Nonlinear Phenomena in Nuclei: The Antisoliton Model for Fission

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Abstract

The non-linear solutions of the Modified Korteweg-de Vries (MKdV) equations travel on the nuclear surface of medium-heavy nuclei and generate highly deformed shapes. The cnoidal and soliton solutions provide the existence of rotons as large amplitude collective oscillations. The dynamics is based on the nonlinear equations and Hamiltonian of a realistic liquid drop model (LDM). The antisoliton solutions are obtained through a general formulation of nonrelativistic quantum mechanics in terms of density and current operators. The quantum averaging of the antisoliton pair rotation describes symmetric or asymmetric fission modes. The nuclear asymmetry near the scission point seems to be in qualitative agreement with the general shapes considered in all other phenomenological fission models.

1 Introduction

The theoretical study of atomic nuclei under extreme conditions includes large nuclear deformations. Interest in these features is complementary to experimental work at new experimental facilities, like radioactive beams. In such nuclei, the outermost neutrons and protons move with respect to the inner shells and the result is a highly deformed shape, or even super-deformed or hyper-deformed nuclei can result. Moreover, such deformations become asymmetric or can lead to a `neck'. This shape is not convex, and the traditional parametrizations (spherical harmonics) are not anymore valid.
In the theory of fission, the nuclear potential energy is of high interest. The dependence of the potential energy on the shape of the nuclear surface allows conclusions about the dynamics of the fission process and informations concerning the barrier. These calculations can further provide estimates of stability against spontaneous fission and predictions of optimal fusion paths for the synthesis of the superheavy nuclei.

The simplest way to understand nuclear fission is provided by the LDM together with shell corrections. However, for such a process involving large deformations one has to go beyond the harmonic approximations. The expansion in multipoles cannot describe fission through the separation into daughter nuclei, no matter the number of harmonics involved. It’s not only the convergence problem (theoretical or numerical) but, fundamentally there is the problem that the radius becomes a multivalued function of the angle if the nuclear neck is sufficiently constricted.

Consequently, the traditional theoretical approaches use different parametrizations to describe highly deformed shapes, like tangent spheres, neck coordinates, Casinian ovals, deformed ellipsoids [1]. A convenient parametrization of the nuclear shape is a decisive factor for the success of the calculations [2]. However, without exception, the known parametrizations are artificially introduced by pure geometrical methods, without physical support.

In the present work we introduce a model which can describe highly deformed nuclear shapes on the way to fission in a dynamic approach. Starting from a nonlinear nuclear hydrodynamic Hamiltonian with effective Skyrme contact $\delta$-interactions, plus shell corrections, and by including nonlinear terms, we obtain a nonlinear Schrödinger equation (NLS) of order three that describes the nuclear density and surface. This equation reduces to a modified KdV (MKdV) equation having soliton and antisoliton solutions. The soliton solutions are just the rotons, introduced in [3]. The antisoliton solutions, coupled in symmetric pairs, produce all the nuclear shapes obtained in other fission models. Loosely speaking, such a deformation is a pair of symmetric holes in the nuclear surface which travel with constant velocity. From the quantum mechanical point of view, these rotating holes ”cut” a channel of lower probability of localizations for nucleons. The corresponding shape is equivalent to a separation into two fragments. Moreover, by using quantum mechanical fragmentation theory or a WKB approximation for the corresponding potential valleys, one can relate the parameters of these antisolitons (height, width and velocity) with the nuclear potential. This antisoliton system represents a new collective excitation of the nucleus towards
fission, with a deep (nonlinear) physical background, which answers, not only the question "how?", but also the question "why?".

