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A diophantine approach to the mass relative abundance of cosmological structures

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We investigate the cosmological problem of predicting rest masses for the most abundant stable structures in the universe, i.e. masses that can be multiply realized by large overlapping of resonant modes. By solving the Klein-Gordon equation for an expanding, symmetric the universe model we find the energy spectrum from the moving boundaries conditions in the form of a four dimensional non-homogeneous quadratic diophantine equation. The number of distinct solutions with same energy of this equation can provide information on the relative abundance of stable masses in the present universe. Even if the ranges for the numbers occurring in this equation are very large (10^{122}) we conjecture that the evolution of the solutions follows a certain pattern which stabilizes for ranges workable by present computers. We present solutions for some numerical ranges and compare them with experimental data.

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1. Introduction

The simplest causal model for predicting stable configurations of matter formed in an expanding universe, regarded as resonant wavefunctions of some equation of state, needs to have two minimal constraints. Namely, the model needs to be relativistically covariant and canonically quantifiable. The great challenge of the problem consists in covering the whole mass spectrum of our observational world, from Planck scale and the elementary particle range to the size of the present universe, which means the formidable task of solving a four-dimensional Minkowskian diophantine equation where its mass parameter ranges from 1 to 10^{122}.

2. The cosmological model

For this model we use the scalar wavefunctions generated by the solutions of the four-dimensional Klein-Gordon equation [1]

$$\triangle\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{m_0^4 c^2}{\hbar^2}\Psi. \tag{1}$$

This is a linear hyperbolic partial differential equation of order two, expressed in the M_4 Minkowskian space [2]. Here $c \sim 2.997 \times 10^8$ m/s is the speed of light in vacuum, and $\hbar = h/(2\pi) \sim 6.582 \times 10^{-16}\ eVs$ is the reduced Planck constant. For small fluctuations of $|\Psi|$ around the vacuum state, Eq. (1) is the linear approximation for the nonlinear sine-Gordon equation, [3], which is known to be a good model for constant curvature metrics (Jackiw-Teitelboim dilaton gravity) and provides a particular solution to the problem of the origin of low dimensional black hole entropy. The steady-states of Eq. (1) are wave functions for a free spin-less particle of rest mass m_0. These wavefunctions, representing scalar density fields $\Psi_n(x_1, x_2, x_3, t)$, depend on four quantum numbers.

We impose cosmological boundary conditions requiring these wave functions to cancel everywhere outside an expanding, at the speed of light, cubic box of size $L = cT$. That is, $\Psi = 0$ outside of the space-time 4-cube $cT \times cT \times cT \times T$. Here we take $T = 1/H$ to be the age of universe considered at the present moment, where H is the Hubble constant [4]. The recent Concordance model, [5], derived $H \sim 72$ km/s/Mpc, yielding an age for the Universe of $T = 13.75 \pm 0.17$ Gyr. The solutions of Eq. (1) take into account all possible information accumulated by the universe from its formation until the present time. Constrained with these boundary conditions the solutions become

$$\Psi_n = \sin\frac{n_1\pi x}{L}\sin\frac{n_2\pi y}{L}\sin\frac{n_3\pi z}{L}\sin\frac{n_0\pi t}{T}, \tag{2}$$

depending on four integer quantum numbers (n_0, n_1, n_2, n_3) fulfilling

$$n_0^2 - n_1^2 - n_2^2 - n_3^2 = \frac{c^4 m_0^2 T^2}{\pi^2\hbar^2} = \left(\frac{m_0}{\mu}\right)^2, \tag{3}$$

where $\mu = \pi\hbar/c^2T$. This value for μ gives the lowest limit for the rest mass scale, and it represents the smallest mass that could be measured in the present universe [6].

