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INVARIANT WEIGHTED WIENER MEASURES AND ALMOST SURE GLOBAL WELL-POSEDNESS FOR THE PERIODIC DERIVATIVE NLS.

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Abstract. In this paper we construct an invariant weighted Wiener measure associated to the periodic derivative nonlinear Schrödinger equation in one dimension and establish global well-posedness for data living in its support. In particular almost surely for data in a Fourier-Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$ with $s \geq \frac{1}{2}$, $2 < r < 4$, $(s-1)r < -1$ and scaling like $H^{\frac{1}{2}-\epsilon}(\mathbb{T})$, for small $\epsilon > 0$. We also show the invariance of this measure.

1. Introduction

In the past few years, methods such as those by J. Bourgain (high-low method; e.g. [5, 6]) on the one hand and by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao (I-method or method of *almost conservation laws* e.g. [15, 16, 17]) on the other, have been applied to study the global in time existence of dispersive equations at regularities which are right below or in between those corresponding to conserved quantities. As it turns out however, for many dispersive equations and systems there still remains a gap between the local in time results and those that could be globally achieved. In those cases, it seems natural to return to one of Bourgain's early approaches for periodic dispersive equations (NLS, KdV, mKdV, Zakharov system) [3, 4, 5, 7, 8, 9] where global in time existence was studied in the almost sure sense via the existence and invariance of the associated Gibbs measure (cf. Lebowitz, Rose and Speer's and Zhidkov's works [30] [48]). More recently this approach has been used for example by N. Tzvetkov [44, 45] for subquintic radial nonlinear wave equation on the disc, N. Burq and N. Tzvetkov [12, 13] for subcubic and subquartic radial nonlinear wave equations on 3d ball, N. Burq, L. Thomann, and N. Tzvetkov [11] for the nonlinear Schrödinger equation with harmonic potential, and by T. Oh [33, 34, 35, 36] for the periodic KdV-type coupled systems, Schrödinger-Benjamin-Ono system and white noise for the KdV equation.

Failure to show global existence by Bourgain's high-low method or the I-method might come from certain 'exceptional' initial data set, and the virtue of the Gibbs measure is that it does not see that exceptional set. At the same time, the invariance of the Gibbs measure, just like the usual conserved quantities, can be used to control the growth in time of those solutions in its support and extend the local in time solutions to global ones almost surely. The difficulty in this approach lies in the actual construction of the associated Gibbs measure and in showing both its invariance under the flow and the almost sure global well-posedness, since, on the one hand, we need invariance to show global well-posedness and on the other hand we need globally defined flow to discuss invariance.

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Our goal in this paper is to construct an invariant weighted Wiener measure associated to the periodic derivative nonlinear Schrödinger equation DNLS in (2.1) in one dimension and establish global well-posedness for data living in its support. In particular almost surely for data in a Fourier-Lebesgue space $\mathcal{F}L^{s,r}$ defined in (2.2) below (c.f. [27, 21, 14, 22]) and scaling like $H^{\frac{1}{2}-\epsilon}(\mathbb{T})$, for small $\epsilon > 0$. The motivation for this paper stems from the fact that by scaling DNLS should be well posed for data in H^σ, $\sigma \geq 0$ but the results so far obtained are much weaker.

Local well-posedness is known for $\sigma \geq 1/2$ for the nonperiodic [40] and periodic [26] cases while global well-posednes is known for $\sigma \geq 1/2$ for the nonperiodic case ($\sigma > 1/2$ in [16] and $\sigma \geq 1/2$ in [31]) and for $\sigma > 1/2$ in the periodic case [47]. Furthermore, in the non periodic case the Cauchy initial value problem for DNLS is ill-posed for data in $H^\sigma(\mathbb{R}), \sigma < \frac{1}{2}$ [40] [2], a strong indication that ill-posedness should also be expected in the periodic case on that range. Grünrock and Herr [22] have recently established local well posedness for the periodic DNLS in Fourier-Lebesgue spaces $\mathcal{F}L^{s,r}$, which for appropriate choices of (s, r) scale like $H^\sigma(\mathbb{T})$ for any $\sigma > \frac{1}{4}$. Their result is the starting point of this work (cf. Section 2 for a more detailed discussion).

The measure we construct is based on the energy functional rather than the Hamiltonian. Hence we simply refer to it as weighted Wiener measure rather than Gibbs measure since the name 'Gibbs measure' has traditionally been reserved for those weighted Wiener measures constructed using the Hamiltonian. By invariance of a measure μ we mean that if $\Phi(t)$ denote the flow map associated to our nonlinear equation then $\Phi(t)$ is defined for all $t \in \mathbb{R}$, μ almost surely and for all $f \in L^1(\mu)$ and all $t \in \mathbb{R}$,

$$\int f(\Phi(t)(\phi))\, \mu(d\phi) \;=\; \int f(\phi)\mu(d\phi).$$

In general terms our aim is to construct a well defined measure μ so that local well posedness of the periodic DNLS holds in some space $\mathcal{B}$ containing the support of μ. Then we show almost sure global well posedness as well as the invariance of μ via a combination of the methods of Bourgain and Zhidkov [48] in the context of NLS, KdV, mKdV. In implementing this scheme however we need to overcome two main obstacles due to the need to gauge the equation to show local well posedness (eg. [40, 26]) and to construct an invariant measure. The symplectic form associated to the periodic gauged derivative nonlinear Schrödinger equation GDNLS in (2.8) does not commute with Fourier modes truncation and so the truncated finite-dimensional systems are not necessarily Hamiltonian. The first (mild) obstacle is to show the conservation of the Lebesgue measure associated to the finite dimensional approximation to the periodic gauged derivative nonlinear Schrödinger equation FGDNLS, defined in (3.1) by hand, rather than by using the Hamiltonian structure. The second obstacle is much more serious and is at the heart of this work. The energy $\mathcal{E}$ defined in (2.16) associated to the gauged periodic DNLS[1] which we prove to be conserved in time, ceases to be so when computed on solutions of the finite dimensional approximation equation; that is $\frac{d}{dt}\mathcal{E}(v^N) \neq 0$, when v^N is a solution to the finite dimensional gauged DNLS (see (4.18)). In other words the finite dimensional weighted Wiener measure is not invariant any longer and unlike Zhidkov's work [48] on KdV we do not have a priori knowledge of global well posedness. We show however that it is *almost* invariant in the sense that we can control the growth in time of $\mathcal{E}(v^N)(t)$. This idea is reminiscent of the I-method. However, while in the I-method one needs to estimate the variation of the energy of solutions to the infinite dimensional equation at time t smoothly projected onto

[1]We emphasize $\mathcal{E}$ is not the Hamiltonian of the gauged DNLS.

frequencies of size up to N; here one needs to control the variation of the energy $\mathcal{E}$ of the solution v^N to the finite dimensional approximation equation FGDNLS. We note that the loss in energy conservation for solutions to the finite dimensional equation is principally due to the manner one chooses to approximate the infinite dimensional gauged equation by using Fourier projections onto the first Nth frequencies. In [3] Bourgain describes an alternative approach that relies on using a discrete system of ODE which seems to preserve the conservation of energy. This approach however entails a number of other difficulties, for one needs to replace the circle $\mathbb{T}$ by the cyclic group $\mathbb{Z}_N$ and carry out the analysis on cyclic groups. We choose not to follow this path here.

We expect the ungauged invariant Wiener measure associated to DNLS (2.1) we obtain in Section 7 to be absolutely continuous with respect to the weighted Wiener measure constructed by Thomann and Tzvetkov [42]. This question is addressed in a forthcoming paper [32].

The paper is organized as follows. In Section 2 we present some general background, notation and results on the derivative nonlinear Schrödinger equation in one dimension. In Section 3 we discuss FGDNLS. In Section 4 we overcome the first two obstacles mentioned above. Namely we prove the invariance of the Lebesgue measure associated to FGDNLS and devote the rest of the section to prove our energy growth estimate Theorem 4.2. In Section 5 we carry out the construction of the weighted Wiener measure associated to the GDNLS. In Section 6 we prove the almost sure global well-posedness result for the GDNLS and the invariance of the measure constructed in section 5. Finally in Section 7 we translate back our results to the ungauged DNLS equation.

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Notation. Whenever we write $a+$ for $a \in \mathbb{R}$ we mean $a+\varepsilon$ for some $\varepsilon > 0$; similarly for $a-$. In addition, we write $A \lesssim B$ to mean there exist some absolute constant $C > 0$ such that $A \leq CB$.

2. The Derivative NLS Equation in one dimension

The initial value problem for DNLS takes the form:

$$\begin{cases} u_t - i\, u_{xx} = \lambda(|u|^2 u)_x \\ u\big|_{t=0} = u_0, \end{cases} \tag{2.1}$$

where either $(x,t) \in \mathbb{R} \times (-T,T)$ or $(x,t) \in \mathbb{T} \times (-T,T)$ and λ is real. In this paper we will take $\lambda = 1$ for convenience. DNLS is a Hamiltonian PDE whose flow conserves also mass and energy; i.e. the following are conserved quantities of time[2] (c.f. [28, 25, 26]) :

[2]In fact, DNLS is completely integrable.

$$\begin{aligned}
\text{Mass:} \qquad M(u)(t) &= \int |u(x,t)|^2\,dx. \\
\text{Energy:} \qquad E(u)(t) &= \int |u_x|^2\,dx + \frac{3}{2}\mathrm{Im}\int u^2\overline{u}\,\overline{u_x}\,dx + \frac{1}{2}\int |u|^6\,dx. \\
\text{Hamiltonian:} \qquad H(u)(t) &= \mathrm{Im}\int u\overline{u}_x\,dx + \frac{1}{2}\int |u|^4\,dx.
\end{aligned}$$

DNLS was introduced as a model for the propagation of circularly polarized Alfvén waves in a magnetized plasma with a constant magnetic field (cf. Sulem-Sulem [39]). The equation is scale invariant for data in L^2; *i.e.* if $u(x,t)$ is a solution then $u_a(x,t) = a^\alpha u(ax, a^2t)$ is also a solution if and only if $\alpha = \frac{1}{2}$. Thus *a priori* one expects some form of existence and uniqueness results for (2.1) for data in $H^\sigma, \sigma \geq 0$. Many results are known for the Cauchy problem with smooth data, including data in H^1, such as those by M. Tsutsumi and I. Fukuda [43], N.Hayashi [23], N. Hayashi and T. Ozawa [24, 25] and T. Ozawa [37] and others (cf. references therein).