2 Models for nuclear fission

One of the most used methods in modeling nuclear fission is quantum mechanical fragmentation theory where collective coordinates of charge and mass asymmetry are introduced [4]. In this model the potential energy of the system is defined as the sum of the liquid drop energies, shell effects, the proximity nuclear potential, the Coulomb interaction and the rotational energy due to the angular momentum

\[
V = \sum_{i=1}^{2} (V_{\text{LDM}}(A_i, Z_i) + \delta U_i) + V_P + V_C + V_l
\]  

for the LDM describes in [5], with empirical shell effects, one has

\[
V = -\sum_{i=1}^{2} \left( a_a A_i - a_s A_i^{2/3} - a_e \frac{Z_i^2}{A_i^{2/3}} - a_a \left( A_i - 2Z_i \right)^2 \right)
\]

\[
+ \sum_{i=1}^{2} \left( C_1 \left\{ (2/A_i)^{2/3} [F(Z_i) + F[N_i]] - C_2 A_i^{1/3} \right\} \right)
\]

\[
+ 4\pi \gamma b < R > \Phi(\xi) + \frac{Z_1Z_2e^2}{R} + \frac{l^2}{2J(R_1, R_2)},
\]

with empirical values \( a_a = a_e K - a_s K A^{-1/3}, \gamma = \gamma_0 (1 - \gamma_1 (N - Z)^2/A^2) \) and

\[
\Phi(\xi) = \begin{cases} 
-0.5(\xi - 2.54)^2 - 0.0852(\xi - 2.54)^3 & \xi \leq 1.2511 \\
-3.437 \exp(\xi/0.75) & \xi \geq 1.2511.
\end{cases}
\]

The shell correction is described by

\[
F(X) = \frac{3}{5} X - M_{i-1}) M_i^{5/3} - M_{i-1}^{5/3} - \frac{3}{5} \left( X^{5/3} - M_{i-1}^{5/3} \right),
\]

[5] where \( M_{i-1} \) is the lowest magic number closest to \( X (=Z_i \text{ or } N_i) \). Since we are interested in spontaneous fission and hence in low spin states, we can approximate the moment of inertia with

\[
J(R_1, R_2) = \frac{\mu}{4} (R_1 + R_2)^2 + \frac{2}{5} A_1 m R_1^2 + \frac{2}{5} A_2 m R_2^2,
\]
where $\mu = A_1A_2/(A_1 + A_2)$ is the reduced mass. The rest of the coefficients in the above equations are fixed numerical values, provided for example in [4].

The next step is the quantization of the classical kinetic energy described by the (charge and mass) asymmetry coordinates, $\eta_Z = (Z_1 - Z_2)/Z$ and $\eta_A = (A_1 - A_2)/A$, respectively. By adopting the Pauli prescription, [1], a Laplace operator in the curvilinear coordinates $\eta_Z, \eta_A$ can be written, similar to the Bohr Hamiltonian for the $\beta - \gamma$ vibrations

$$T = -\frac{\hbar^2}{2B}\frac{\partial^2}{\partial\eta_Z^2} - \frac{\hbar^2}{2B}\frac{\partial^2}{\partial\eta_A^2} - \frac{\hbar^2}{2B}\frac{\partial^2}{\partial\eta_Z\partial\eta_A},$$

where the mass parameters $B$ can be calculated in the cranking model or in the hydrodynamic model. From eq.(2) one can calculate the potential energy surface against the parameters of the model. As the nucleus proceeds towards fission, there are a number of valleys, indicating the preferred binary mass split-ups; these are usually characterized by the formation of one daughter nucleus close to magic numbers. The main characteristic in all these representations is that the valleys have an almost constant profile as the separation coordinate increases closer to scission. This indicates that the mass distribution of the fragments has been determined early in the fission process.

The mass distributions expected are calculated by solving the collective Schrödinger equation in the asymmetry degrees of freedom, for the Hamiltonian given in eq.(5). This model can reproduce the experimental data resonably well, if the asymmetry is not coupled with the relative motion.

A similar approach is given in the Two-Center Shell Model [7]. This model differs from the previous by the fact that the geometric separation is introduced by an anharmonic (quadratic) oscillator potential. Also, the neck parameter is determined only indirectly by interpolation of the equipotential surfaces with a harmonic oscillator.

### 3 The antisoliton model

The microscopic theories have shown that the harmonic approximations have only a very rough validity in the limit of very small vibrations. For collective motions with larger amplitudes one has to take into account either differ-
ent anharmonic terms or a coherent combination of these, like in nonlinear systems.