When the quantum number quadruplets take arbitrary values the expression in Eq. (3) generates all possible masses m_0 of resonant structures in the universe. However, it is

very likely that the same values for m_0 will be generated by a multitude of combinations of four integers. The number of distinct combinations (also called degeneracy of the wave function) can be associated with the abundance of the structure with rest mass m_0 predicted in this model. Namely, the larger this number is, the larger the stability of this structure. The number of solutions for a given value of E is proportional to the probability of existence of a particular mass (the one corresponding to the value of E used) in the universe[6]. Consequently, the goal of this study is to find the number of distinct solutions to Eq. (3) as a function of m_0, and to compare this number predicted spectrum with the relative distribution of masses in the universe, obtained from current astronomical measurements.

3. Physical limitations

For convenience we denote the right hand side of Eq. (3) by E. There are a few properties of the equation $E = n_0^2 - n_1^2 - n_2^2 - n_3^2$ that make our problem difficult. Our problem would be greatly simplified if our equation would be $E = n_0^2 + n_1^2 + n_2^2 + n_3^2$ because of Lagrange's Four-Square Theorem, which states that every positive integer is the sum of four squares [7].

Moreover, the number of distinct solutions to Eq. (3) may be estimated by using the symmetry of the equation namely by separating it into two simpler problems, and combining the solutions in some way.

$$n_0^2 - A = E_0, \text{ where } n_1^2 + n_2^2 + n_3^2 = A.$$

Since the sum of three squares does not have the product identity, calculating the number of solutions to this equation is not trivial. Even if we would consider the known solutions of the sum of the squares function in terms of Jacobi elliptic (generating) functions, [7], there are two more physical conditions that break the symmetry of the problem and make these symmetry approaches impractical.

Both the temporal quantum number n_0 and the sum of squares of the squares spatial quantum numbers n_i for $i = 1, 2, 3$ have upper bounds. If (n_0, n_1, n_2, n_3) is a solution of Eq. (3), then the maximum value n_0 is obtained when the spatial numbers are all zero. This maximum value cannot exceed the estimated mass of the observable universe. Therefore

$$n_0 \leq \frac{m_0}{\mu} \leq \frac{M_U}{\mu}, \tag{4}$$

where M_U is the mass of the observable universe estimated in the range $10^{35} - 10^{60}$ [9]. The right hand side of this first physical limitation has the range $0 - 10^{122}$.

Also, from quantum mechanics we know that the sum of the squares of the spatial quantum numbers is related to the average value of the linear momentum P, and the momentum can be expressed in terms of the wavelength

$$n_1^2 + n_2^2 + n_3^2 = \left(\frac{PL}{\pi\hbar}\right)^2 = \left(\frac{L}{\frac{\lambda}{2}}\right)^2 \leq \left(\frac{2L}{\lambda_P}\right)^2. \tag{5}$$

Here we estimated the average value of the momentum with $P = \hbar/\lambda$, λ being the de Broglie wavelength of the steady-state. This wavelength cannot be less than the Planck length

$$\lambda_P = \sqrt{\frac{\hbar G}{c^3}} \sim 1.61 \times 10^{35} m,$$

with G the universal constant of gravitation, which evaluates the right hand side of the second limitation, Eq. (5), in the range $0 - 10^{122}$.

The great challenge of the problem consists in covering the whole mass spectrum of our observational world, from the Planck scale and the elementary particle range ($m_0 = \mu$), to the size of the present universe ($m_0/\mu \sim 10^{122}$). This involves solving Eq. (3) for all values of the right hand side in the range $0 - 10^{122}$, with the limitations, Eqs. (4,5), in the same numerical range.

4. Solutions of the diophantine equation

In this section we present a number theoretic approach for finding the number of distinct solutions for Eq. (3) under constraints given by Eqs. (4,5). In order to study the solutions of this diophantine equation we introduce the notations E, N_1, N for the right hand terms of Eqs. (3,4,5), respectively. We formulate the following mathematical problem.