In looking for solutions to (2.1) we face a derivative loss arising from the nonlinear term $(|u|^2u)_x = u^2\,\overline{u}_x + 2\,|u|^2\,u_x$ and hence for low regularity data the key is to somehow make up for this loss.

For the non-periodic case ($x \in \mathbb{R}$) Takaoka [40] proved sharp local well-posedness (LWP) in $H^{\frac{1}{2}}(\mathbb{R})$ relying on the gauge transformation used by Hayashi and Ozawa [24, 25] and the so-called Fourier restriction norm method. Then, Colliander-Keel-Staffilani-Takaoka and Tao [15, 16] established global well-posedness (GWP) in $H^\sigma(\mathbb{R})$, $\sigma > \frac{1}{2}$ of small L^2 norm using the so-called I-method on the gauge equivalent equation (see also [41]). Here, small in L^2 just means less than an appropriate constant $\sqrt{\frac{2\pi}{\lambda}}$ which forces the associated 'energy' to be positive via Gagliardo-Nirenberg inequality. This result was recently improved by Miao, Wu and Xu to $\sigma \geq 1/2$. The Cauchy initial value problem for DNLS is ill-posed for data in H^σ and $\sigma < \frac{1}{2}$ (data map fails to be C^3 or uniformly C^0 [40] [2].) In [21] A. Grünrock proved that the non-periodic DNLS is locally well posed in the Fourier-Lebesgue spaces $\mathcal{F}L^{s,r}(\mathbb{R})$ which for appropriate choices of (s,r) scale like $H^\sigma(\mathbb{R})$ for any $\sigma > 0$ (c.f. (2.2) below.)

In the periodic setting, S. Herr [26] showed that the Cauchy problem associated to periodic DNLS is locally well-posed for initial data $u(0) \in H^\sigma(\mathbb{T})$, if $\sigma \geq \frac{1}{2}$ in the sense of local existence, uniqueness and continuity of the flow map. Herr's proof is based on an adaptation to the periodic setting of the gauge transformation introduced by Hayashi [23] Hayashi and Ozawa [24, 25] on $\mathbb{R}$, in conjunction with sharp multilinear estimates for the gauged equivalent equation in periodic Fourier restriction norm spaces $X^{s,b}$ that yield local well posedness for the gauged equation. Moreover, by use of conservation laws, the problem is also shown to be globally well-posed for $\sigma \geq 1$ and data which is small in L^2-as in [15, 16]- [26]. More recently, Win [47] applied the I-method to prove GWP in $H^\sigma(\mathbb{T})$ for $\sigma > \frac{1}{2}$.

A. Grünrock and S. Herr [22] showed that the Cauchy problem associated to (DNLS) is locally well-posed for initial data $u_0 \in \mathcal{F}L^{s,r}(\mathbb{T})$ with $2 < r < 4$ and $s \geq \frac{1}{2}$ where

$$\|u_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} := \|\,\langle n\rangle^s\,\widehat{u}_0\,\|_{\ell^r_n(\mathbb{Z})}. \tag{2.2}$$

These spaces scale like the Sobolev $H^\sigma(\mathbb{T})$ ones where $\sigma = s + \frac{1}{r} - \frac{1}{2}$. Their proof is based on Herr's adapted periodic gauge transformation and new multilinear estimates for

the gauged equivalent equation in an appropriate variant of Fourier restriction norm spaces $X^{s,b}_{r,q}$ introduced by Grünrock-Herr [22][3].

For $s, b \in \mathbb{R}$, $r, q \geq 1$ we define the space $X^{s,b}_{r,q}$ as the completion of the Schwartz space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ with respect to the norm

$$\|u\|_{X^{s,b}_{r,q}} := \|\langle n\rangle^s \langle \tau + n^2\rangle^b \widehat{u}(n,\tau)\|_{\ell^r_n L^q_\tau}$$

where first we take the L^q_τ norm and then the ℓ^r_n one. We also define the space

$$\|u\|_{X^{s,b}_{r,q;-}} := \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(n,\tau)\|_{l^r_n L^q_\tau},$$

and note that $u \in X^{s,b}_{r,q}$ if and only if $\overline{u} \in X^{s,b}_{r,q;-}$.

For $\delta > 0$ fixed, we define the restriction space $X^{s,b}_{r,q}(\delta)$ of all $v = u\big|_{[-\delta,\delta]}$ for some $u \in X^{s,b}_{r,q}$ with norm

$$\|v\|_{X^{s,b}_{r,q}(\delta)} := \inf\{\|u\|_{X^{s,b}_{r,q}} \ : \ u \in X^{s,b}_{r,q} \ \text{ and } \ v = u\big|_{[-\delta,\delta]}\}. \tag{2.3}$$

When we take $q = 2$ we simply write $X^{s,b}_{r,2} = X^{s,b}_r$. Note $X^{s,b}_{2,2} = X^{s,b}$. Later we will also use the space

$$Z^s_r(\delta) \ := \ X^{s,\frac{1}{2}}_{r,2}(\delta) \cap X^{s,0}_{r,1}(\delta). \tag{2.4}$$

Some simple embeddings are as follows. For $s, b_1, b_2 \in \mathbb{R}$, $r \geq 1$ and $b_1 > b_2 + \frac{1}{2}$

$$X^{s,b_1}_{r,2} \subset X^{s,b_2}_{r,1} \qquad \text{and} \qquad X^{s,0}_{r,1} \subset C(\mathbb{R}, \mathcal{F}L^{s,r})$$

which follow by Cauchy-Schwarz with respect to the L^1_τ norm and by $\mathcal{F}^{-1}L^1 \subset L^\infty$ respectively. In particular

$$Z^s_r(\delta) \subset C([-\delta,\delta], \mathcal{F}L^{s,r}).$$

We finally recall the following estimate[4] heavily used in the proof of Theorem 4.2 below.

Lemma 2.1 (Lemma 5.1 [22]). *Let $\frac{1}{3} < b < \frac{1}{2}$ and $s > 3\left(\frac{1}{2} - b\right)$. Then*

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{0,b}}.$$

In particular if $b = \frac{1}{2}-$, then

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{\epsilon,\frac{1}{2}-}}\|v\|_{X^{\epsilon,\frac{1}{2}-}}\|w\|_{X^{0,\frac{1}{2}-}}, \tag{2.5}$$

for small $\epsilon > 0$; while when $b = \frac{1}{3}+$

$$\|uv\overline{w}\|_{L^2_{xt}} \lesssim \|u\|_{X^{\frac{1}{2}-,\frac{1}{3}+}}\|v\|_{X^{\frac{1}{2}-,\frac{1}{3}+}}\|w\|_{X^{0,\frac{1}{3}+}}. \tag{2.6}$$

[3]Note that in our notation the indices (r, q) are the dual of the corresponding ones in Grünrock-Herr [22]

[4]This is a trilinear refinement of Bourgain's $L^6(\mathbb{T})$ Strichartz estimate [10].

2.1. **The Periodic Gauged Derivative NLS Equation.** We first recall S. Herr's gauge transformation. For $f \in L^2(\mathbb{T})$, let

$$G(f)(x) := \exp(-iJ(f))\, f(x)$$

where

$$J(f)(x) := \frac{1}{2\pi}\int_0^{2\pi}\int_\theta^x |f(y)|^2 - \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{T})}\, dy\, d\theta. \tag{2.7}$$

Note $G(f)$ is 2π-periodic since its integrand has zero mean value. Then for $u \in C([-T,T]; L^2(\mathbb{T}))$ and $m(u) := \frac{1}{2\pi}\int_{\mathbb{T}} |u(x,0)|^2 dx$ the adapted periodic gauge is defined as[5]

$$\mathcal{G}(u)(t,x) := G(u(t))(x - 2\,t\,m(u)).$$

Note the L^2 norm of $\mathcal{G}(u)(t,x)$ is still conserved since the torus is invariant under translation.
We have that

$$\mathcal{G} : C([-T,T]; H^\sigma(\mathbb{T})) \to C([-T,T]; H^\sigma(\mathbb{T}))$$

is a homeomorphism for any $\sigma \geq 0$ and locally bi-Lipschitz on subsets of $C([-T,T]; H^\sigma(\mathbb{T}))$ with prescribed $\|u(0)\|_{L^2}$ ([26]). Moreover the same is true if we replace $H^\sigma(\mathbb{T})$ by $\mathcal{F}L^{s,r}$ with $s > \frac{1}{2} - \frac{1}{r}$ when $2 < r < \infty$ and $s \geq 0$ when $r = 2$ ([22]).

Then if u is a solution to DNLS (2.1) and $v := \mathcal{G}(u)$ we have that v solves the gauged DNLS equation (GDNLS):

$$v_t - iv_{xx} = -v^2\overline{v}_x + \frac{i}{2}|v|^4 v - i\psi(v)v - im(v)|v|^2 v \tag{2.8}$$

with initial data $v(0) = \mathcal{G}(u(0))$, where

$$m(v)(t) := \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,t)|^2 dx \qquad \text{and} \tag{2.9}$$

$$\psi(v)(t) := -\frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Im}(v\overline{v}_x)\, dx + \frac{1}{4\pi}\int_{\mathbb{T}} |v|^4 dx - m(v)^2. \tag{2.10}$$

Note that $m(v)$ is conserved in time; more precisely $m(v)(t) = \frac{1}{2\pi}\int_{\mathbb{T}} |v(x,0)|^2 dx = m(u)$ and that both $m(v)$ and $\psi(v)$ are real.

The initial value problem associated to (2.8) with data in $\mathcal{F}L^{s,r}(\mathbb{T})$ is locally well-posed in $Z^s_r(\delta)$, $2 < r < 4$, $s \geq \frac{1}{2}$, for some $\delta > 0$. This was proved in Theorem 7.2 of [22].

Remark 2.2. Local well-posedness for (GDNLS) (2.8) implies local existence, uniqueness and continuity of the flow map for DNLS (2.1) [26, 22]. One cannot however carry back to solutions to DNLS all the auxiliary estimates coming from the local well posedness result for GDNLS.

Now we show how the energy $E(u)$ and $H(u)$ transform under the gauge. Let u be the solution to (DNLS) (2.1) and define

$$w = e^{-iJ(u)}u.$$

Then w solves (GDNLS) (2.8) with the extra $m(w)w_x$ term in the linear part of the equation [26]. So the gauge transform is, properly speaking the transformation $w = e^{-iJ(u)}u$ followed by the transformation

$$v(x,t) = w(t, x - 2m(w)t)$$

[5]Recall $m(u)(t)$ is conserved under the flow of (2.1).