We have recently introduced a nonlinear model, [3, 6], that is a new type of surface nuclear excitation, called a 'roton', in order to explain higher deformations of nuclei through emission of clusters. Rotons are localized waves that propagate with little change in form on the surface of droplets, shells, or bubbles. They behave like solitons, but arise from normal modes of spheroids that obey nonlinear dynamics. Our model describes a new large amplitude collective motion in nuclei described by equations of the KdV type and their traveling wave solutions. This model yields a unifying dynamical picture of these modes: solutions simulate harmonic oscillations that are driven into anharmonic ones by nonlinear terms in the Hamiltonian, and ultimately cnoidal wave forms of the latter develop into rotating solitary waves.

The roton model brings one new thing into the physics of the liquid drop (besides the novelty of obtaining the KdV equation on a sphere, without gravity). It generates a large set of highly deformed shapes in a dynamical way. That is it explains the mechanism for the formation of patterns and provides analytical solutions. Rotons were observed experimentally when the amplitude of the shape oscillations of a droplet became substantial.

However, the roton model cannot describe or explain the formation of 'necks' or concave shapes on the surface. The solitons are always convex. However, nonlinear equations also have antisoliton solutions, that is solitons with negative amplitude.

There are many opportunities to introduce nonlinear fluid dynamics concepts in nuclear physics. Since the nonlinear LDM approach was by and large presented in, [3, 6], we introduce in this paper a quantum hydrodynamic approach with effective an Skyrme interaction. [8]. In the usual second-quantization formalism, a nucleus can be described by a nonrelativistic Hamiltonian with a local two-body potential

$$H = \frac{\hbar^2}{2m} \int \nabla \Psi^+(x) \nabla \Psi(x) d^3x$$

$$+ \int \Psi^+(x) \Psi(x) U(x-y) \Psi^+(y) \Psi(y) d^3xd^3y,$$  \hspace{1cm} (6)

where the nucleon fields $\Psi^+(x), \Psi(x)$ are canonically conjugated and fulfill equal time anticommutation relations. Here $x$ is the space-time 4-vector. We
will describe the dynamics of the nucleus in a restricted space of collective variables representing the density and nucleon current of the system

\[ \rho(x) = \Psi^+(x)\Psi(x). \]

\[ j_k(x) = \frac{\hbar}{2mi} \left( \Psi^+(x) \frac{\partial}{\partial x_k} \Psi(x) - \frac{\partial}{\partial x_k} \Psi^+(x) \Psi(x) \right). \] (7)

The operators \( \rho, j \) fulfill commutation relations in the Heisenberg representation. The corresponding Hamilton equations are

\[ \frac{\partial \rho(x)}{\partial t} = \frac{1}{i\hbar} [\rho(x), H] = -\sum_{k=1}^{3} \frac{\partial}{\partial x_k} j_k(x), \]

\[ \frac{\partial j_k(x)}{\partial t} = \frac{1}{i\hbar} [j_k(x), H] = -\frac{\hbar^2}{2m^2} \sum_{n=1}^{3} \frac{\partial}{\partial x_n} \left( T_{nk} - \frac{1}{2} \frac{\partial^2 \rho(x)}{\partial x_n \partial x_k} \right) - \frac{2}{m} \rho(x) \frac{\partial}{\partial x_k} \int U(x-y) \rho(y) d^3y, \] (8)

where the kinetic energy tensor operator is

\[ T_{nk}(x) = \frac{\partial}{\partial x_n} \Psi^+(x) \frac{\partial}{\partial x_k} \Psi(x) + \frac{\partial}{\partial x_k} \Psi^+(x) \frac{\partial}{\partial x_n} \Psi(x). \] (9)

Finally, the Hamiltonian in eq.(6) becomes

\[ H = \frac{m}{2} \sum_{k=1}^{3} \int j_k \frac{1}{\rho} j_k d^3x + \frac{\hbar^2}{8m} + \int \frac{[\nabla \rho]^2}{\rho} d^3x \]

\[ \int \rho(x) U(x-y) \rho(y) d^3x d^3y, \] (10)

which is similar to the hydrodynamic Hamiltonian in [3]. Within this equivalence, the equation of motion eqs.(8) are formally just the continuity and Euler operator equations for a quantum fluid.