For any given set of three integers $N_1, N, E \in \mathbf{N}$, find all the quadruples of integers (n_0, n_1, n_2, n_3), $n_i \in \mathbf{Z}, i = 0, \ldots, 3$ that are solutions of the problem

$$n_0^2 - n_1^2 - n_2^2 - n_3^2 = E, \tag{6}$$

within the restrictions

$$n_0 \leq N_1, \quad n_1^2 + n_2^2 + n_3^2 \leq N. \tag{7}$$

For given E, N_1, N we denote the number the solutions fulfilling Eq. (6) and the constraints Eqs. (7) by $\tilde{G}(E, N_1, N)$.

By direct substitution it is easy to see that for odd E the quadruple $(n^2+(E+1)/2, n^2+(E-1)/2, n, n)$ is a solution of Eq. (6) for all $n \in \mathbf{Z}$. Similarly, for even E, if E is an even multiple of 2, the quadruple $(2n^2 + (E+4)/4, 2n^2 + (E-4)/4, 2n, 2n)$ is a solution of Eq. (6) for all $n \in \mathbf{Z}$, and if E is an odd multiple of 2, $(2n^2 + 2n + (E+6)/4, 2n^2 + 2n + (E-2)/4, 2n+1, 2n+1)$ is a solution for all $n \in \mathbf{Z}$.

Let us suppose (a, b, c, d) is a solution of Eq. (6) subjected to the constraints in Eqs. (7). Then since $a^2 - E = b^2 + c^2 + d^2$, $a^2 - E \leq N$ so $|a| \leq \sqrt{N+E}$. With the restriction that $a \leq \min(N_1, \sqrt{N+E})$, both restrictions in Eq. (7) will be satisfied. By this argument, we need only consider bounds such that $N_1 = \sqrt{N+E}$. Consequently, we introduce the notation $\tilde{G}(E, N_1, N) = G(E, N)$.

We have shown that there are infinitely many solutions to Eq. (6) for any E. However, subject to the bounds on n_0 and $n_1^2 + n_2^2 + n_3^2$ these solutions are restricted. For example, if E is odd, $(n^2 + (E+1)/2, n^2 + (E-1)/2, n, n)$ is a solution for all $n \in \mathbf{Z}$, but now $n^2 + (E+1)/2 \leq \sqrt{N+E}$. Thus there are $\lfloor \sqrt{\sqrt{N+E} - (E+1)/2} \rfloor$ (integer part) of these solutions counted in $G(E, N)$. Since the given solutions are not an exhaustive set, this is an interesting count but does not give a total value for $G(E, N)$.

We now use number theoretic properties to develop a formula to generate $G(E, N)$. First, we discuss a few necessary conditions for the solutions.

Proposition 1

The square of the greatest common divisor of the four quantum numbers in a solution should divides E.

Proof

Assume that (a, b, c, d) is a solution to Eq. (6). Since $D = \gcd(a, b, c, d)$, it is clear that $D|a$, $D|b$, $D|c$, and $D|d$. Therefore, there exist integers k_0, k_1, k_2, and k_3 such that $a = k_0 D$, $b = k_1 D$, $c = k_2 D$, and $d = k_3 D$. By substitution, we know that

$$\begin{aligned} E &= a^2 - b^2 - c^2 - d^2 \\ &= (k_0 D)^2 - (k_1 D)^2 - (k_2 D)^2 - (k_3 D)^2 \\ &= k_0^2 D^2 - k_1^2 D^2 - k_2^2 D^2 - k_3^2 D^2 \\ &= D^2 (k_0^2 - k_1^2 - k_2^2 - k_3^2). \end{aligned}$$

Since $k_0^2 - k_1^2 - k_2^2 - k_3^2 \in \mathbb{Z}$, we conclude that $D^2|E$. ■

By the above proposition, we see that the only solutions to our equation will be the ones for which the square of the greatest common divisor of (n_0, n_1, n_2, n_3), denoted $\gcd(n_0, n_1, n_2, n_3)$, appears as a factor of E. We can also show that each of the solutions for a given E with $\gcd(n_0, n_1, n_2, n_3) > 1$ corresponds to a solution for a smaller value of E. In particular, we have:

Proposition 2

There is a one-to-one correspondence between the solution for Eq. (6) over the natural numbers with $E = E_0$ and $\gcd(n_0, n_1, n_2, n_3) = k$ and solutions of the same equation with $E = E_0/k^2$ with $\gcd(n_0, n_1, n_2, n_3) = 1$.

