possible, such as one in which there are $d-2$ easy directions rather than a single easy direction. The full Hamiltonian is then

$$\beta H_E = \beta H_\psi + \beta H_A + \beta H_{\text{ext}}\ . \qquad (4.3)$$

For this Hamiltonian only δn_2 is hydrodynamic, and the only critical propagator is

$$D_{ij}(\vec{q}) = \delta_{i2}\delta_{j2}(K_1q_2^2 + K_2q_E^2 + K_3q_1^2)\ , \qquad (4.4)$$

where $q^2=q_1^2+q_2^2+q_E^2$. For computational convenience we set $K_2=K_3=K$. This does not affect our results because we can show that they are driven to the same value under the RG. βH_E can be cast in a gauge-invariant form by following exactly the same steps outlined in Sec. II: Change variables from ψ and $\delta\vec{n}$ to ψ' and $\vec{A}$ and then relax the constraint $\vec{\nabla}\cdot\vec{A}=0$. The resulting $\vec{A}$-dependent term in βH_E will depend only on $\vec{\nabla}\times\vec{A}$ but with a coupling that is more complicated and anisotropic than the $\overline{K}_{ij}$ of Eq. (2.13). Gauge-dependent critical properties can now be calculated in exactly the same way as in Sec. II with D_{ij}^{θ} satisfying Eq.(3.6) with D_{ij} given by Eq.(4.4).

B. Recursion relations

We proceed exactly as in Sec. III A except that we use a slightly different anisotropic rescaling procedure: q_2 scales as $e^{-l(1+\mu)}$ while q_1 and q_E scale as e^{-l}. This new definition of μ requires $\eta_A=\epsilon+\mu$ for q_0 to remain constant under rescaling. The recursion relations are evaluated in the Appendix and we obtain

$$\frac{dr}{dl}=(2-\eta_{SC})r + \tfrac{1}{2}(n+2)C_4\frac{u}{1+r} + C_4 q_0^2 \frac{2\sqrt{K_1}+\sqrt{K}}{\sqrt{K}(\sqrt{K_1}+\sqrt{K})^2} ,$$

$$\frac{du}{dl}=(\epsilon-2\eta_{SC}+\mu)u - \tfrac{1}{2}(n+8)C_4u^2 - q_0^4 C_4\frac{2}{\sqrt{K}(\sqrt{K_1}+\sqrt{K})^3}\left[1+3\left[\frac{K_1}{K}\right]^{1/2}+\frac{K_1}{K}\right] ,$$

$$\eta_{SC}(l)=-\tfrac{1}{3}C_4q_0^2\frac{3\sqrt{K_1}+\sqrt{K}}{(\sqrt{K_1}+\sqrt{K})^3} ,$$

$$\mu(l)=-4C_4q_0^2\frac{\sqrt{K}}{(\sqrt{K_1}+\sqrt{K})^3}\left[\tfrac{1}{3}+\left[\frac{K_1}{K}\right]^{1/2}+\frac{K_1}{K}\right] ,$$

$$\frac{dK_1}{dl}=-(\eta_A+2\mu)K_1 , \tag{4.5}$$

$$\frac{dK}{dl}=-\eta_A K+\tfrac{1}{6}nC_4q_0^2 ,$$

$$\eta_A(l)=\epsilon+\mu(l) ,$$

$$\Delta\eta(\theta,l)=q_0^2\int\frac{d\Omega_4}{(2\pi)^4}[1-f(\theta,\hat{q})f^*(\theta,\hat{q})] \times\frac{\cos^2\gamma}{K_1\cos^2\gamma+K\sin^2\gamma} .$$

The gauge-invariant recursion relations above are the same as those obtained by Halsey and Nelson[5] in the LC gauge. We refer the reader to Ref. 9 for an analysis of the fixed points of these equations. Note that

$$\Delta\eta(\theta=0)=+q_0C_4^2\frac{1}{(\sqrt{K_1} + \sqrt{K})^2} \tag{4.6}$$

is perfectly finite when $K_1=0$. This explains why it is possible to carry out all calculations directly in the LC gauge when $\vec{E}\neq0$ whereas it is not when $\vec{E}=0$. $\eta_{SC}+\Delta\eta(\theta=0)$ is identical to η_{LC} calculated in Ref. 9.

