SU(1,1) Algebraic Description of One-dimensional Potentials within The R-matrix Theory

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SU(1,1) algebraic description of one-dimensional potentials within the R-matrix theory

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Abstract

The eigenstates of a particle in a rectangular-well potential with appropriate boundary conditions are proved to be the standard basis of an irreducible representation of the su(1,1) Lie algebra. The algebra operators are constructed explicitly and the energy levels and the R-function are calculated. Due to the general connection between the generators of su(1,1) we can algebraically relate a wide class of one-dimensional potentials to the su(1,1) Lie algebra in this framework. This algebraic approach allows us to write an algebraic parametrization for the R-function.

Short title: SU(1,1) algebraic description of potentials.

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1 Introduction

Lie algebras are among the basic tools of modern physics. The algebraic approaches to the problems of nuclear, atomic or molecular physics have been extensively employed in the last decades. The algebraic techniques proved to be successful in the Interacting Boson Model in treating quadrupole collective motion and also with other degrees of freedom [1]. Group theoretical descriptions of dipole rotations and vibrations [2] have been applied also to nuclear molecular states [3], to the vibron model [4], applications of spectrum generating algebras techniques to scattering problems by using U(4) symmetry [5] as well as to other systems relevant to molecular, nuclear and hadronic spectroscopy [6].

Many applications find place for the Algebraic Scattering Theory [7] especially concerning their success in classification of exactly solvable potentials and concerning their relation with the potential picture, study of particles in central potentials subjected to magnetic fields [8] and algebraic description of atom-molecule interactions [9]. One-dimensional atom-molecule collisions have been studied by using a combination of differential and algebraic techniques for a variety of potentials [10]. We mention also the recent introduction of supersymmetric quantum mechanics (SUSY) which allows pairing of isospectral potentials [11] and the Natanzon classification of potentials [12]. New directions appeared with the developing of the “quantized” or q-deformed Lie algebras (quantum groups) [13], like the exactly one-dimensional solvable potentials [14]. And the list of applications is far from being closed. Solvable potentials generated by su(1, 1) and su(2) Lie algebras are also analysed in [15] and [16], respectively, and recently in [17].

Such approaches opened the question of the geometrical interpretation of the algebraic Hamiltonian which is written as one of the Lie algebra operators or a bilinear combination in these operators. The geometric-algebraic connection is well known for the Coulomb problem [18] and Morse or Pöschl-Teller potential [7]. For the last case the algebraic interpretation is realized taking an auxiliary coordinate. By using such an auxiliary coordinate, Kais and Levine [19] gave an algebraic interpretation of the states in the infinitely deep rectangular-well.
In the following we shall give an algebraic realization for the rectangular-well potential with special boundary conditions (the boundary conditions that are useful to describe the scattering in the R-function [20] framework) without using an auxiliary coordinate. Also, we describe in the algebraic framework the problem of the oscillator potential on the real semiaxis. This approach will allow us to obtain algebraically a parametrization of the R-function [20] which contains all the informations we need to describe the scattering [21]. This parametrization could be considered as an extension of the algebraic scattering theory proposed by Iachello and co-workers [6,7,22]

2 Rectangular-well potential with boundary conditions

We consider the problem of a particle in a one-dimensional rectangular-well potential \( V(r) = -V_0 \) if \( r \in [0, a] \) where \( V_0 \) is the potential depth and \( a \) its radius. The eigenfunctions and the spectrum of the Hamiltonian \( H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - V_0 \) with the boundary conditions

\[
\Phi(r = 0) = 0 \\
\frac{d\Phi}{dr}(r = a) = 0
\]

are well known

\[
|\mu> = \Phi_\mu(r) = \sqrt{\frac{2}{a}} \sin \left(\mu + \frac{1}{4}\right) 2\pi \cdot \frac{r}{a} \\
E_\mu = \frac{\hbar^2}{2m} \left(\frac{2\pi}{a}\right)^2 \left(\mu + \frac{1}{4}\right)^2 - V_0
\]

where \( \mu \in \mathbb{Z} \). The functions \( \Phi_\mu \) are normalized. \( \int_0^a \Phi_{\mu_1}^* (r) \Phi_{\mu_2} (r) dr = \delta_{\mu_1\mu_2} \)