But all the terms involved in the conserved quantities we considered are invariant under this second transformation $w \to v$ (the torus is invariant under translation). We also notice that $m(u) = m(w) = m(v)$, hence below we will be simply using m for this quantity.

Since

$$u = e^{iJ(w)} w$$

we have

$$u_x = e^{iJ(w)}(w_x + iJ(w)_x w)$$

with $J(w)_x = |w|^2 - m$.

We have

$$\begin{aligned} H(u) &= \mathrm{Im} \int_{\mathbb{T}} u\overline{u}_x \, dx + \frac{1}{2}\int_{\mathbb{T}} |u|^4 \, dx. \\ &= \mathrm{Im} \int_{\mathbb{T}} w \left(\overline{w}_x - iJ(w)_x \overline{w}\right) \, dx + \frac{1}{2}\int_{\mathbb{T}} |w|^4 \, dx. \\ &= \mathrm{Im} \int_{\mathbb{T}} w\overline{w}_x - \frac{1}{2}\int_{\mathbb{T}} |w|^4 \, dx + 2\pi m^2 =: \mathscr{H}(w) \end{aligned} \tag{2.11}$$

In addition we have

$$\begin{aligned} u_x\overline{u_x} &= (w_x + iJ(w)_x w)\left(\overline{w_x} - iJ(w)_x \overline{w}\right) \\ &= w_x\overline{w_x} + iJ(w)_x(w\overline{w_x} - \overline{w}w_x) + J(w)_x^2|w|^2 \\ &= w_x\overline{w_x} - 2\mathrm{Im} J(w)_x w\overline{w_x} + \left(|w|^6 - 2m|w|^4 + m^2\,|w|^2\right) \\ &= w_x\overline{w_x} - 2\mathrm{Im}\, w^2\overline{ww_x} + 2m\,\mathrm{Im}\, w\overline{w_x} + \left(|w|^6 - 2m|w|^4 + m^2\,|w|^2\right) \end{aligned} \tag{2.12}$$

By the same calculations we also have

$$u^2\overline{uu_x} = w^2\overline{ww_x} - i|w|^6 + im\,|w|^4. \tag{2.13}$$

We now recall that

$$E(u)(t) = \int |u_x|^2 \, dx + \frac{3}{2}\mathrm{Im}\int u^2\overline{u}\,\overline{u_x}\, dx + \frac{1}{2}\int |u|^6 \, dx, \tag{2.14}$$

hence by using (2.14), (2.12), (2.13) we find

$$E(u) = \int w_x\overline{w_x}\, dx - \frac{1}{2}\mathrm{Im}\int w^2\overline{ww_x}\, dx + 2m\,\mathrm{Im}\int w\overline{w_x}\, dx - \frac{1}{2}m\int |w|^4\, dx + 2\pi m^3.$$

If we define

$$\mathscr{E}(w) := \int_{\mathbb{T}} |w_x|^2 \, dx - \frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} w^2\overline{w}\,\overline{w_x}\, dx + \frac{1}{4\pi}\left(\int_{\mathbb{T}} |w(t)|^2 \, dx\right)\left(\int_{\mathbb{T}} |w(t)|^4 \, dx\right), \tag{2.15}$$

then $E(u)$ can be rewritten as

$$E(u) = \mathscr{E}(w) + 2m\,\mathscr{H}(w) - 2\pi\, m^3 =: \mathcal{E}(w). \tag{2.16}$$

Remark 2.3. We observe that $H(u)(t) = \mathscr{H}(w)(t)$ and $\frac{d}{dt}H(u)(t) = 0$ since H is the Hamiltonian for (DNLS) (2.1), hence it follows that $\frac{d}{dt}\mathscr{H}(w)(t) = 0$. On the other hand, we also know that $\frac{d}{dt}E(u)(t) = 0$, hence $\frac{d}{dt}\mathscr{E}(w)(t) = 0$. By the translation invariance of integration over $\mathbb{T}$, we have that (2.16) holds with v in place of w and

$$\frac{d}{dt}\mathscr{H}(v)(t) = 0 = \frac{d}{dt}\mathscr{E}(v)(t).$$

3. Finite dimensional approximation of (GDNLS)

We denote by $P_N f = \sum_{|n|\leq N} \widehat{f}(n)e^{inx}$ the finite dimensional projection onto the first $2N+1$ modes and $P_N^{\perp} := I - P_N$. Then the finite dimensional approximation (FGDNLS) is:

$$v_t^N = iv_{xx}^N - P_N((v^N)^2\overline{v_x^N}) + \frac{i}{2}P_N(|v^N|^4 v^N) - i\psi(v^N)(t)v^N - im(v^N)P_N(|v^N|^2 v^N) \tag{3.1}$$

with initial data

$$v_0^N = P_N\, v_0, \tag{3.2}$$

where m and ψ are as defined in (2.9) and (2.10) respectively.

Lemma 3.1. *We have that*

$$\frac{d}{dt}m(v^N)(t) := \frac{d}{dt}\frac{1}{2\pi}\int_{\mathbb{T}} |v^N(x,t)|^2 dx = 0.$$

Proof. Indeed for simplicity let us momentarily denote by $w := v^N$ a solution to (3.1); note $P_N w = w$. Then using that for any F, $\int P_N(F(v^N))\overline{v^N}dx = \int F(v^N)P_N\overline{v^N}dx = \int F(v^N)\overline{v^N}dx$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\big(2\pi m(w)\big) = 2\mathrm{Re}\int w_t\overline{w}\,dx \\
&= 2\mathrm{Re}\Big(-i\int |w_x|^2 - \int P_N(w^2\overline{w}_x)\overline{w} + \frac{i}{2}\int P_N(|w|^4 w)\overline{w} - \\
&\qquad - i\psi(w)(t)\int |w|^2 - im(w)(t)\int P_N(|w|^2 w)\overline{w}\Big) \\
&= 2\mathrm{Re}\Big(-\int (w^2\overline{w}_x)\overline{w} + \frac{i}{2}\int |w|^6 - i\psi(w)\int |w|^2 - im(w)\int |w|^4\Big) \\
&= -\int w^2\overline{w}\overline{w}_x - \int w w_x \overline{w}^2 = -\frac{1}{2}\int \partial_x(|w|^4) = 0.
\end{aligned}$$

□

Theorem 3.2 (Local well-posedness)**.** *Let $2 < r < 4$ and $s \geq \frac{1}{2}$. Then for every*

$$v_0^N \in B_R := \{v_0^N \in \mathcal{F}L^{s,r}(\mathbb{T}) \,/\, \|v_0^N\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < R\} \tag{3.3}$$

and $\delta \lesssim R^{-\gamma}$, for some $\gamma > 0$, there exists a unique solution

$$v^N \in Z_r^s(\delta) \subset C([-\delta,\delta]; \mathcal{F}L^{s,r}(\mathbb{T})) \tag{3.4}$$

of (3.1) *and* (3.2)*. Moreover the map*

$$\big(B_R, \|\cdot\|_{\mathcal{F}L^{s,r}(\mathbb{T})}\big) \longrightarrow C([-\delta,\delta]; \mathcal{F}L^{s,r}(\mathbb{T})) : \ v_0^N \to v^N$$

is real analytic.

Proof. The proof follows the argument in [22], Theorem 7.2 since P_N acts on a multilinear nonlinearity and it is a bounded operator in $L^p, 1 < p < \infty$ and commutes with D^s. Also, although the proof in [22] is presented for $s = \frac{1}{2}$, a simple argument of persistence of regularity gives the result for any $s \geq \frac{1}{2}$. □

The following lemma gives control on how close the finite dimensional approximations are to the solution of (2.8). Our proof is a variation of Bourgain's Lemma 2.27 in [3] (see also [12]).

Lemma 3.3 (Approximation lemma)**.** *Let $v_0 \in \mathcal{F}L^{s,r}(\mathbb{T}), s > \frac{1}{2}$, $2 < r < 4$ be such that $\|v_0\|_{\mathcal{F}L^{s,r}(\mathbb{T})} < A$, for some $A > 0$, and let N be a large integer. Assume the solution v^N of* (3.1) *with initial data $v_0^N(x) := P_N(v_0)$ satisfies the bound*

$$\|v^N(t)\|_{\mathcal{F}L^{s,r}(\mathbb{T})} \leq A, \text{ for all } t \in [-T, T], \tag{3.5}$$

for some given $T > 0$. Then the IVP (GDNLS) (2.8) *with initial data v_0 is well-posed on $[-T, T]$ and there exists $C_0, C_1 > 0$, such that its solution $v(t)$ satisfies the following estimate:*

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \lesssim \exp[C_0(1 + A)^{C_1} T] N^{s_1 - s}, \tag{3.6}$$

for all $t \in [-T, T], \frac{1}{2} \leq s_1 < s$.