Note that the recursion relations for r, u, η_{SC}, μ, and K_3 in Eqs. (3.11) with $K_2=\infty$ and $K_1=0$ ($K_1=\infty$) are identical to those for r, u, η_{SC}, μ, and K in Eqs. (4.5) when $K_1=0$ ($K_1=\infty$). Since $K_1=0$ or $K_1=\infty$ at the only accessible fixed points, the N-A transition of the negative anisotropy de Gennes model in an external field is in the same universality class as that of the fieldless de Gennes model with $K_2=\infty$. They have the same thermodynamic exponents $\nu_\perp$ and $\nu_\parallel=(1+\mu)\nu_\perp$. Correlations of the physical order parameter in the LC gauge differ, however, in the two cases when $K_1^*=0$. For the transition in a field, $\eta=\eta_{SC}+\Delta\eta$ is perfectly finite, and $G(\vec{x},0)$ decays algebraically at the critical point. For the $K_2=\infty$ transition with $\vec{E}=0$, $\Delta\eta(0)$ is infinite and G does not exhibit power-law decay. Instead, it decays exponentially with an exponent controlled by the dangerous irrelevant variable K_1 as for the case with K_2 finite.[6,7] The properties of G in the N and A phases and at the critical point for $K_2=\infty$ can be calculated using the techniques discussed in Ref. 7. Since they are so similar to those for $K_2\neq\infty$, we will not discuss them here.

ACKNOWLEDGMENTS

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APPENDIX

The diagrams that contribute to the renormalization of the potentials are shown in Fig. 2. The propagators are

$$D_{ij}^{\theta}(\vec{q})=P_{ik}^{\theta}(\vec{q})P_{lj}^{\theta}(-\vec{q})\,(\hat{e}_\perp)_k(\hat{e}_\perp)_l\frac{1}{(K_1q_\perp^2 + K_3q_\parallel^2)} + (\hat{e}_t)_i(\hat{e}_t)_j\frac{1}{(K_2q_\perp^2 + K_3q_\parallel^2)} \tag{A1}$$

and

$$G(\vec{q})=\frac{1}{1+q^2} , \tag{A2}$$

where

$$\delta_{ij}=n_{0i}n_{0j}+(\hat{e}_\perp)_i(\hat{e}_\perp)_j+(\hat{e}_t)_i(\hat{e}_t)_j \tag{A3}$$

with

$$\begin{aligned}\hat{q}&=(\cos\gamma,\sin\gamma\,\cos\beta,\sin\gamma\,\sin\beta\cos\alpha,\sin\gamma\,\sin\beta\,\sin\alpha) ,\\ \hat{e}_\perp&=(0,\cos\beta,\sin\beta\cos\alpha,\sin\beta\,\sin\alpha) ,\\ \hat{e}_t&=(0,-\sin\beta,\cos\beta\cos\alpha,\cos\beta\,\sin\alpha) ,\\ \hat{e}_t{}'&=(0,0,-\sin\alpha,\cos\alpha)\end{aligned} \tag{A4}$$

in four dimensions. γ and β run from zero to π and α runs from zero to 2π.