We concern now with the problem of R-matrix description of the scattering problem for this potential [6,7]. In this approach, the dynamics in the internal region is taken into account through the R-matrix. For the potential scattering, the R-matrix reduces to the R-function which can be written in terms of the internal spectrum and
the values at the boundary \( r = a \) of the normalized eigenfunctions

\[ R(E) = \sum_{\mu} \frac{\gamma_{\mu}^2}{E_{\mu} - E} \] (4)

where \( \gamma_{\mu} = \sqrt{\frac{\hbar^2}{2ma}} \Phi_{\mu}(a) \) and \( \gamma_{\mu}^2 \) is the reduced width of the level \( \mu \). The R-function together with the logarithmic derivative of the outgoing wave-function at \( r = a \) allow a complete description of the scattering.

Our aim is to give an algebraic description of the eigenvalue problem for the Hamiltonian

\[ H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - V_0 \] (5)

with the boundary conditions (1).

In order to obtain the appropriate algebraic approach we begin by defining the operator \( \sigma \), acting on \( \mathcal{R} \), as \( \sigma x = -x \). We prolonge its action on smooth functions defined on \( \mathcal{R} \) by \( \sigma f(x) = f(-x) \). This operator can be realised in a differential form by: \( \sigma = e^{\pm i\pi x \partial} \), where we denote \( \partial = \frac{\partial}{\partial x} \). It is easy to verify that we have the correct action of \( \sigma \), by using the dilatation operator [23] \( e^{\alpha x \partial} f(x) = f(e^\alpha x) \) for \( \alpha = \pm i\pi \). In this realisation the operator \( \sigma \) generates, together with the operators: \( t = \partial, d = x\partial \) and \( e = x^2\partial \) (the translation, dilatation and expansion operators respectively, which define the Lie algebra which invariates the line, i.e. the affine group) a quadratic associative algebra having \( su(1,1) \) as a subalgebra (generated by \( t, d, e \)). We can write the commutators:

\[ [\sigma, t] = 2\sigma t, \quad [\sigma, e] = 2\sigma e, \quad [\sigma, d] = 0 \] (6)

\[ [e, t] = -2d, \quad [d, t] = -t, \quad [d, e] = e \] (7)

It is nice to remark that, since \( \sigma = e^{i\pi d} \), we can write the first, second and fourth commutators in eqs.(6-7) in the form:

\[ [\sigma, t] = 2\sigma t, \quad [\sigma, e] = -e2\sigma, \quad [e, t] = -\frac{2i}{\pi} \ln \sigma \] (8)

which represent a special case of the quadratic deformed algebra introduced in [24] with linear deformation in the first two commutators: \( G(\sigma) = 2\sigma \) and having for the
third commutator: $F(\sigma) = -2i/\pi \ln \sigma$. The unirreps of these algebras are studied for general functions $G$ and $F$ and their Hopf algebra (as a q-deformed algebra) structure was proved [25]. This nonlinear algebra, eqs.(6,7), is in itself interesting but in the following, because we want to analyse the dynamics of the particle in the interval $[0,a]$, we need to use another operator, instead of $\sigma$, which invariates this interval. We introduce the operator $T$, defined by its action: $Tf(r) = f(a-r)$, where we write $r$ for $x$ when $x \in [0,a]$. The corresponding differential realisation of $T$ is given by: $T = e^{(-a+i\pi r)\partial_r} = e^{-at}\sigma$. Consequently we introduce the operators:

\[
J_z = \frac{a}{2\pi} T \frac{d}{dr} - \frac{1}{4}
\]

\[
J_\pm = \left( \cos 2\pi \frac{r}{a} \pm \sin 2\pi \frac{r}{a} \cdot T \right) \cdot \left( \frac{a}{2\pi} T \frac{d}{dr} - \frac{1}{4} \pm \frac{1}{2} \right)
\]

which act on the space of complex valued functions $L = \{ f : [0, a] \rightarrow C | f \in L^2([0, a]), f \in C^1([0, a]), f(0) = 0 \}$. Taking into account the definition of the operators $J_z, J_\pm$ and the obvious relations

\[
T \cdot \frac{d}{dr} = -\frac{d}{dr} \cdot T
\]

\[
T \cdot \cos 2\pi \frac{r}{a} = \cos 2\pi \frac{r}{a} \cdot T
\]

\[
T \cdot \sin 2\pi \frac{r}{a} = -\sin 2\pi \frac{r}{a} \cdot T
\]

we can establish the following commutation relations

\[
[J_z, J_\pm] = \pm J_\pm
\]

\[
[J_+, J_-] = -2J_z
\]