Proof. We first observe that from the local well-posedness theory ([22] and Theorem 3.2), (GDNLS) (2.8) with initial data v_0 and (FGDNLS) (3.1) with initial data v_0^N are both well-posed in $[-\delta, \delta], \delta \sim (1 + A)^{-\gamma}$. Let $w := v - v^N$, then w satisfies the equation

$$w_t - iw_{xx} = F(v) - P_N F(v^N) = P_N[F(v) - F(v^N)] + (1 - P_N)F(v), \tag{3.7}$$

where $F(\cdot)$ is the nonlinearity of (2.8). By the Duhamel principle we have

$$w(t) = S(t)[v_0 - v_0^N] + \int_0^t S(t - t')(P_N[F(v) - F(v^N)](t') + (1 - P_N)F(v)(t'))\, dt',$$

where $S(t) = e^{it\Delta}$, and from the proof of Theorem 7.2 in [22] we have the bound

$$\begin{aligned}
\|w\|_{Z_r^{s_1}(\delta)} &\lesssim \|v_0 - v_0^N\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} + \delta^\gamma(1 + \|v^N\|_{Z_r^{s_1}(\delta)} + \|v\|_{Z_r^{s_1}(\delta)})^4 \|w\|_{Z_r^{s_1}(\delta)} \\
&+ \left\| (1 - P_N) \int_0^t S(t - t')F(v)(t')\, dt' \right\|_{Z_r^{s_1}(\delta)} \\
&\lesssim AN^{s_1 - s} + \delta^\gamma(1 + \|v^N\|_{Z_r^{s_1}(\delta)} + \|v\|_{Z_r^{s_1}(\delta)})^4 \|w\|_{Z_r^{s_1}(\delta)} + N^{s_1 - s}\delta^\gamma(1 + \|v\|_{Z_r^{s}(\delta)})^5. \qquad (3.8)
\end{aligned}$$

By choosing a smaller δ if necessary we obtain from (3.8)

$$\|w\|_{Z_r^{s_1}(\delta)} \leq CAN^{s_1 - s} + \frac{1}{2}\|w\|_{Z_r^{s_1}(\delta)},$$

for some absolute constant $C > 0$, from where

$$\|v(t) - v^N(t)\|_{\mathcal{F}L^{s_1,r}(\mathbb{T})} \leq 2CAN^{s_1 - s}, \quad \text{for all } t \in [-\delta, \delta] \tag{3.9}$$

and by iteration (3.6) follows. □

4. Analysis of the Finite Dimensional Equation (FGDNLS)

Recall that equation (DNLS) is Hamiltonian and hence its gauge equivalent formulation should stay Hamiltonian (change of coordinates). However, the gauge transformation is not a 'canonical map' and the symplectic form in the new coordinates depends on v; that is we lose the simple expression the symplectic form (namely ∂_x) had in the original coordinates. Two problems arise from the lack of commutativity between the gauged skew-selfadjoint form J and P_N:

(1) The conservation of Lebesgue measure associated to (FGDNLS) is not obvious as before. We must prove that this is indeed the case; see Subsection 4.1 below.

and more seriously:

(2) We lose the conservation of the energy $\mathcal{E}(v^N)$ for the finite dimensional approximations; that is $\dfrac{d\mathcal{E}(v^N)}{dt} \neq 0$. In particular we lose the invariance of μ_N, the associated finite dimensional weighted Wiener measure. However we have an estimate controlling its growth, namely Theorem 4.2 below.

4.1. **Invariance of the Lebesgue measure.** If we rewrite (FGDNLS) (3.1) as a system of complex ODE's for the Fourier coefficients $c_k \equiv \widehat{v^N}(k)$ we obtain a set of $2N+1$ complex equations of the form $\frac{d}{dt}c_k = F_k(\{c_j, \bar{c}_j\})$, or equivalently $4N+2$ equations $\frac{d}{dt}a_k = \mathrm{Re}F_k(\{c_j, \bar{c}_j\})$, and $\frac{d}{dt}b_k = \mathrm{Im}F_k(\{c_j, \bar{c}_j\})$ for the real functions $a_k = \mathrm{Re}F_k$ and $b_k = \mathrm{Im}F_k$.

To show that this set of equations preserves volume we need to verify that the divergence of the vector field vanishes, i.e.,

$$\sum_k \frac{\partial \mathrm{Re}F_k}{\partial a_k} + \frac{\partial \mathrm{Im}F_k}{\partial b_k} = 0.$$

This is easily shown to be equivalent to

$$\sum_k \frac{\partial F_k}{\partial c_k} + \frac{\partial \bar{F}_k}{\partial \bar{c}_k} = 0.$$

And indeed we have

Lemma 4.1. *The Lebesgue measure $\prod_{|j|\leq N} da_j db_j$ is invariant under the flow of the system of ODE's (4.1).*

Proof. The (FGDNLS) (3.1) as a system of complex ODE's for the Fourier coefficients c_k takes the form

$$\begin{aligned} \frac{d}{dt}c_k &= -ik^2 c_k + i \sum_{n_1,n_2,n_3} n_3 c_{n_1} c_{n_2} \bar{c}_{n_3} \delta_{n_1+n_2-n_3-k} \\ &\quad + \frac{i}{2} \sum_{n_1,n_2,n_3,n_4,n_5} c_{n_1} c_{n_2} c_{n_3} \bar{c}_{n_4} \bar{c}_{n_5} \delta_{n_1+n_2+n_3-n_4-n_5-k} \\ &\quad - i\psi(\{c_j, \bar{c}_j\}) c_k - i m(\{c_j, \bar{c}_j\}) \sum_{n_1,n_2,n_3} c_{n_1} c_{n_2} \bar{c}_{n_3} \delta_{n_1+n_2-n_3-k} \end{aligned} \tag{4.1}$$

with $m(\{c_j, \bar{c}_j\}) = \sum_j |c_j|^2$ and

$$\psi(\{c_j, \bar{c}_j\}) = -2\sum_k k|c_k|^2 + \frac{1}{2} \sum_{n_1,n_2,n_3,n_4} c_{n_1} c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{n_1+n_2-n_3-n_4} - \left(\sum_j |c_j|^2\right)^2. \tag{4.2}$$

To show that this set equation preserve volume we need to verify

$$\sum_k \frac{\partial F_k}{\partial c_k} + \frac{\partial \bar{F}_k}{\partial \bar{c}_k} = 0. \tag{4.3}$$

The vector field F_k consists of several terms which we analyze separately.

(1) $F_k^{(1)} = -ik^2 c_k$. Then $\frac{\partial F_k^{(1)}}{\partial c_k} + \frac{\partial \bar{F}_k^{(1)}}{\partial \bar{c}_k} = -ik^2 + ik^2 = 0$.

(2) $F_k^{(2)} = i\sum_{n_1,n_2,n_3} n_3 c_{n_1} c_{n_2} \bar{c}_{n_3} \delta_{n_1+n_2-n_3-k}$. To differentiate we consider the terms with $n_1 = k$ and $n_2 = k$ and obtain

$$\frac{\partial F_k^{(2)}}{\partial c_k} = i2\pi \sum_{n_2,n_3} n_3 c_{n_2} \bar{c}_{n_3} \delta_{n_2-n_3} + i2\pi \sum_{n_1,n_3} n_3 c_{n_1} \bar{c}_{n_3} \delta_{n_1-n_3} = i4\pi \sum_n n|c_n|^2 \tag{4.4}$$

and similarly

$$\frac{\partial \bar{F}_k^{(2)}}{\partial \bar{c}_k} = -i4\pi \sum_n n|c_n|^2 \tag{4.5}$$

and thus all the contributions of this term to the divergence disappear.

(3) $F_k^{(3)} = \frac{i}{2}\sum_{n_1,n_2,n_3,n_4,n_5} c_{n_1} c_{n_2} c_{n_3} \bar{c}_{n_4} \bar{c}_{n_5} \delta_{n_1+n_2+n_3-n_4-n_5-k}$. This term is treated similarly as (2) and is left to the reader.

(4) $F_k^{(4)} = 2i(\sum_j j|c_j|^2)c_k$. We have

$$\frac{\partial F_k^{(4)}}{\partial c_k} = 2ik|c_k|^2 + 2i\sum_j j|c_j|^2 \tag{4.6}$$

and

$$\frac{\partial \bar{F}_k^{(4)}}{\partial \bar{c}_k} = -2ik|c_k|^2 - 2i\sum_j j|c_j|^2 \tag{4.7}$$

and so these terms do not contribute to the divergence.

(5) $F_k^{(5)} = i(\sum_j |c_j|^2)^2 c_k$. We have

$$\frac{\partial F_k^{(5)}}{\partial c_k} = 2i(\sum_j |c_j|^2)|c_k|^2 + i(\sum_j |c_j|^2)^2 \tag{4.8}$$

and again we have $\frac{\partial F_k^{(5)}}{\partial c_k} + \frac{\partial \bar{F}_k^{(5)}}{\partial \bar{c}_k} = 0$.

(6) $F_k^{(6)} = -\frac{i}{2}\sum_{n_1,n_2,n_3,n_4} c_{n_1} c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{n_1+n_2-n_3-n_4} c_k$. We have

$$\begin{aligned} \frac{\partial F_k^{(6)}}{\partial c_k} &= -\frac{i}{2}\sum_{n_1,n_2,n_3,n_4} c_{n_1} c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{n_1+n_2-n_3-n_4} \\ &\quad -i\sum_{n_2,n_3,n_4} c_k c_{n_2} \bar{c}_{n_3} \bar{c}_{n_4} \delta_{k+n_2-n_3-n_4} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \frac{\partial \bar{F}_k^{(6)}}{\partial \bar{c}_k} &= +\frac{i}{2}\sum_{n_1,n_2,n_3,n_4} \bar{c}_{n_1} \bar{c}_{n_2} c_{n_3} c_{n_4} \delta_{n_1+n_2-n_3-n_4} \\ &\quad +i\sum_{n_2,n_3,n_4} \bar{c}_k \bar{c}_{n_2} c_{n_3} c_{n_4} \delta_{k+n_2-n_3-n_4}. \end{aligned} \tag{4.10}$$

The first terms in (4.9) and (4.10) cancel for each k. By summing the second terms in (4.9) and (4.10) over k, we see that they do not contribute to the divergence.

(7) $F_k^{(7)} = -i\sum_j |c_j|^2 \sum_{n_1,n_2,n_3} c_{n_1}c_{n_2}\bar{c}_{n_3}\delta_{n_1+n_2-n_3-k}$. We have

$$\frac{\partial F_k^{(7)}}{\partial c_k} = -i\sum_{n_1,n_2,n_3} c_{n_1}c_{n_2}\bar{c}_{n_3}\bar{c}_k\delta_{n_1+n_2-n_3-k} - 2i(\sum_j |c_j|^2)^2 \tag{4.11}$$

and

$$\frac{\partial \bar{F}_k^{(7)}}{\partial \bar{c}_k} = i\sum_{n_1,n_2,n_3} \bar{c}_{n_1}\bar{c}_{n_2}c_{n_3}c_k\delta_{n_1+n_2-n_3-k} + 2i(\sum_j |c_j|^2)^2. \tag{4.12}$$

The second terms add to 0 for each k while the first terms cancel if we sum over all k.

□

4.2. **Energy growth estimate.**

Theorem 4.2. *Let $v^N(t)$ be a solution to (FGDNLS) (3.1) in $[-\delta,\delta]$, and let $K>0$ be such that $\|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)} \le K$. Then there exists $\beta>0$ such that*

$$|\mathcal{E}(v^N(\delta)) - \mathcal{E}(v^N(0))| = \Big|\int_0^\delta \frac{d}{dt}\mathcal{E}(v^N)(t)dt\Big| \lesssim C(\delta)N^{-\beta}\max(K^6,K^8). \tag{4.13}$$

Remark 4.3. It is possible that the estimate (4.13) may still hold for a different choice of $X_r^{s,\frac{1}{2}}(\delta)$ norm, with $s\ge\frac{1}{2}$, $2<r<4$ so that local well-posedness holds. On the other hand the pair (s,r) should also be such that $(s-1)\cdot r<-1$ in order for $\mathcal{F}L^{s,r}$ to contain the support of the Wiener measure (c.f. Section 5). Our choice of $s=\frac{2}{3}-$ and $r=3$ allows us to prove (4.13) while satisfying the conditions for local well-posedness and the support of the measure. Note that $\mathcal{F}L^{\frac{2}{3}-,3}$ scales like $H^{\frac{1}{2}-}$.