As in the roton model, we restrict ourselves to irrotational motion and introduce a velocity potential operator \( \Phi(x) \) of the form [9],

\[ j_k(x) = \frac{1}{2} \left[ \rho(x), \frac{\partial}{\partial x_k} \Phi(x) \right]_+, \] (11)

which projects the commutation relations between \( \rho \) and \( \vec{j} \) into a linear representation of the Heisenberg H(1) Lie algebra generated by \( \rho(x), \Phi(x) \).
Due to the irrotational condition we can use the following substitution for the local density and the velocity potential

$$
\rho(x,t) = |u(x,t)|^2, \quad \Phi(x,t) = \frac{\hbar}{m} \arg(u(x,t)).
$$

(12)

In the semiclassical approach eqs. (8) can be reduced through the irrotationality condition, eq.(11), to a NLS equation

$$
i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \nabla u + \tilde{U}[|u|^2]u.
$$

(13)

where we have introduced $\tilde{U}[\rho]$ as a non-linear effective potential like in the Hartree-Fock approach [1]. With this choice $\tilde{U}[\rho]$ is the unique term responsible for the nonlinearities in the dynamical equation.

In order to reduce eqs.(12-13) to a solvable nuclear model we choose a $\delta$-function interaction.

$$
U(x - y) = -k \int \delta(x - y) d^3x d^3y \to \tilde{U}.
$$

(14)

where from now on we denote by $x$ only the space component of the 4-vector. Consequently, we obtain the NLS equation

$$
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla \Psi - k|\Psi|^2\Psi,
$$

(15)

for the dynamics of the system. If we add higher order terms in density in the $\delta$-interaction we will have higher order nonlinearities in the NLS, respectively. Similar equations have been successfully used to describe other nonlinear systems (plasma, solids, etc.).

The most general localized stable solution of Eq.(15)

$$
u(x,t) = -\frac{\hbar a}{\sqrt{km}} \exp\left[ivx/2 - i\hbar v^2t/8m + i\hbar a^2t/2m + i\chi\right] \cosh a(x - b + \hbar vt/2m),
$$

(16)

is an antisoliton with negative amplitude $-a \leq 0$ and velocity $v$. Also, $b, \chi$ are free parameters in the solution.

We can obtain the same soliton/antisoliton solution from eq.(15) by introducing a functional transform

$$
\Psi(x,t) = P(x,t) e^{iS(x,t)},
$$

(17)
where $P$ and $S$ are real functions. In addition, since the particles should be free at infinity (we take into account only localized solutions) we can consider $S$ to be a phase factor, $S(x,t) = kx - \omega t + S_0$. Introducing this substitution in eq.(15) we obtain a MKdV equation for $P$

$$\frac{\partial P}{\partial t} - 6P^2 \frac{\partial P}{\partial x} - k \frac{\partial^3 P}{\partial x^3} = 0,$$

with a traveling soliton/antisoliton solution

$$P(x,t) = \pm P_0 Sech(2P_0 x - 8P_0^3 k^3 t + f_0),$$

where $P_0$ is a free parameter describing the amplitude of the soliton/antisoliton ($P_0 > 0$ or $P_0 < 0$, respectively) and $f_0$ is a free phase factor. The solution has a half-width $L = 1/2P_0$ and travels with the velocity $v = 4P_0$. The antisolitons have special shape-motion dependence. The larger the amplitude the narrower the width and the large the velocity. This relation can be used to experimentally distinguish them from other linear modes. Since this solution is very well localized, any linear combination of soliton or antisoliton, shifted with a distance larger than $2L$, is still an approximate solution.

Eq.(15) can be projected on one spherical coordinate ($x \rightarrow \phi$), [3]. A solution of eq.(15) described by two identical antisolitons shifted with $\pi$ is presented in Fig.1.

From the classical point of view, this solution represents two identical holes in the surface (actually two gaps in density) placed in opposition and moving with the same angular velocity. From the quantum mechanics point of view, the probability of localization of these antisolitons is equally distributed on their whole path, which is a circle. Hence, the average quantum effect of this solution is a narrowing of the surface precisely under their path. This is equivalent with the occurrence of a neck.