Also, we have the hermiticity properties

\[
J_z^\dagger = J_z
\]

\[
J_\pm^\dagger = J_\mp
\]

As an example, we prove the relation (12). We have

\[
\left( f_1, (J_z + \frac{1}{4})f_2 \right) = -\int_0^a f_1^*(r) \frac{a}{2\pi} \frac{df_2(a-r)}{dr} dr = \\
-\frac{a}{2\pi} f_1^*(r)f_2(a-r)|_0^a + \int_0^a \left( \frac{a}{2\pi} \frac{df_1(r)}{dr} \right)^* f_2(a-r) dr =
\]
\[ \int_{0}^{a} \left( \frac{a}{2\pi} \frac{df_{1}(a-r)}{d(a-r)} \right)^{*} f_{2}(r) dr = \left( (J_{z} + \frac{1}{4}) f_{1}, f_{2} \right) \]

where we used the condition \( f_{1,2}(r = 0) = 0 \). The relations (13) is now obvious. Therefore, the operators \( J_{z}, J_{\pm} \) are closed under commutation relations (11) and express a realization of the Lie algebra \( su(1, 1) \). It is a straightforward exercise to prove that the action of the operators (9) on the eigenfunctions (2) can be written

\[
J_{z}|\mu> = \mu|\mu>
J_{\pm}|\mu> = \left( \mu \pm \frac{1}{2} \right) |\mu \pm 1>
\]  

The \( su(1, 1) \) Casimir operator is \( C = J_{z}^{2} - J_{z} - J_{+}J_{-} \) and we have

\[
C|\mu> = -\frac{1}{4} |\mu>
\]

The relations (14) and (15) mean that the internal eigenstates \( |\mu> \) can be taken as the standard basis of the irreducible representation of \( su(1, 1) \) with \( j = -\frac{1}{2} \) or \( j(j+1) = -\frac{1}{4} \) and \( \mu \in \mathbb{Z} \). In fact, the relations (14) are the \( j = -\frac{1}{2} \) case of the generator action on the standard basis.

Now, we can write the Hamiltonian (5) in the above algebraic framework as

\[
H_{0} = \frac{\hbar^{2}}{2m} \left( \frac{2\pi}{a} \right)^{2} \left( J_{z} + \frac{1}{4} \right)^{2} - V_{0}
\]  

Thus, the \( su(1, 1) \) Lie algebra is the dynamical algebra for the internal motion. The appropriate realization is given in terms of a single variable and it is a differential realization, i.e. \( T = \sum_{n=0}^{\infty} \frac{(a-2r)^{n}}{n!} \frac{d^{n}}{dr^{n}} \), it is an infinite order differential operator.

The next step is to obtain a purely algebraic relation for the reduced widths, for the above case. The reduced widths being proportional to the wave-function values at the boundary, it seems that a purely algebraic description is not possible. Therefore, by taking into account the realization (9) we obtain the asymptotic connection:

\[
\lim_{r \to a} J_{\pm}\varphi(r) = \lim_{r \to a} \left( J_{z} \pm \frac{1}{2} \right) \varphi(r)
\]

where \( \varphi(r) \) is an arbitrary function of class \( C^{1} \). If \( \varphi(r) = \Phi_{\mu}(r) \) then, taking into account (14), we have a simple recurrence relation:

\[
\left( \mu \pm \frac{1}{2} \right) \Phi_{\mu \pm 1}(a) = \left( \mu \pm \frac{1}{2} \right) \Phi_{\mu}(a)
\]
Consequently the reduced widths do not depend on the quantum number $\mu$ as we have $\mu \in \mathbb{Z}$. Therefore, all the $\gamma_\mu$ are equal, and we can chose a certain state, e.g. $\gamma_0$. We have algebraically obtained, up to a multiplicative factor, the R-function in the form:

$$R(E) = \sum_{\mu \in \mathbb{Z}} \frac{\gamma_0^2}{\hbar^2 \left( \frac{2\pi}{a} \right)^2 \left( \mu + \frac{1}{4} \right)^2 - V_0 - E}$$

$$= \frac{a\gamma_0^2}{\hbar} \sqrt{\frac{m}{2(V_0 + E)}} \tan \frac{a}{\hbar} \sqrt{2m(V_0 + E)}$$

The coefficient $\gamma_0$ can be given only by the analytical expression of one single state, like $|0\rangle$.