4.3. **Preparation for the proof of Theorem 4.2.** Let v^N denote the solution of (FGDNLS) (3.1) which we rewrite as

$$v_t^N = \mathcal{L}v^N + P_N^\perp((v^N)^2\overline{v_x^N}) - \frac{i}{2}P_N^\perp(|v^N|^4v^N) + im(v^N)P_N^\perp(|v^N|^2v^N),$$

where

$$\mathcal{L}v^N := iv_{xx}^N - (v^N)^2\overline{v_x^N} + \frac{i}{2}|v^N|^4v^N - i\psi(v^N)v^N - im(v^N)|v^N|^2v^N. \tag{4.14}$$

We first observe that from (2.16) and Lemma 3.1 we have

$$\frac{d}{dt}\mathcal{E}(v^N) = \frac{d}{dt}\mathscr{E}(v^N) + 2m_N\frac{d}{dt}\mathscr{H}(v^N), \tag{4.15}$$

where $m_N := m(v^N)$.

Lemma 4.4. *With the above notations we have*

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(v^N)(t) =& -2Im\int v^N\overline{v^Nv_x^N}P_N^\perp((v^N)^2\overline{v_x^N})\,dx + Re\int v^N\overline{v^Nv_x^N}P_N^\perp(|v^N|^4v^N)\,dx\\ &- 2m_N Re\int v^N\overline{v^Nv_x^N}P_N^\perp(|v^N|^2v^N)\,dx + 2m_N Re\int v^N\overline{v^N}^2P_N^\perp((v^N)^2\overline{v_x^N})\,dx\\ &+ m_N Im\int v^N\overline{v^N}^2P_N^\perp(|v^N|^4v^N)\,dx - 2m_N^2 Im\int v^N\overline{v^N}^2P_N^\perp(|v^N|^2v^N)\,dx,\end{aligned} \tag{4.16}$$

$$\frac{d}{dt}\mathscr{H}(v^N)(t) = -2Re\int_{\mathbb{T}}(\overline{v^N})^2 v^N P_N^{\perp}((v^N)^2\overline{v_x^N})\,dx + Im\int v^N(\overline{v^N})^2 P_N^{\perp}(|v^N|^4 v^N)\,dx$$
$$-2m_N Im\int v^N(\overline{v^N})^2 P_N^{\perp}(|v^N|^2 v^N)\,dx, \tag{4.17}$$

and

$$\frac{d}{dt}\mathcal{E}(v^N)(t) = -2Im\int v^N\overline{v^N v_x^N}P_N^{\perp}((v^N)^2\overline{v_x^N})\,dx + Re\int v^N\overline{v^N v_x^N}P_N^{\perp}(|v^N|^4 v^N)\,dx$$
$$-2m_N Re\int v^N\overline{v^N v_x^N}P_N^{\perp}(|v^N|^2 v^N)\,dx - 2m_N Re\int_{\mathbb{T}}(\overline{v^N})^2 v^N P_N^{\perp}((v^N)^2\overline{v_x^N})\,dx \tag{4.18}$$
$$+3m_N Im\int v^N(\overline{v^N})^2 P_N^{\perp}(|v^N|^4 v^N)\,dx \;-6m_N Im\int v^N(\overline{v^N})^2 P_N^{\perp}(|v^N|^2 v^N)\,dx.$$

Proof. From (2.15) and integration by parts we have that

$$\frac{d}{dt}\mathscr{E}(v^N)(t) = -2\mathrm{Re}\int v_t^N\overline{v_{xx}^N}\,dx - 2\mathrm{Im}\int v^N v_t^N\overline{v^N v_x^N}\,dx + 2m_N\mathrm{Re}\int v^N v_t^N\overline{v^N}^2\,dx. \tag{4.19}$$

Due to the energy conservation for the (GDNLS) (infinite system), one can see that the contribution in (4.19) from $\mathcal{L}v^N$ defined in (4.14) is zero. On the other hand by orthogonality we also have

$$-2\mathrm{Re}\int\overline{v_{xx}^N}\Big(P_N^{\perp}((v^N)^2\overline{v_x^N}) - \frac{i}{2}P_N^{\perp}(|v^N|^4 v^N) + im(v^N)P_N^{\perp}(|v^N|^2 v^N)\Big)\,dx = 0.$$

Hence (4.16) follows. By a similar argument we obtain (4.17) as well. The lemma follows by substituting (4.16) and (4.17) into (4.15).

□

Remark 4.5. To establish Theorem 4.2 we need to estimate the terms in (4.18). In doing so we will ignore absolute constants and weather we are looking at the real or imaginary parts of the terms.

The first term in (4.18) gives a contribution to (4.13) which is essentially:

$$I_1 = \int_0^{\delta}\int v^N\overline{v^N v_x^N}P_N^{\perp}((v^N)^2\overline{v_x^N})\,dx\,dt. \tag{4.20}$$

This term is the hardest to control since it has two derivatives, so we will treat this one first. We start by discussing how to absorb the rough time cut-off. Assume ϕ is any function in $X_3^{\frac{2}{3}-,\frac{1}{2}}$ such that

$$\phi|_{[-\delta,\delta]} = v^N. \tag{4.21}$$

Then we write

$$I_1 = \int_{\mathbb{T}\times\mathbb{R}}\chi_{[0,\delta]}(t)\;P_N^{\perp}((v^N)^2\partial_x\overline{v^N})\,v^N\overline{v^N v_x^N}dxdt$$
$$= \int_{\mathbb{T}\times\mathbb{R}}\;P_N^{\perp}((\chi_{[0,\delta]}\phi^N)^2\,\chi_{[0,\delta]}\overline{\phi_x^N})\;\chi_{[0,\delta]}\phi^N\;\chi_{[0,\delta]}\overline{\phi^N}\;\chi_{[0,\delta]}\overline{\phi_x^N}dxdt$$

and by denoting

$$w := \chi_{[0,\delta]}\phi, \qquad w = P_N(w), \tag{4.22}$$

we will in fact show that

$$|I_1| = \left| \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}((w)^2 \partial_x \overline{w})\, w\overline{ww_x} dxdt \right| \leq C(\delta) N^{-\beta} \|w\|^6_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}. \tag{4.23}$$

To go back to v^N we use the following lemma:

Lemma 4.6 (Time-Cutoff)**.** *Let $b < b_1 < 1/2$. Then the exists $C'(\delta) > 0$ such that*

$$\|w\|_{X_3^{\frac{2}{3}-,b}} \leq C'(\delta)\, \|\phi\|_{X_3^{\frac{2}{3}-,b_1}} \leq C'(\delta)\, \|v^N\|_{X_3^{\frac{2}{3}-,\frac{1}{2}}(\delta)}$$

where w, ϕ and v^N are as above.

Proof. Since the regularity in x does not play any role, without any loss of generality we ignore the power $s = \frac{2}{3}-$. Then,

$$\begin{aligned} \|w\|_{X_3^{0,b}} &= \left(\sum_n \left(\int |\widehat{\chi_{[0,\delta]}\phi}(n,\tau)|^2 \langle \tau + n^2 \rangle^{2b}\, d\tau \right)^{\frac{3}{2}} \right)^{\frac{1}{3}} \\ &= \left(\sum_n \left(\int \left| \int_{\tau_1} \widehat{\chi_{[0,\delta]}}(\tau - \tau_1)\, \widehat{\phi}(n,\tau_1)\, d\tau_1 \right|^2 \langle \tau + n^2 \rangle^{2b}\, d\tau \right)^{\frac{3}{2}} \right)^{\frac{1}{3}}. \end{aligned} \tag{4.24}$$

Writing $\tau + n^2 = (\tau - \tau_1) + (\tau_1 + n^2)$ we bound (4.24) by

$$\lesssim \left(\sum_n \left(\int \left| \int_{\tau_1} \widehat{\chi_{[0,\delta]}}(\tau - \tau_1)\, \langle \tau - \tau_1 \rangle^b\, \widehat{\phi}(n,\tau_1)\, d\tau_1 \right|^2 d\tau \right)^{\frac{3}{2}} \right)^{\frac{1}{3}} \tag{4.25}$$

$$+ \left(\sum_n \left(\int \left| \int_{\tau_1} \widehat{\chi_{[0,\delta]}}(\tau - \tau_1)\, \widehat{\phi}(n,\tau_1)\, \langle \tau_1 + n^2 \rangle^b\, d\tau_1 \right|^2 d\tau \right)^{\frac{3}{2}} \right)^{\frac{1}{3}}. \tag{4.26}$$

We treat the first sum (4.25), the second one (4.26) being similar. If $\langle \tau - \tau_1 \rangle < \langle \tau_1 + n^2 \rangle$ then by Young's inequality (4.25) can be bounded by

$$\lesssim \|\frac{\widehat{\chi_{[0,\delta]}}(\tau)}{\langle \tau \rangle^{\varepsilon}}\|_{L^1} \left\| \|\widehat{\phi}(\tau,n) \langle \tau + n^2 \rangle^{b+\varepsilon}\|_{L^2} \right\|_{\ell^3} \lesssim \|\chi\|_{H^\beta}\, \|\phi\|_{X_3^{0,b_1}}$$

by Cauchy-Schwarz on the $\widehat{\chi}$ term provided $\beta + \varepsilon > \frac{1}{2}, \beta < \frac{1}{2}$ and where $b_1 := b + \varepsilon < \frac{1}{2}$.

On the other hand if $\langle \tau - \tau_1 \rangle \geq \langle \tau_1 + n^2 \rangle$, then again by Young's inequality (4.25) can be bounded by

$$\lesssim \|\widehat{\chi_{[0,\delta]}(\tau)} \langle \tau \rangle^{b+\varepsilon}\|_{L^2} \left\| \|\widehat{\phi}(\tau,n) \langle \tau + n^2 \rangle^{-\varepsilon}\|_{L^1} \right\|_{\ell^3} \lesssim \|\chi\|_{H^{b+\varepsilon}}\, \|\phi\|_{X_3^{0,b_1}}$$

by Cauchy-Schwarz on the $\widehat{\phi}$ term provided $b_1 + \varepsilon > \frac{1}{2}, b_1 < \frac{1}{2}$. Finally by taking infimum and using the definition of $X_3^{0,\frac{1}{2}}(\delta)$ a bound in terms of $\|v^N\|_{X_3^{0,\frac{1}{2}}(\delta)}$ follows.