All diagrams can be evaluated in terms of the general integrals

$$L_s^{mnp}(K_3,K_1)=\int\frac{d\Omega_4}{(2\pi)^4}\frac{\cos^{2m}\gamma\,\sin^{2n}\gamma\,\cos^{2p}\beta}{(K_3\cos^2\gamma+K_1\sin^2\gamma)^s} \tag{A5}$$

and

$$M_s^{mnp}(\theta,K_3,K_1)=\int\frac{d\Omega_4}{(2\pi)^4}f(\theta,\hat{q})f^*(\theta,\hat{q}) \times\frac{\cos^{2m}\gamma\,\sin^{2n}\gamma\,\cos^{2p}\beta}{(K_3\cos^2\gamma+K_1\sin^2\gamma)^s} . \tag{A6}$$

The method of evaluating integrals of this form and most of those that we need in this calculation are given in Ref. 5. We take the momentum of the external legs to be $\vec{k}=(k_1,k_2,0,0)$. Then the integrals I_j corresponding to the diagram j of Fig. 2 are

$$I_2=\int_{e^{-l}}^{1}\frac{d^4q}{(2\pi)^4}D_{ii}^{\theta}(\vec{q}) =lL_1^{000}(K_3,K_1)+2lL_1^{000}(K_3,K_2)-lM_1^{010}(\theta,K_3,K_1) , \tag{A7}$$

$$I_3=\int_{e^{-l}}^{1}\frac{d^4q}{(2\pi)^4}(2k+q)_i D_{ij}^{\theta}(\vec{q})(2k+q)_j G(\vec{k}+\vec{q})$$

$$=lL_1^{010}(K_3,K_1)-lM_1^{010}(K_3,K_1)\frac{1}{1+r}$$

$$-\ lk^2[L_1^{010}(K_3,K_1)-M_1^{010}(K_3,K_1)]$$

$$+4lk_2^2[L_1^{201}(K_3,K_1)+L_1^{000}(K_3,K_2)-L_1^{001}(K_3,K_2)]$$

$$+4lk_1^2L_1^{110}(K_3,K_1)\ , \tag{A8}$$

$$I_5=\int_{e^{-l}}^{1}\frac{d^4q}{(2\pi)^4}q_iD_{ij}^{\theta}(\vec{q})q_j$$

$$=lL_1^{010}(K_3,K_1)-lM_1^{010}(K_3,K_1)\ , \tag{A9}$$

$$I_\alpha=\int_{e^{-l}}^{1}\frac{d^4q}{(2\pi)^4}\{\ D_{ij}^{\theta}(\vec{q})D_{ij}^{\theta}(\vec{q})+[q_iD_{ij}^{\theta}(\vec{q})q_j)]^2$$

$$-2D_{ij}^{\theta}(\vec{q})q_jD_{ik}^{\theta}(\vec{q})q_k\ \}$$

$$=lL_2^{200}(K_3,K_1)+2lL_2^{000}(K_3,K_2)\ , \tag{A10}$$

where $I_\alpha=I_6+I_7-2I_8$ and where we have only retained terms to $O(k^2)$,$O(l)$, and $1/(1+r)$. The 4-point function is evaluated at zero external momentum. These integrals and the procedure outlined in Sec. III lead to the following recursion relations:

$$\frac{dr}{dl}=(2-\eta)r\ +\ \tfrac{1}{2}(n+2)C_4\frac{u}{1+r}$$

$$+\ q_0^2[L_1^{000}(K_3,K_1)-M_1^{010}(K_3,K_1)+2L_1^{000}(K_3,K_2)]$$

$$-\ q_0^2\frac{1}{1+r}[L_1^{010}(K_3,K_1)-M_1^{010}(K_3,K_1)]\ , \tag{A11a}$$

$$\frac{du}{dl}=(\epsilon-2\eta-\mu)u\ +\ \tfrac{1}{2}(n+8)C_4\frac{u^2}{1+r}$$

$$+\ 2uq_0^2[L_1^{010}(K_3,K_1)-M_1^{010}(K_3,K_1)]$$

$$-2q_0^4[L_2^{100}(K_3,K_1)\ +\ 2L_1^{000}(K_3,K_2)]\ , \tag{A11b}$$

$$\frac{dK_1}{dl}=-(\epsilon-\mu)K_1\ , \tag{A11c}$$

$$\frac{dK_2}{dl}=-(\epsilon-\mu)k_2\ +\ \tfrac{1}{6}nC_4q_0^2\ , \tag{A11d}$$

$$\frac{dK_3}{dl}=-(\epsilon-\mu)k_3\ +\ \tfrac{1}{6}nC_4q_0^2\ , \tag{A11e}$$

$$(\eta+2\mu)=q_0^2[M_1^{010}(K_3,K_1)-L_1^{010}(K_3,K_1)]$$

$$+\ 4q_0^2L_1^{110}(K_3,K_1)\ , \tag{A11f}$$

$$\eta=4q_0^2[L_1^{201}(K_3,K_1)+L_1^{000}(K_3,K_2)-L_1^{001}(K_3,K_2)]$$

$$+q_0^2[M_1^{010}(K_3,K_1)-L_1^{010}(K_3,K_1)]\ . \tag{A11g}$$

The first term in the expression for η is just η_{SC} calculated by Lubensky and Chen and we define the second term to be $\Delta\eta(\theta)$. If we substitute for η in equations a through f and expand $1/(1+r)=1-r$ (correct to order ϵ) we find that all the θ-dependent parts vanish. Evaluation of the integrals gives the recursion relations of Sec. III B.

In the presence of an external field the propagators are

$$D_{ij}^{\theta}=P_{ik}^{\theta}(\vec{q})P_{ij}^{\theta}(-\vec{q})\hat{e}_{2k}\hat{e}_{2l}\frac{1}{K_1q_2^2+K_2q_E^2+K_3q_1^2}\ , \tag{A12}$$

$$G(\vec{q})=\frac{1}{r+q^2}\ . \tag{A13}$$

We make the new definition

$$\hat{q}=(\sin\gamma\,\cos\beta,\cos\gamma,-\sin\gamma\,\sin\beta\cos\alpha,-\sin\gamma\,\sin\beta\sin\alpha) \tag{A14}$$

and set the external momentum to $\vec{k}=(k_1,k_2,k_3,k_4)$. If we set $K_2=K_3=K$ the analysis proceeds exactly as before. The diagrams are all the same and the recursion relations obtained are

$$\frac{dr}{dl}=(2-\eta)r+\tfrac{1}{2}C_4(n+2)\frac{u}{1+r}$$

$$-q_0^2\frac{1}{1+r}[L_1^{100}(K_1,K)-M_1^{100}(K_1,K)]\ ,$$

$$+q_0^2[L_1^{000}(K_1,K)-M_1^{100}(K_1,K)] \tag{A15a}$$

$$\frac{du}{dl}=(\epsilon-2\eta-\mu)u-\tfrac{1}{2}(n+8)C_4u^2$$

$$+2uq_0^2[L_1^{100}(K_1,K)-M_1^{100}(K_1,K)]$$

$$-2q_0^4L_2^{020}(K_1,K)\ , \tag{A15b}$$

$$\frac{dK_1}{dl}=-(\eta_A+2\mu)K_1\ , \tag{A15c}$$

$$\frac{dK}{dl}=-\eta_AK+\tfrac{1}{6}nC_4q_0^2\ , \tag{A15d}$$

$$\eta+2\mu=-q_0^2[4L_1^{000}(K_1,K)-9L_1^{100}(K_1,K)$$

$$+M_1^{100}(K_1,K)+4L_1^{200}(K_1,K)]\ , \tag{A15e}$$

$$\eta=-q_0^2[M_1^{100}(K_1,K)-L_1^{100}(K_1,K)]$$

$$-\tfrac{4}{3}q_0^2L_1^{110}(K_1,K)\ . \tag{A15f}$$

The last term in the expression for η is what we call η_{SC} and the rest of the expression is $\Delta\eta(\theta)$. If we substitute for η in equations (A15a) through (A15e) and expand $1/(1+r)=1-r$ we again find that all the θ dependence vanishes. The evaluation of the integrals leads to the recursion relations of Sec. IV B.

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