So far we have been interested in the simplest problem of the rectangular potential. In the following we shall extend the algebraic description to other potentials.

### 3 Algebraization of other potentials

The realization (9) allows us to write

$$\cos 2\pi \frac{r}{a} = (2J_z - 1)^{-1}J_+ + (2J_z + 1)^{-1}J_- = J_+(2J_z + 1)^{-1} + J_-(2J_z - 1)^{-1}$$

(19)

It is important to note that the expression (19) is not in the $su(1, 1)$ universal enveloping algebra but it can be considered to be well defined on a dense subspace of the $su(1, 1)$ representation Hilbert space. We could write algebraically $\cos n2\pi \frac{r}{a}$, $n \in \mathbb{N}$ using the expression of $\cos nx$ as an $n$-order polinomial in $\cos x$. For example, using the particular value of the Casimir operator, we have

$$\cos 4\pi \frac{r}{a} = 2(2J_z - 1)^{-1}(2J_z - 3)^{-1}J_+^2 + 2(2J_z + 1)^{-1}(2J_z + 3)^{-1}J_-^2$$

(20)

To write algebraically a certain potential which can be written as a Fourier series we also need an expression for $\sin 2\pi \frac{r}{a}$, in terms of the operators $J_z$, $J_\pm$. Such an expression seems to be a very complicated one and we succeeded to obtain it using the explicit
realization of the standard basis (2). One can write
\[
\sin 2\pi \frac{r}{a} = \frac{1}{\pi} (2J_z + \frac{3}{2})^{-1} (2J_z - \frac{1}{2})^{-1} + \\
\frac{1}{\pi} \sum_{n=1}^{\infty} (2J_z + \frac{3}{2} - n)^{-1} (2J_z - \frac{1}{2} - n)^{-1} \left[ (J_z - \frac{1}{2})^{-1} J_+^n \right] + \\
\frac{1}{\pi} \sum_{n=1}^{\infty} (2J_z + \frac{3}{2} + n)^{-1} (2J_z - \frac{1}{2} + n)^{-1} \left[ (J_z + \frac{1}{2})^{-1} J_-^n \right]
\] (21)

Therefore, in principle, every potential developed in a Fourier series can be written in an algebraic form. As an example we shall study the Hamiltonian
\[
H = H_0 + \alpha \cos 2\pi \frac{r}{a} = \\
-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - V_0 + 2\alpha \cos 2\pi \frac{r}{a} = \\
\frac{\hbar^2}{2m} \left( \frac{2\pi}{a} \right)^2 (J_z + \frac{1}{4})^2 - V_0 + 2\alpha \{(2J_z - 1)^{-1} J_+ + (2J_z + 1)^{-1} J_- \}
\] (22)

where \( \alpha \) is a coupling parameter. We would like to obtain the eigenstates of the algebraic Hamiltonian (22) in terms of the \( H_0 \) eigenstates, i.e. in terms of the \( su(1,1) \) standard basis (14).

We suppose that the \( H \) eigenstate \( |E> \) corresponds to the unknown energy \( E \), \( H|E>= |E> \). We can write
\[
|E> = \sum_{\mu} C_{\mu}^E |\mu>
\] (23)

We admit that the action of \( H \) on the above sum commutes with the sum. In the case \( \alpha = 0 \), this assumption gives
\[
\left( E - \frac{\hbar^2}{2m} \left( \frac{2\pi}{a} \right)^2 \left( \mu + \frac{1}{4} \right)^2 + V_0 \right) C_{\mu}^E = 0
\]
for every integer \( \mu \) and one obtains that the energy \( E \) must be \( E_\mu \). Therefore, we reobtained the \( H_0 \) spectrum and its eigenstates.