□

4.4. **Proof of Theorem 4.2.** Returning to (4.23) we write

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}\times\mathbb{R}} P_N^{\perp}(w^2\partial_x\overline{w})\, w\, \overline{ww_x}dxdt \\
&= \int_\tau \sum_{|n|>N} (\widehat{w^2\overline{w_x}})(n,\tau)\overline{(\widehat{\overline{w}ww_x})}(n,\tau)d\tau \\
&= \int \sum_{|n|>N} \Big(\int_{\tau=\tau_1+\tau_2-\tau_3} \sum_{n=n_1+n_2-n_3,\, |n_j|\le N} \widehat{w}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)(-in_3)\overline{\widehat{w}}(n_3,\tau_3)d\tau_1 d\tau_2 \Big) \\
&\quad \times \Big(\int_{-\tau=\tau_4-\tau_5-\tau_6} \sum_{-n=n_4-n_5-n_6,\, |n_j|\le N} \widehat{w}(n_4,\tau_4)\overline{\widehat{w}}(n_5,\tau_5)(-in_6)\overline{\widehat{w}}(n_6,\tau_6)d\tau_4 d\tau_5 \Big) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \int \sum_{N<|n|\le 3N} \Big(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3,\, |n_j|\le N} \widehat{w}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)(in_3)\overline{\widehat{w}}(n_3,\tau_3)d\tau_1 d\tau_2 \Big) \\
&\quad \times \Big(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6,\, |n_j|\le N} \widehat{w}(n_4,\tau_4)\overline{\widehat{w}}(n_5,\tau_5)(in_6)\overline{\widehat{w}}(n_6,\tau_6)d\tau_4 d\tau_5 \Big) d\tau \\
&= \int \sum_{N<|n|\le 3N} \Big(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3,\, |n_j|\le N} \widehat{w_1}(n_1,\tau_1)\widehat{w_2}(n_2,\tau_2)(in_3)\overline{\widehat{w_3}}(n_3,\tau_3)d\tau_1 d\tau_2 \Big) \\
&\quad \times \Big(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6,\, |n_j|\le N} \widehat{w_4}(n_4,\tau_4)\overline{\widehat{w_5}}(n_5,\tau_5)(in_6)\overline{\widehat{w_6}}(n_6,\tau_6)d\tau_4 d\tau_5 \Big) d\tau,
\end{aligned}
$$

where $w_1 = w_2 = w_4 = w$ and $\overline{w_3} = \overline{w_5} = \overline{w_6} = \overline{w}$.

Remark 4.7. In what follows we always think of N_j, N as dyadic; more precisely $N_j := 2^{K_j}$, $N := 2^K$ where $K_j < K$ since $n_j \in \mathbb{Z}$. By a slight abuse of notation we then denote by N_j both $|n_j|$ and the dyadic interval $[2^{K_j}, 2^{K_j+1})$ $|n_j|$ belongs to when $n_j \neq 0$. Moreover we denote by w_{N_j} the function such that $\widehat{w_{N_j}}(n_j) = \chi_{\{|n_j|\sim N_j\}}\widehat{w_j}(n_j)$

From the expression above we then have,

$$|n_j| \le N, \qquad N \le |n| \le 3N, \qquad n = n_1+n_2+n_3, \quad \text{and} \quad -n = n_4+n_5+n_6, \tag{4.27}$$

$$N \sim \max(N_1,N_2,N_3) \sim \max(N_4,N_5,N_6), \tag{4.28}$$

$$\tau + n^2 - (\tau_1 + n_1^2) - (\tau_2 + n_2^2) - (\tau_3 - n_3^2) = 2(n-n_1)(n-n_2) \tag{4.29}$$

and

$$\tau + n^2 + (\tau_4 + n_4^2) + (\tau_5 - n_5^2) + (\tau_6 - n_6^2) = 2(n+n_5)(n+n_6). \tag{4.30}$$

So if we let $\tilde{\sigma}_j := \tau_j \pm n_j^2$ and $\sigma_j := \langle \tau_j \pm n_j^2 \rangle$ we have by subtracting (4.29) from (4.30)

$$\sum_{j=1}^{6} \tilde{\sigma}_j = -2\,(\, n\,(n_1+n_2+n_5+n_6) - n_1n_2 + n_5n_6\,). \tag{4.31}$$

This in turn can also be rewritten using $n_1+n_2+n_3+n_4+n_5+n_6 = 0$ or $n = n_1+n_2+n_3$ and $-n = n_4+n_5+n_6$ as:

$$\sum_{j=1}^{6} \tilde{\sigma}_j = 2(\, n\,(n_3+n_4) + n_1 n_2 - n_5 n_6\,). \tag{4.32}$$

In addition, since $\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=0$, adding and subtracting n_j^2, $j=1,\dots,6$ in the appropriate fashion, we obtain:

$$\sum_{j=1}^{6} \tilde{\sigma}_j = (n_3^2+n_5^2+n_6^2) - (n_1^2+n_2^2+n_4^2). \tag{4.33}$$

Hence we need to estimate

(4.34)

$$\begin{aligned}
|I_1| =& \Big| \sum_{N_i\le N;\, i=1,\dots 6} \int_{\mathbb{R}}\int_{\mathbb{T}} P_N^{\perp}\Big(w_{N_1}\, w_{N_2}\, \partial_x \overline{w_{N_3}}\Big)\, w_{N_4}\, \overline{w_{N_5}}\, \partial_x \overline{w_{N_6}}\, dxdt \Big| \\
=& \Big| \sum_{N_i\le N;\, i=1,\dots 6} \sum_{|n|\ge N} \int_{\tau} \Big(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3} \widehat{w_{N_1}}\, \widehat{w_{N_2}}\,(in_3)\, \widehat{\overline{w_{N_3}}}\; d\tau_1 d\tau_2 \Big) \quad \times \\
& \qquad \Big(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6} \widehat{w_{N_4}}\, \widehat{\overline{w_{N_5}}}\,(in_6)\, \widehat{\overline{w_{N_6}}}\; d\tau_4 d\tau_5 \Big)\, d\tau \Big|
\end{aligned}$$

(4.35)

$$\begin{aligned}
\le& \sum_{N\le|n|\le 3N} \sum_{N_i\le N;\, i=1,\dots 6} \int_{\tau} \Big(\int_{\tau=\tau_1+\tau_2+\tau_3} \sum_{n=n_1+n_2+n_3} |\widehat{w_{N_1}}|\,|\widehat{w_{N_2}}|\,|n_3|\,|\widehat{\overline{w_{N_3}}}|\; d\tau_1 d\tau_2 \Big) \quad \times \\
& \qquad \Big(\int_{-\tau=\tau_4+\tau_5+\tau_6} \sum_{-n=n_4+n_5+n_6} |\widehat{w_{N_4}}|\,|\widehat{\overline{w_{N_5}}}|\,|n_6|\,|\widehat{\overline{w_{N_6}}}|\; d\tau_4 d\tau_5 \Big)\, d\tau.
\end{aligned}$$

Remark 4.8. This expression (4.35) will be our point of departure in beginning our estimate. In what follows we will abuse notation and write w_{N_j} for $\widetilde{|\widehat{w_{N_j}}|}$ and $\overline{w_{N_k}}$ for $\widetilde{|\widehat{\overline{w_{N_k}}}|}$ since at the end we will estimate all functions in the $X_r^{s,b}$ norms which depend solely on the absolute value of the Fourier transform.

We start by laying out all possible cases and organizing them according to the sizes of the two derivative terms.

Types:

I. $N_3\sim N$, $N_6\sim N$

II. $N_3\sim N$ and $N_6\ll N$

III. $N_6\sim N$ and $N_3\ll N$

IV. $N_3\ll N$; $N_6\ll N$

Now we subdivide into all subcases in each situation and group them according to how many low frequencies (ie. $N_j\ll N$) we have overall taking into account (4.28).

All Cases for each type:

IA. $N_3\sim N$, $N_6\sim N$ and 4 lows: $N_1, N_2, N_4, N_5 \ll N$

IB. $N_3 \sim N$, $N_6 \sim N$ and 3 lows
- (i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$
- (ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$
- (iii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$
- (iv) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$

IC. $N_3 \sim N$, $N_6 \sim N$ and 2 lows
- (i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$
- (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$
- (iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$
- (iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$
- (v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$
- (vi) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$

ID. $N_3 \sim N$, $N_6 \sim N$ and 1 low
- (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$
- (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$
- (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IE. $N_3 \sim N$, $N_6 \sim N$ and $N_1, N_2, N_4, N_5 \sim N$

IIA. $N_3 \sim N$ and $N_6 \ll N$ and 3 lows
- (i) $N_1, N_2, N_4 \ll N$ and $N_5 \sim N$
- (ii) $N_1, N_2, N_5 \ll N$ and $N_4 \sim N$

IIB. $N_3 \sim N$ and $N_6 \ll N$ and 2 lows
- (i) $N_1, N_2 \ll N$ and $N_4, N_5 \sim N$
- (ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$
- (iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$
- (iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$
- (v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IIC. $N_3 \sim N$ and $N_6 \ll N$ and 1 low
- (i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$
- (ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$
- (iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$
- (iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IID. $N_3 \sim N$ and $N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

IIIA. $N_6 \sim N$ and $N_3 \ll N$ and 3 lows
- (i) $N_2, N_4, N_5 \ll N$ and $N_1 \sim N$
- (ii) $N_1, N_4, N_5 \ll N$ and $N_2 \sim N$

IIIB. $N_6 \sim N$ and $N_3 \ll N$ and 2 lows

(i) $N_4, N_5 \ll N$ and $N_1, N_2 \sim N$

(ii) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$

(iii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$

(iv) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$

(v) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IIIC. $N_6 \sim N$ and $N_3 \ll N$ and 1 low

(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$

(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$

(iii) $N_4 \ll N$ and $N_1, N_2, N_5 \sim N$

(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IIID. $N_6 \sim N$ and $N_3 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