Proof

Let S be the set of solutions of Eq. (6) with $E = E_0$, and let T be the set of solutions for Eq. (6) with $E = E_0/k^2$. Since

$$\left(\frac{a}{k}\right)^2 + \left(\frac{b}{k}\right)^2 + \left(\frac{c}{k}\right)^2 + \left(\frac{d}{k}\right)^2 = \frac{a^2 + b^2 + c^2 + d^2}{k^2},$$

and

$$\gcd\left(\frac{a}{k}, \frac{b}{k}, \frac{c}{k}, \frac{d}{k}\right) = \frac{1}{k} \gcd(a, b, c, d) = 1,$$

it is clear that we can define a map $f : S \to T$ such that $f(a, b, c, d) = (a/k, b/k, c/k, d/k)$. Suppose $f(a, b, c, d) = f(s, t, u, v)$ and $\gcd(a, b, c, d) = \gcd(s, t, u, v) = k$. Then $(a/k, b/k, c/k, d/k) = (s/k, t/k, u/k, v/k)$. Multiplying by the scalar k, we see that $(a, b, c, d) = (s, t, u, v)$. Therefore f is one-to-one.

Suppose now $(x, y, z, w) \in T$. Then $\gcd(x, y, z, w) = 1$ and $x^2 - (y^2 + z^2 + w^2) = E/k^2$. Then $(kx)^2 - [(ky)^2 + (kz)^2 + (kw)^2] = E$ and $\gcd(kx, ky, kz, kw) = k$.Thus $(kx, ky, kz, kw) \in S$ and f is onto. So f is a one-to-one correspondence. ■

In the light of the above proposition we define functions $G_k(E,N)$ to be the number of solutions of Eqs. (6,7) such that $\gcd(n_0,n_1,n_2,n_3)=k$. Note that $G_k(E,N)=0$ if k^2 does not divide E. By applying Proposition 1, we see that $G(E,N)=\sum_{k^2|E} G_k(E,N)$. Now by Proposition 2, every solution counted by $G_k(E,N)$ comes from a solution of Eqs. (6,7) with the right hand side of Eqs. (6) equal to E/k^2. In particular, taking the bounds $n_0 \le \sqrt{N+E}$ and $\sum_{i=1}^3 n_i^2 \le N$ into consideration we see that the solution (a,b,c,d) corresponds to $(a/k,b/k,c/k,d/k)$, where

$$\frac{a}{k} \le \frac{\sqrt{N+E}}{k} \quad \text{and} \quad \left(\frac{b}{k}\right)^2 + \left(\frac{c}{k}\right)^2 + \left(\frac{d}{k}\right)^2 \le \frac{N}{k^2}.$$

Note that the first bound can be written as

$$\frac{a}{k} \le \sqrt{\frac{N}{k^2} + \frac{E}{k^2}}.$$

Thus $G_k(E,N) = G_1(E/k^2, N/k^2)$. We now have

$$G(E,N) = \sum_{k^2|E} G_1\left(\frac{E}{k^2} + \frac{N}{k^2}\right). \tag{8}$$

We have shown that we need only consider solutions to Eqs. (6,7) for which $\gcd(n_0,n_1,n_2,n_3) = 1$ subject to various constraints.

5. Discussion of the solutions and their degeneracy

We now present some computer generated data for $G(E,N)$ in the range $1 \le E \le 64$ and $1 \le k \le 147$. We assumed, without loss of generality, that $n_1 \ge n_2 \ge n_3$ when we generated solutions. We also considered only nonnegative integers since $(\pm n_0, \pm n_1, \pm n_2, \pm n_3)$ is a solution whenever (n_0,n_1,n_2,n_3) is a solution.