In the above assumption concerning the Hamiltonian (22) we obtain the following recursion relation
\[
\alpha C_{\mu-1}^E + \left[ \frac{\hbar^2}{2m} \left( \frac{2\pi}{a} \right)^2 \left( \mu + \frac{1}{4} \right)^2 - (E + V_0) \right] C_{\mu}^E + \alpha C_{\mu+1}^E = 0 \quad , \quad \mu \in \mathbb{Z}
\] (24)

By writing \( C_{\mu+1}^E = \xi_{1,\mu} C_{\mu}^E \), the relation (24) reads
\[
\alpha \frac{1}{\xi_{\mu-1}^E} + \left[ \frac{\hbar^2}{2m} \left( \frac{2\pi}{a} \right)^2 \left( \mu + \frac{1}{4} \right)^2 - (E - V_0) \right] + \alpha \xi_{\mu}^E = 0
\] (25)
For the convergence of the series (23) at the boundary, where $|\mu| \to \text{const}$, it is necessary to have $\lim_{\mu \to \infty} \xi_\mu = 0$ and $\lim_{\mu \to -\infty} \frac{1}{\xi_\mu} = 0$. Taking into account the relation (25) viewed as a definition relation for $\xi^E_\mu$, we can write

$$\xi^E_0 = \frac{\epsilon - c \left(\frac{1}{4}\right)^2}{\alpha} - \frac{1}{\epsilon - c \left(\frac{1-1}{4}\right)^2 - c \left(\frac{2-\frac{1}{4}}{4}\right)^2 - \cdots} \quad (26)$$

We can consider as well the relation (25) as a relation which gives $\xi^E_{\mu-1}$ and we obtain

$$\xi^E_0 = \frac{1}{\epsilon - c \left(\frac{1+\frac{1}{4}}{4}\right)^2 - c \left(\frac{2+\frac{1}{4}}{4}\right)^2 - \cdots} \quad (27)$$

where we used the notation $\epsilon = E + V_0$ and $c = \frac{\hbar^2}{2m} \left(\frac{2\pi}{a}\right)^2$. Therefore, the continuous fractions (26) and (27) must be identical and this condition gives an equation for the unknown energy $E = \epsilon - V_0$. The lowest order approximation for the continuous fractions yields

$$\epsilon - c \left(\frac{1}{4}\right)^2 = 0$$

which represents the energy of the first $H_0$ eigenstate, i.e. the first unperturbed state. The next order approximation gives

$$\left[\epsilon - c \left(\frac{1}{4}\right)^2\right] \left[\epsilon - c \left(\frac{3}{4}\right)^2\right] \left[\epsilon - c \left(\frac{5}{4}\right)^2\right] - \alpha^2 \left[\epsilon - c \left(\frac{3}{4}\right)^2\right] - \alpha^2 \left[\epsilon - c \left(\frac{5}{4}\right)^2\right] = 0 \quad (28)$$

If we take $\alpha = 0$ in the above equation, then we obtain the energies of three unperturbed levels, $\epsilon_0$, $\epsilon_{\pm 1}$. We note that this approximation is equivalent with $\frac{1}{\xi^E_2} = 0$ or $C_{-2} = 0$ and $\frac{1}{\xi^E_1} = 0$ or $C_2 = 0$. Therefore, equation (28) is equivalent with the truncation of the Hilbert space of states to the first three unperturbed levels. In the first order in $\alpha^2$ the solutions of the equation (28) are

$$\tilde{\epsilon}_0 = \epsilon_0 + \frac{\alpha^2}{\epsilon_0 - \epsilon_1} + \frac{\alpha^2}{\epsilon_0 - \epsilon_{-1}} \quad (29)$$

$$\tilde{\epsilon}_{-1} = \epsilon_{-1} + \frac{\alpha^2}{\epsilon_{-1} - \epsilon_0} \quad (30)$$
\[ \tilde{\epsilon}_1 = \epsilon_1 + \frac{\alpha^2}{\epsilon_1 - \epsilon_0} \]  

(31)

In the second order in \( \alpha^2 \) the lowest level can be written

\[ \tilde{\epsilon}_0 = \epsilon_0 + \frac{\alpha^2}{\epsilon_0 - \epsilon_1} + \frac{\alpha^2}{\epsilon_0 - \epsilon_{-1}} \]

\[ -\alpha^4 \left[ (\epsilon_0 - \epsilon_{-1}) + (\epsilon_0 - \epsilon_1) \right] \left[ (\epsilon_0 - \epsilon_{-1})^2 + (\epsilon_0 - \epsilon_1)^2 \right] \]