Depending on the antisoliton parameter $P_0$ and on the shift $\Delta \phi$ between the two antisolitons (if it is not quite $\pi$) we can describe different nuclear shapes. We can relate the nonlinear parameters of the model with the typical parameters required to describe the fission process in traditional models.

The elongation coordinate, which describes the length of the major semi-axis at the beginning of the fission, and approaches the distance between the separated fragments can be related to the half-width of the antisoliton pair, $L$. 


The neck coordinate, which describes the thickness of the neck between the fragments can be related with the amplitude \( P_0 \) of the soliton or antisoliton pairs and also with the velocity \( v \) which controls the probability of delocalization.

The fragmentation coordinate, which measures the deviation from symmetry in the mass distribution is also related to a combination of the amplitude and half-width.

In Fig. 2 we present some shapes in a suitable parametrization. All figures in the frame are cross-sections of a two antisoliton solution on a sphere, like in Fig. 1. The continuous line represents two antisolitons that are shifted by \( \pi \) and the dashed line two antisolitons with a different shift (but same amplitude and half-width) to simulate the mass asymmetry. In the fragmentation theory language, the \( b \) axis represents the elongation coordinate and the \( a \) axis represents the neck coordinate. We have found a pretty good fit between all the traditional nuclear shapes and the two antisolitons, [1]. In the lower row we present, for comparison, soliton solutions (rotons) with positive amplitude.

A proper quantization of the antisolitons and obtaining of the exact wavefunctions of the resulting Schrödinger equation are the next step towards experimental comparison. Once the surface is given by the antisoliton solution (in one or many pairs), we can calculate the total nuclear energy by using eq. (2) for all the shell effect corrections. With pinned initial and final states (the fission channel of a given parent nucleus) in the space of the parameters \( (P_0, L, V) \) we can calculate the quantum penetrability by using the Gamov formula

\[
S = \exp \left[ -\frac{2}{\hbar} \int_{\tau_{\text{parent}}}^{\tau_{\text{fragments}}} \sqrt{\mu V(P_0(\tau), L(\tau), v(\tau))} \, d\tau \right],
\]

which will provide the lifetime for spontaneous fission. The potential valley along the path, in the two antisoliton model, should have an additional minimum with respect to the same path in the LDP, Fig. 3. The fission probability is given by the ratio of the two wave amplitudes in the corresponding wells evaluated with the help of the barrier penetrability between these two minima.
4 Conclusions

The antisoliton model introduced in this paper, brings correction to the fission calculation within the LDM. In general, by the introduction of arbitrary parameters to describe the geometry of the fission, one can obtain a good fit with the experiment. However, these parameters are only abstract geometrical entities artificially introduced and without a physical background. The roton and antisoliton model predict the same shapes, starting from a physical Hamiltonian of the LDM or nuclear hydrodynamics with effective Skyrme contact $\delta$-interactions. It has been shown that, in a certain order of approximation, these Hamiltonian reduces to the NLS and MKdV (or KdV) equations respectively, allowing the existence of roton solutions (solitons on the nuclear surface) and antisolitons. Next, by quantizing the stationary motion of an antisoliton pair, one can ”cut” a quantum channel in the initial nucleus, hence producing a suplimentary valley in the potential energy profile. The antisoliton model, which is complementary to the roton model, can be also used in various nonlinear dynamical phenomena in heavy-ion collisions besides other semiclassical methods (TDHF, the dynamical Thomas-Fermi theory, etc.)

5 Acknowledgements

Supported by the U.S. National Science Foundation through a regular grant, No. 9603006, and a Cooperative Agreement, No. EPS-9550481, that includes matching from the Louisiana Board of Regents Support Fund.

References


Figure 1: Two antisoliton solution on a spherical surface.


Figure 2: Cross-sections of a two-antisoliton solution on a sphere. With continuous line are drawn the two-antisolitons shifted with $\pi$. With dashed line there are two antisolitons with a different shift, in order to simulate the mass asymmetry. In the fragmentation theory language, the $b$ axis represents the elongation coordinate and the $a$ axis represents the neck coordinate. Last row shows different soliton solutions.
Figure 3: Schematic representation of the liquid drop fission barrier. Continuous for the linear theory, dashed for the antisoliton model.