In Figs. (1,3) we illustrate some properties of $G(E,N)$ noticed from our computer generated data. If we compute the ratio of the number of solutions for say $E=4$ to the number of solutions for $E=1$, we see that this ratio seems to be approaching a particular number (roughly 1.73) as we increase the limit N. Even though our computers cannot compute the number of solutions to our equation when $N \approx 10^{122}$ as we would hope, it seems that we may be able to use ratios of numbers of solutions for different values of E to make predictions about how many more solutions one value of E has than another value of E in the range of $N \approx 10^{122}$. In Fig. (1) we present the seemly constant ratio of

$$\frac{G(E,N)}{G(E,N')},$$

from where...exemplify that the bounds do not change the nature of the number of solutions function $G(E,N)$. From this figure we conjecture that the ratio of the number of solutions for the same value for E, and two arbitrary different bound values N, N' approaches asymptotically for large values of E the limit

$$G(E,N) = \frac{N}{N'} G(E,N). \tag{9}$$

In Fig. (3) we present the asymptotic limit of ...

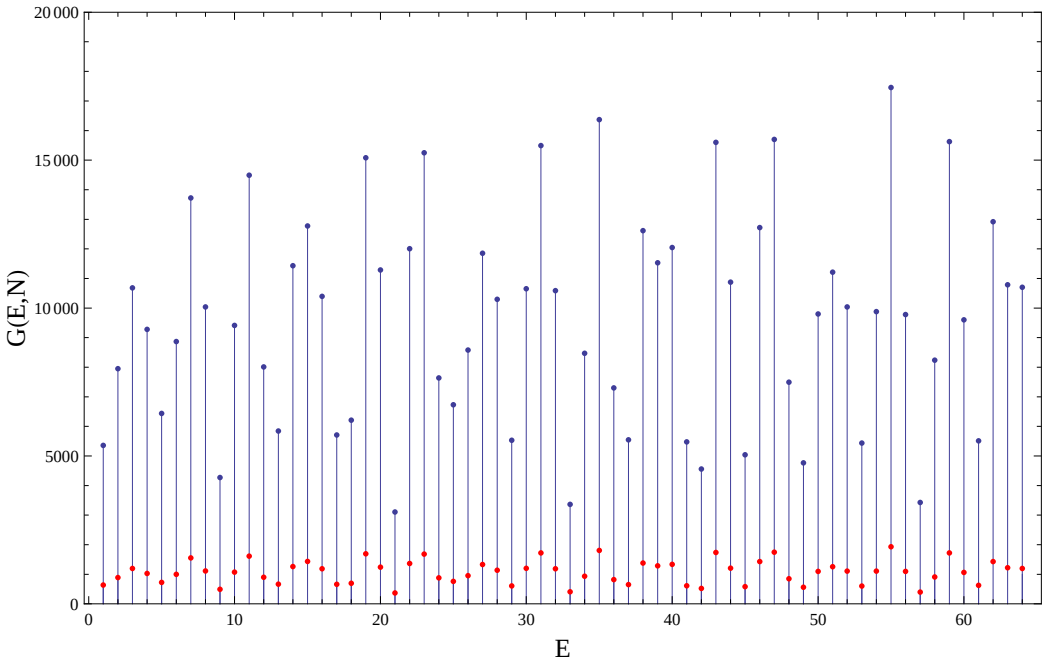

Figure 1. We present an example of the distribution of the number of solutions $G(E, N)$, for fixed limiting values $N = 16,000$ (red dots) and $N = 148,000$ (blue dots), versus E in the range $0, \ldots, 64$.