(32)

We want to stress few observations. The solution (29) for the energy of the lowest level is exactly the result of the second order perturbation theory and the contribution \( \tilde{\epsilon}_0 - \epsilon_0 \) is negative (a general result of the perturbation theory). In the same order of approximation for the continuous fractions we obtained the solution (32) which contains a term corresponding to an order higher than two in the perturbation theory.

By using the solution \( \tilde{\epsilon}_0 \) (or higher order approximations) and the same order approximation in (27) we obtain

\[ \xi_{\tilde{\epsilon}_0} = \frac{\alpha}{\epsilon_0 - \epsilon_1} \]  

(33)

In the lowest order in \( \alpha \) we have

\[ \xi_{\tilde{\epsilon}_0} = \frac{\alpha}{\epsilon_0 - \epsilon_1} \]  

(34)

We can choose \( C_{\tilde{\epsilon}_0}^\mu = 1 \) and using (34) we obtain \( C_{\tilde{\epsilon}_0}^\mu = \frac{\alpha}{\epsilon_0 - \epsilon_1} \). Now, the relation (24) allows us to obtain \( C_{\tilde{\epsilon}_0}^\mu \) for all integers \( \mu \). To obtain a normalized state we need to calculate the norm \( N = \sqrt{\sum |C_{\tilde{\epsilon}_0}^\mu|^2} \) and the partial width will be in the form

\[ \tilde{\gamma}_0 = \frac{1}{N} \left( \sum_{\mu} C_{\tilde{\epsilon}_0}^\mu \right) \cdot \gamma \]  

(35)

For example, if \( \alpha = \frac{\alpha}{16} \) and we use \( C_{\tilde{\epsilon}_0}^\mu \) for \( \mu = -2, -1, 0, 1, 2 \) in the formula (35) we obtain \( \tilde{\gamma}_0 = 0.8 \cdot \gamma_0 \).

The simplicity of the above example is due to the particular form (22) of the Hamiltonian which we considered. If \( H \) contains, for example, a term in \( \cos 2 \cdot 2\pi \frac{r}{a} \) then a recursion relations similar to (24) can be written but it will contain five coefficients.
4 Oscillator potential with boundary condition in origin

In the following we shall study the eigenfunction problem for the harmonic oscillator Hamiltonian on the real semiaxis with the boundary condition

\[ \frac{d\Phi}{dx}(x = 0) = 0 \]  \hfill (36)

We take \( m = \frac{1}{2}, \omega = 2 \) and

\[ H_0 = p^2 + x^2 = -\hbar^2 \frac{d^2}{dx^2} + x^2, \ x \in [0, \infty) \]  \hfill (37)

The eigenfunctions of the above Hamiltonian are

\[ |k> = \Psi_k(x) = \sqrt{\frac{1}{\pi \hbar}} \frac{1}{\sqrt{2^{2k}(2k)!}} e^{-\frac{1}{2} \hbar x^2} H_{2k} \left( \sqrt{\frac{1}{\hbar}} x \right) \]  \hfill (38)

where \( k = 0, 1, 2, \cdots \) and \( H_n \) are the Hermite polynomials. The corresponding energies are

\[ E_k = 4\hbar \left( k + \frac{1}{4} \right) \]  \hfill (39)

In the following we want to realise the \( su(1,1) \) Lie algebra in terms of creation and annihilation operators. The Hamiltonian of the harmonic oscillator can be written in terms of the creation and annihilation operators, \( a_+ \) and \( a_- \)

\[ a_\pm = \frac{1}{\sqrt{2\hbar}} \left( -i\hbar \frac{d}{dx} \pm ix \right) \]  \hfill (40)

which satisfies the commutation relation

\[ [a_-, a_+] = 1 \]  \hfill (41)

The operators defined bellow

\[ J_z = \frac{1}{4}(a_+ a_- + a_- a_+) \]

\[ J_\pm = \frac{1}{2} a_\pm \]  \hfill (42)
satisfies the \(su(1,1)\) commutation relations of \(su(1,1)\), eq.(11), Also, we have the hermiticity properties \(J_+^\dagger = J_z\) and \(J_+^\dagger = J_-\). For the realization (42) we have the Casimir operator

\[
C = J_z^2 - J_z - J_+J_- = -\frac{3}{16}
\]

and \(H = 4\hbar J_z\).