IVA. $N_3 \ll N, N_6 \ll N$ and 2 lows

(i) $N_1, N_4 \ll N$ and $N_2, N_5 \sim N$

(ii) $N_1, N_5 \ll N$ and $N_2, N_4 \sim N$

(iii) $N_2, N_4 \ll N$ and $N_1, N_5 \sim N$

(iv) $N_2, N_5 \ll N$ and $N_1, N_4 \sim N$

IVB. $N_3 \ll N, N_6 \ll N$ and 1 low

(i) $N_1 \ll N$ and $N_2, N_4, N_5 \sim N$

(ii) $N_2 \ll N$ and $N_1, N_4, N_5 \sim N$

(iii) $N_4 \ll N$ and $N_1, N_4, N_5 \sim N$

(iv) $N_5 \ll N$ and $N_1, N_2, N_4 \sim N$

IVC. $N_3 \ll N, N_6 \ll N$ and $N_1, N_2, N_4, N_5 \sim N$

In what follows we will use the following estimates repeatedly:

Lemma 4.9. *Let w_{N_i} be as above. Then*

$$\|w_{N_i}\|_{X^{0+,\frac{1}{2}-}} \leq N_i^{-\frac{1}{2}+} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \tag{4.36}$$

$$\|w_{N_i}\|_{X^{\frac{1}{2}-,\frac{1}{3}+}} \leq \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}. \tag{4.37}$$

We also have that

$$\|w_{N_i}\|_{L^8_{xt}} \leq \|w_{N_i}\|_{X_3^{\frac{13}{24}+,\frac{3}{8}+}}. \tag{4.38}$$

If we assume that $\sigma_i \lesssim N^\gamma$, for any $\gamma > 0$, then

$$\|w_{N_i}\|_{L^\infty_{xt}} \leq N^{0+} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}. \tag{4.39}$$

Proof. The estimates (4.36) and (4.37) are a consequence of frequency localization and Hölder's inequality. The estimate (4.39) is a consequence of Sobolev embedding together with the assumption that $\sigma_i \leq N^\gamma$. □

Lemma 4.10. *Let $0 < \beta < 2$, $\rho \geq 0$ and $\delta > 0$. Let $M > 0$ and w_M be such that* $\operatorname{supp} w_M(\cdot, x) \subset [-\delta, \delta]$, $x \in \mathbb{T}$. *Then if we define*

$$\widehat{J_\beta w_M}(\tau, n) := \chi_{\{|n|\sim M\}} \chi_{\{|\tau+n^2|\leq M^\beta\}} |\widehat{w_M}(\tau, n)|,$$

we have

$$\|J_\beta w_M\|_{X^{0,\rho}} \lesssim C_\delta\, A(\beta, M)^{\frac{1}{6}} M^{\rho\beta+} \|w_M\|_{X_3^{0,\frac{1}{6}}}, \tag{4.40}$$

where $A(M, \beta)$ defined below is bounded by $1 + M^{\beta-1}$.

Proof. We write

$$\begin{aligned}
\|J_\beta w_M\|^2_{X^{0,\rho}} &= \sum_{|n|\sim M} \int_{|\tau+n^2|\leq M^\beta} |\widehat{w_M}(\tau, n)|^2 \langle \tau + n^2\rangle^{2\rho}\, d\tau \\
&\leq M^{2\rho\beta} \int_\tau \Big(\sum_{|n|\sim M,\, |\tau+n^2|\leq M^\beta} |\widehat{w_M}(\tau, n)|^2 \Big)\, d\tau \\
&\leq M^{2\rho\beta} \int_\tau \Big[\sum_{|n|\sim M,\, |\tau+n^2|\leq M^\beta} |\widehat{w_M}(\tau, n)|^3 \Big]^{\frac{2}{3}} |S(\tau, M, \beta)|^{\frac{1}{3}}\, d\tau, \qquad (4.41)
\end{aligned}$$

where

$$S(\tau, M, \beta) := \{n \in \mathbb{Z} : |n| \sim M \text{ and } |\tau + n^2| \leq M^\beta\}. \tag{4.42}$$

and $|S|$ represents the counting measure of the set.

We will show below that

$$A(M, \beta) := \sup_\tau |S(\tau, M, \beta)| \leq 1 + M^{\beta-1} \tag{4.43}$$

Hence (4.41) is less than or equal to

$$\begin{aligned}
& A(M, \beta)^{\frac{1}{3}} M^{2\rho\beta} \int_\tau \Big[\sum_n \chi_{\{|n|\sim M\}}(n),\, \chi_{\{|\tau+n^2|\leq M^\beta\}}(\tau, n) |\widehat{w_M}(\tau, n)|^3 \Big]^{\frac{2}{3}}\, d\tau \\
= \;& A(M, \beta)^{\frac{1}{3}} M^{2\rho\beta} \int_\tau \Big\| \Big\{ \chi_{\{|\tau+n^2|\leq M^\beta\}}(\tau, n) \widehat{w_M}(\tau, n) \Big\}_n \Big\|^2_{\ell^3(|n|\sim M)}\, d\tau \\
\sim \;& A(M, \beta)^{\frac{1}{3}} M^{2\rho\beta} \int_t \Big\| \mathcal{F}_\tau^{-1} \Big(\Big\{ \chi_{\{|\tau+n^2|\leq M^\beta\}}(\tau, n) \widehat{w_M}(\tau, n) \Big\}_n \Big)(t) \Big\|^2_{\ell^3(|n|\sim M)}\, dt \\
= \;& A(M, \beta)^{\frac{1}{3}} M^{2\rho\beta} \int_t \Big\| \Big\{ \mathcal{F}_\tau^{-1} \Big(\chi_{\{|\tau+n^2|\leq M^\beta\}}(\tau, n) \Big) * \mathcal{F}_\tau^{-1} \Big(\widehat{w_M}(\tau, n) \Big) \Big\}_n (t) \Big\|^2_{\ell^3(|n|\sim M)}\, dt.
\end{aligned}$$

Note that $\mathcal{F}_\tau^{-1}\Big(\widehat{w_M}(\cdot, n)\Big)(t)$ is still supported on $[-\delta, \delta]$ for all n and

$$\mathcal{F}_\tau^{-1}\Big(\chi_{\{|\tau+n^2|\leq M^\beta\}}(\cdot, n)\Big)(t) = 2e^{-itn^2} \frac{\sin(M^\beta t)}{t}. \tag{4.44}$$

We then continue the chain of inequalities from the expression above with

$$
\begin{aligned}
&= A(M,\beta)^{\frac{1}{3}} M^{2\rho\beta} \int_t \left\| \int_{\mathbb{R}} \chi_{[-\delta,\delta]}(t') \mathcal{F}_n(w_M(t',\cdot))(n) e^{-i(t-t')n^2} \frac{\sin(M^\beta(t-t'))}{t-t'} dt' \right\|^2_{\ell^3(|n|\sim M)} dt \\
&\leq A(M,\beta)^{\frac{1}{3}} M^{2\rho\beta} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi_{[-\delta,\delta]}(t') \|\mathcal{F}_n(w_M(t',\cdot))(n)\|_{\ell^3(|n|\sim M)} \left| \frac{\sin(M^\beta(t-t'))}{t-t'} \right| dt' \right]^2 dt.
\end{aligned}
$$

Let $p = 2-$ and $q = 1+$, then we compute

$$
\begin{aligned}
\left\| \frac{\sin(M^\beta(t-t'))}{t-t'} \right\|_{L^q_t} &= M^\beta \left(\int_{\mathbb{R}} \left| \frac{\sin(M^\beta t)}{tM^\beta} \right|^q dt \right)^{\frac{1}{q}} \\
&= M^\beta M^{-\frac{\beta}{q}} \left(\int_{\mathbb{R}} \left| \frac{\sin(r)}{r} \right|^q dr \right)^{\frac{1}{q}} \lesssim M^{0+}. \qquad (4.45)
\end{aligned}
$$

On the other hand for $\frac{1}{\gamma} = \frac{1}{p} - \frac{1}{3}$

$$
\begin{aligned}
\left\| \chi_{[-\delta,\delta]}(\cdot) \|\mathcal{F}_n(w_M(t,\cdot))(n)\|_{\ell^3(|n|\sim M)} \right\|^2_{L^p_t} &\lesssim \delta^{\frac{2}{\gamma}} \left\| \|\mathcal{F}_n(w_M(t,\cdot))(n)\|_{\ell^3(|n|\sim M)} \right\|^2_{L^3_t} \\
&\lesssim \delta^{\frac{2}{\gamma}} \left\| \left\| e^{itn^2} \mathcal{F}_n(w_M(t,\cdot))(n) \right\|_{\ell^3(|n|\sim M)} \right\|^2_{L^3_t} \\
&= \delta^{\frac{2}{\gamma}} \left\| \left\| e^{itn^2} \mathcal{F}_n(w_M(t,\cdot))(n) \right\|_{L^3_t} \right\|^2_{\ell^3(|n|\sim M)} \\
&\lesssim \delta^{\frac{2}{\gamma}} \left\| \left\| e^{itn^2} \mathcal{F}_n(w_M(t,\cdot))(n) \right\|_{H^{\frac{1}{6}}_t} \right\|^2_{\ell^3(|n|\sim M)} \\
&= \delta^{\frac{2}{\gamma}} \|w_M\|^2_{X_3^{0,\frac{1}{6}}}, \qquad (4.46)
\end{aligned}
$$

where we used the Sobolev theorem and the definition of $X_r^{s,b}$. Finally by Young's inequality, (4.45) and (4.46) we have the desired estimate.

It remains to show (4.43). We use an argument similar to [18]. For fixed τ let $S := S(\tau, M, \beta) \neq \emptyset$, then there exists $n_0 \in S$ and hence

(4.47)
$$|S| \leq 1 + |\{l \in \mathbb{Z} / |\, n_0 + l \,| \sim M,\ |\tau + (n_0+l)^2| \leq M^\beta\}| \leq 1 + |\{l \in \mathbb{Z} \,/\, |l| \leq M,\ |2n_0 l + l^2| \lesssim M^\beta\}|.$$

(4.48)
$$|2n_0 l + l^2| = |(l+n_0)^2 - n_0^2| \lesssim M^\beta \quad \text{if and only if} \quad -CM^\beta + n_0^2 \leq (l+n_0)^2 \leq n_0^2 + CM^\beta$$

Hence we need $|\, l \,| \leq M$ to satisfy

$$
\begin{aligned}
&-\sqrt{n_0^2 + CM^\beta} \leq (l+n_0) \leq \sqrt{n_0^2 + CM^\beta}, \\
&(l+n_0) \geq \sqrt{n_0^2 - CM^\beta} \quad \text{or} \quad (l+n_0) \leq -\sqrt{n_0^2 - CM^\beta}.
\end{aligned}
$$

In other words we need to know the size of

$$[-\sqrt{n_0^2 + CM^\beta}, -\sqrt{n_0^2 - CM^\beta}] \cup [\sqrt{n_0^2 - CM^\beta}, \sqrt{n_0^2 + CM^\beta}]$$

which is of the order of $\frac{M^\beta}{|n_0|}$. Hence since $|n_0| \sim M$ we have that

$$|S| \leq 1 + M^{\beta-1} \tag{4.49}$$

which implies (4.43) by taking $\sup_\tau$.