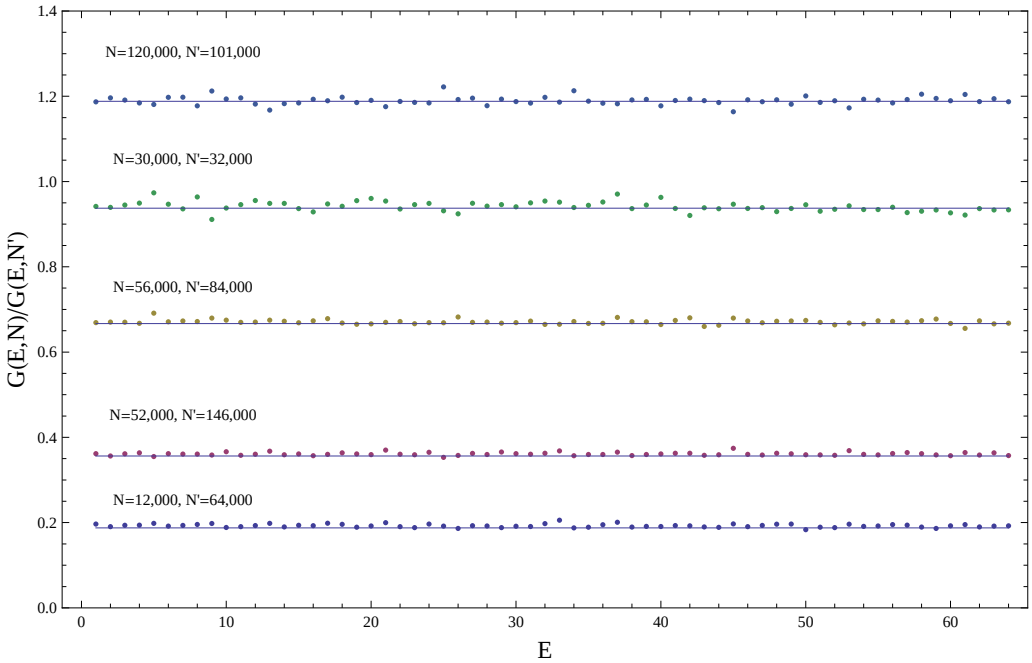

Figure 2. We plot the ratio $G(E,N)/G(E,N')$ versus E for different arbitrary values of N, N' (dots) and the ratios N/N' for comparison.

6. Conclusion

From nuclear and atomic data we can evaluate the abundance of different stable structures in the observable universe. In Fig. (4) we present such a distribution of relative abundances plotted versus the rest mass of these structures.

In the physical sense, this would give us an idea about the relative abundances of particular masses.

Successes:

1. Explained the increasing abundance of masses of structures, because of their recursive formation using same building blocks.

2. Explained that the structure of solutions at smaller N is the same, asymptotic approach, as the structure of solutions at larger N. Both important as saving computer numerical calculations, and following the constructive nature of building blocks.

To be obtained:

1. Wider gaps followed by stripes of solutions

2. Some match with very stable known rest masses

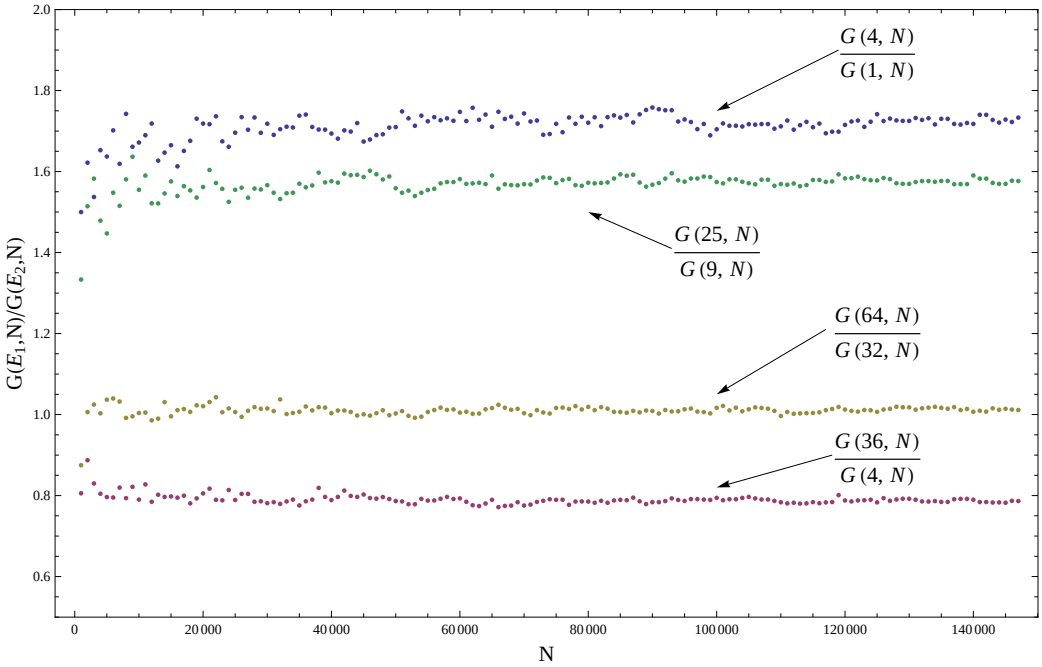

Figure 3. The dependence of the ratio of the numbers of solutions for two different values of E versus the value of the upper bound N. We noticed that for any two arbitrary values $E_1 \geq E_2$, the ratio $G(E_1, N)/G(E_2, N)$ approaches very soon a constant asymptotical limit of small finite value. Hence we conjecture that the number of solutions $G(E, N)$ can be extrapolated even for very large limits of N with the behavior at small values.

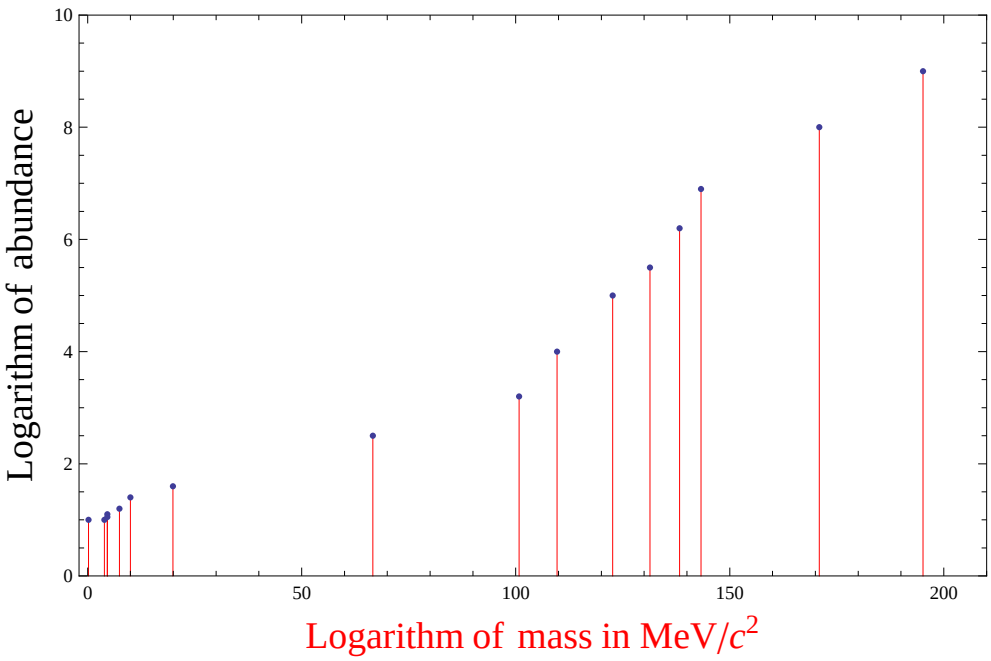

Figure 4. Relative abundance of electron, quark, muon, proton, boson, carbon atom, lead atom, biological cell, macromolecule, dust grain, human scale, Mercury-like planets, Jupiter-like planets, Sun-like star, larger stars versus the logarithm of their rest mass.

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