Therefore, the eigenfunctions (38) represent the standard basis of the \(su(1,1)\) representation with \(j = -\frac{1}{4}\). This representation belongs to the complementary series of representation and the \(J_z\) spectrum is \(m = \frac{1}{2} + k, k = 0, 1, 2, \cdots\). We have the action of the \(J_z, J_\pm\) operators on the \(J_z\) eigenfunctions

\[
J_z |m \rangle = m |m \rangle \\
J_\pm |m \rangle = \sqrt{(m \pm \frac{1}{4})(m \pm \frac{3}{4})} |m \pm 1 \rangle
\]

As in the rectangular well potential case, using the realization (42) for the \(su(1,1)\) Lie algebra operators, we can write the asymptotic connection

\[
\lim_{x \to 0} J_+ f(x) = \lim_{x \to 0} (J_z + \frac{1}{4}) f(x)
\]

In the coordinate realisation, \(|m \rangle = f(x)\), by taking into account the relation (44), we obtain a recursion relation

\[
\lim_{x \to 0} |m + 1 \rangle = \sqrt{\frac{m + \frac{1}{4}}{m + \frac{3}{4}}} \lim_{x \to 0} |m \rangle
\]

The above recursion relation can be solved and it gives

\[
\left(\lim_{x \to 0} |m \rangle \right)^2 = \frac{\Gamma(m + \frac{1}{4})}{\Gamma(m + \frac{3}{4})} \cdot \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} \cdot \left(\lim_{x \to 0} |\frac{1}{4} \rangle \right)^2
\]

We define the R-function as ([20])

\[
R(E) = \sum_{m} \frac{\gamma_m^2}{E_m - E}
\]

where \(\gamma_m^2 = \hbar^2 (\lim_{x \to 0} |m \rangle)^2\). Therefore, we obtain

\[
\gamma_m^2 = \frac{\Gamma(m + \frac{1}{4})}{\Gamma(m + \frac{3}{4})} \frac{1}{\Gamma(\frac{1}{2}) \gamma_{\frac{1}{4}}^2}
\]
and the above defined R-function is

\[ R(E) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left(k + \frac{1}{4}\right)}{4\hbar(k + \frac{1}{4}) - E} \cdot \gamma_{\frac{1}{4}}^2 \]  

(49)

where \( \gamma_{\frac{1}{4}} \) can not be fixed by algebraic means, like in the previous rectangular-well previous case.

## 5 Other algebraic Hamiltonians

We have been proved the connection between the \( su(1,1) \) Lie algebra and the harmonic oscillator with the boundary condition (36). The appropriate operator realization is constructed as second-order differential operators, see (9). We shall see in the following the extension of the algebraic treatment for other potentials defined on the real semiaxis. The key point in the following is the relation

\[ J_+ - 2J_z + J_- = -\frac{x^2}{\hbar} \]  

(50)

which can be obtained from equations (40,42).

Therefore, using the relation (50), we can write algebraically all potentials defined on the positive semiaxis which can be developed in a Taylor series in \( x^2 \) and the series converges on the entire semiaxis. To illustrate the algebraic treatment for such a potential, let us take a simple case

\[ H = H_0 + \alpha x^2 = p^2 + (1 + \alpha)x^2 \]

\[ = 4\hbar J_z - \hbar\alpha(J_+ - 2J_z + J_-) \]  

(51)

The Hamiltonian (51) represents a harmonic oscillator with \( \omega = 2\sqrt{1 + \alpha} \) and we can compare the results obtained for the spectrum of this Hamiltonian with the well-known spectrum of the harmonic oscillator. We shall follow the same way we used in the case of the rectangular well potential. Thus, we obtain the recursion relation

\[ \alpha\sqrt{k\left(k - \frac{1}{2}\right)}C_{k-1}^E + \left[\epsilon - 2\left(k + \frac{1}{4}\right)(2 + \alpha)\right]C_k^E \]
\( + \alpha \sqrt{\left( k + \frac{1}{2} \right)(k + 1)} C_{k+1}^E = 0 \)  

(52)

where \(|E| = \sum_{k=0}^{\infty} C_k^E |k> \) and \( E = \hbar \epsilon \). The relation (52) is true for \( k = 0, 1, 2, \cdots \). The coefficient \( C_{-1} \) does not appear in the relation (52) and the expressions \( k(k - \frac{1}{2}) \) and \( (k + \frac{1}{2})(k + 1) \) are always positive. If \( \alpha = 0 \), i.e. \( H = H_0 \), we reobtain from (52) the spectrum (39). The relation (52), taken for \( k = 0 \), yields

\[
\left[ \epsilon - (1 + \frac{\alpha^2}{2}) \right] C_0^E + \frac{\alpha}{\sqrt{2}} C_1^E = 0
\]

If we write again \( C_{k+1}^E = \xi_k^E C_k^E \) for \( k = 0, 1, 2, \cdots \) we have

\[
\xi_0^E = \frac{\sqrt{2}}{2} \left[ (1 + \frac{\alpha}{2}) - \epsilon \right]
\]

(53)

On the other hand, the relations (52-53) allow us to write for \( k = 1, 2, \cdots \)

\[
\alpha \sqrt{k(k - \frac{1}{2})} \frac{1}{\xi_{k-1}^E} + [\epsilon - (4k + 1)(1 + \frac{\alpha}{2})] + \alpha \sqrt{(k + \frac{1}{2})(k + 1)} \xi_k^E = 0
\]

(54)

which is similar to the recurrence relation (25). Using the above relation we can write \( \xi_0^E \) as a continous fraction and the condition of equality of that fraction with (53) results into an equation for the unknown energy \( E = \hbar \epsilon \). The lowest order approximation for the continous fraction yields the equation

\[
\left[ \left( 1 + \frac{\alpha}{2} \right) - \epsilon \right] \left[ 5 \left( 1 + \frac{\alpha}{2} \right) - \epsilon \right] + \frac{\alpha^2}{2} = 0
\]

(55)

For \( \alpha = 0 \) in the above equation we reobtain the first two unperturbed level of the considered harmonic oscillator. For non-zero \( \alpha \), in the second order in \( \alpha \), we have

\[
\tilde{\epsilon}_0 = \left( 1 + \frac{\alpha}{2} \right) + \frac{\alpha^2}{8}
\]

\[
\tilde{\epsilon}_1 = 5 \left( 1 + \frac{\alpha}{2} \right) - \frac{\alpha^2}{8}
\]

(56)

This values can be compared with the exact values of the oscillator with \( \omega = 2\sqrt{1+\alpha} \) and they are closed for large values of the parameter \( \alpha \). This approach can also be used to obtain the values at \( x = 0 \) of the perturbed wave functions and therefore, to obtain R-function in the same way we used for the rectangular-well potential in Section 2.
6 Conclusions

We have described two particular one-dimensional problems in an algebraic framework, i.e. the rectangular-well and the harmonic oscillator potentials, and we have calculated the corresponding energy spectra and the R-functions. The present realization of the $\text{su}(1,1)$ Lie algebra operators, eqs.(20,21) and eq.(50), allow us to write a large class of potentials in algebraic form, i.e. any potential defined on a bounded domain and expressed in its Fourier series or defined on the positive semiaxis and expressed in Taylor series, in the even powers of $x$, only.

An example for a trigonometric-like potential is worked out. For more complicated potentials, than the above presented examples, their algebraic expression in terms of $J_\pm, J_0$ can be obtain by generalising eqs.(19,21) to higher $n$-values. In this respect one can obtain an algebraic Hamiltonian and the full exact solutions of its dynamical symmetry, for the above presented boundary conditions. We have proved the possibility to obtain a parametrization of the R-function without using the explicite form of the wave function in a manner close to the spirit of the algebraic scattering theory [6,7]. The technique developed in Section 3 allows in principle the algebraic analysis and the exact solvability for any $L_2$, smooth potential defined on a bounded domain or any even potential defined on $\mathcal{R}^+$. 
References


[12] Natanzon G A 1979 *Teor. Mat. Fiz.* **38** 146