□

In what follows we are under the assumption that $\sigma_j \lesssim N^7$ for all $j = 1, \dots, 6$. Towards the end of the proof we remove this assumption. We begin by treating all cases with at least two high frequencies in the non derivative terms. All cases in [IC], [ID] [IE] [IIB] [IIC] [IID] [IIIB] [IIIC] [IIID] [IVA] [IVB] [IVC] follow from the following lemma applied with the exponent σ appearing below set equal to 0.

Lemma 4.11. *Assume there are $i, j \in \{1, 2, 4, 5\}$ such that $N_i \geq N^{1-\sigma}$ for $0 \leq \sigma < \frac{1}{6}$ and $N_j \sim N$ then (4.35) can be estimated by $N^{-\frac{1}{12}+\frac{\sigma}{2}} \prod_{i=1}^6 \|w_i\|_{X_3^{s,b}}$.*

Proof. By Plancherel we have that (4.35) is less than or equal to

$$\sum_{N_j \sim N; N_i \geq N^{1-\sigma};\, N_k \leq N,\, 1 \leq k \leq 6} \int_{\mathbb{R}} \int_{\mathbb{T}} N_3\, N_6 \;\; w_{N_1}\, w_{N_2}\, \overline{w_{N_3}}\, w_{N_4}\, \overline{w_{N_5}}\, \overline{w_{N_6}} \;\, dx\, dt. \tag{4.50}$$

Let $0 < \beta < 1$ to be determined below. Assume

$$\sigma_3 \leq N_3^\beta. \tag{4.51}$$

By Cauchy-Schwarz's inequality, grouping the first three functions in (4.50) in L^2_{xt} and the last three in L^2_{xt} and using (2.5) we have that (4.50) is less than or equal to

$$\sum_{N_j \sim N; N_i \geq N^{1-\sigma};\, N_k \leq N} N_3\, N_6 \prod_{i=1}^6 \|w_{N_i}\|_{X^{\epsilon, \frac{1}{2}-}}. \tag{4.52}$$

Note now that by (4.51) w_{N_3} is equal to $J_\beta w_{N_3}$ as defined in Lemma 4.10 above. Then we have

$$\|w_{N_3}\|_{X^{\epsilon, \frac{1}{2}-}} \leq C_\delta\, N_3^{\frac{1}{2}\beta+} \|w_{N_3}\|_{X_3^{0, \frac{1}{6}+}}. \tag{4.53}$$

Hence by (4.36), (4.53) we have that (4.52) is less than or equal to

$$\sum_{N_j \sim N; N_i \geq N^{1-\sigma};\, N_k \leq N} N_3\, N_6 N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_3^{\frac{1}{2}\beta+} N_3^{-\frac{2}{3}} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} N_6^{-\frac{1}{2}+} \left(\prod_{i=1}^6 \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right). \tag{4.54}$$

$$\lesssim \sum_{N_j \sim N; N_i \geq N^{1-\sigma};\, N_k \leq N} N_3^{\frac{1}{3}+\frac{\beta}{2}+} N^{\frac{1}{2}+} N^{-1+\frac{\sigma}{2}} \left(\prod_{i=1}^6 \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right). \tag{4.55}$$

From here we apply Hölder's inequality with $r = 3, r' = \frac{3}{2}$ to sum in N_j, N_i, N_k (multiply and divide by $N_j^{-\epsilon}$ with a loss of N^ϵ for each term). For example,

$$\sum_{N_j \leq N} \|w_{N_j}\|_{X_3^{s,b}} \;\; = \;\; \sum_{N_j \leq N} \left\| \|\langle n_j \rangle^s \langle \tau + n_j^2 \rangle^b\, \widehat{w}_{N_j}(\tau, n_j)\|_{L^2_\tau} \right\|_{\ell^3}. \tag{4.56}$$

Set $Y_{N_j}(n_j) := \|\langle n_j\rangle^s \langle \tau - n_j^2\rangle^b \, \widehat{w}_{N_j}(\tau, n_j)\|_{L^2_\tau}$, then the expression in (4.56) equals

$$\begin{aligned}
\sum_{N_j \le N} N_j^{\varepsilon} N_j^{-\varepsilon} \|Y_{N_j}\|_{\ell^3} &\le N^{\varepsilon} \Big(\sum_{N_j \le N} N_j^{-\frac{3}{2}\varepsilon} \Big)^{\frac{2}{3}} \Big(\sum_{N_j \le N} \|Y_{N_j}\|_{\ell^3}^3 \Big)^{\frac{1}{3}} \\
&\lesssim N^{\varepsilon} \Big(\sum_{N_j \le N} \sum_{|n_j| \sim N_j} \|\langle n_j\rangle^s \langle \tau + n_j^2\rangle^b \, \widehat{w}_j(\tau, n_j)\|_{L^2_\tau}^3 \Big)^{\frac{1}{3}} \qquad (4.57) \\
&\sim N^{\varepsilon} \|w_j\|_{X_3^{s,b}}.
\end{aligned}$$

Note then that all in all we get at worst a factor of $N^{-\frac{1}{6}+\frac{\beta}{2}+\frac{\sigma}{2}+}$.

Now assume that

$$\sigma_3 \ge N_3^{\beta}. \qquad (4.58)$$

Then rewrite (4.50) as

$$\sum_{N_j \sim N; N_i \ge N^{1-\sigma};\, N_k \le N} \int_{\mathbb{R}} \int_{\mathbb{T}} N_3 \, N_6 \, |\sigma_3|^{-\frac{1}{2}+} \; w_{N_1} \, w_{N_2} \, |\sigma_3|^{\frac{1}{2}-} \overline{w_{N_3}} \, w_{N_4} \, \overline{w_{N_5}} \, \overline{w_{N_6}} \; dx \, dt. \qquad (4.59)$$

We do Hölder by placing $|\sigma_3|^{\frac{1}{2}-} \overline{w_{N_3}}$ in $L^2_{x,t}$, the product of $\overline{w_{N_6}}$ with the two largest among w_{N_1}, w_{N_2}, w_{N_4}, $\overline{w_{N_5}}$ in $L^2_{x,t}$ while the remaining ones in $L^\infty_{x,t}$. Then by (4.36) and (4.39), we bound (4.59) by

$$\begin{aligned}
&\lesssim \sum_{N_j \sim N; N_i \ge N^{1-\sigma};\, N_k \le N} N_3 N_3^{-\frac{1}{2}-\frac{\beta}{2}+} N_6 N_6^{-\frac{1}{2}+} N^{-\frac{1}{2}+} N^{-\frac{1}{2}+\frac{\sigma}{2}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \Big) \\
&\lesssim \sum_{N_j \sim N; N_i \ge N^{1-\sigma};\, N_k \le N} N_3^{\frac{1}{2}-\frac{\beta}{2}+} N^{-\frac{1}{2}+\frac{\sigma}{2}+} \Big(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \Big).
\end{aligned}$$

We want that $\beta > \sigma$ to conclude by Hölder the desired inequality with a decay in N. We now impose that

$$-\frac{1}{6} + \frac{\beta}{2} + \frac{\sigma}{2} = -\frac{\beta}{2} + \frac{\sigma}{2},$$

whence $\beta = \frac{1}{6}$ and provided $0 < \sigma < \frac{1}{6}$ the lemma follows. □

It remains then to treat cases [IA], [IB], [IIA] and [IIIA]. Before starting we note the following support condition that will be used throughout in what follows.

Support Condition. By (4.27) and (4.28) the triplet $(w_{N_1}, w_{N_2}, \overline{w_{N_3}})$ satisfies $n = n_1 + n_2 + n_3$, $|n_j| \le N$, $N \le |n| \le 3N$ and $N \sim \max(N_1, N_2, N_3)$.

Suppose that -say- $\max(N_1, N_2) \le N^\theta$ for some $0 < \theta < 1$. Without any loss of generality assume $n > 0$. Then, we have that $N \le n \le (n_1 + n_2) + n_3 \le 2N^\theta + N$ and hence $n = N + k$ where $0 \le k \le 2N^\theta$. Next observe that $n_3 = n - (n_1 + n_2) = N + k - (n_1 + n_2)$ with $|k - (n_1 + n_2)| \le 4N^\theta$, whence $n_3 = N + O(N^\theta)$. In other words, we have that whenever $\max(N_1, N_2) \le N^\theta$ the support of $\widehat{\overline{w_{N_3}}}$ is of size $O(N^\theta)$. Note that we could have just as well said that the support of $\widehat{\overline{w_{N_3}}}$ is of size $O(\max(N_1, N_2))$ in lieu of $O(N^\theta)$.

When we are in this situation we say we have the *support condition* on $\overline{w_{N_3}}$. This argument is symmetric with respect to w_{N_1}, w_{N_2} or $\overline{w_{N_3}}$. The exact same analysis

holds for $(w_{N_4}, \overline{w_{N_5}}, \overline{w_{N_6}})$. By abuse of notation we still write for example, $\widehat{\overline{w_{N_3}}}(n_3)$ for $\widehat{\overline{w_{N_3}}}(n_3)\chi_{I_3}(n_3)$, where $I_3(n_3)$ is the support of $\widehat{\overline{w_{N_3}}}$ when the support condition holds.

Remark 4.12. As a consequence of the support condition, estimate (4.36) can be improved. For example if we have the support condition on $\widehat{\overline{w_{N_3}}}$ then

$$\|w_{N_3}\|_{X^{0+,\frac{1}{2}-}} \lesssim |I_3|^{\frac{1}{6}} \|w_{N_3}\|_{X_3^{0+,\frac{1}{2}-}}$$

Case [IIIA]. Note that (i) and (ii) are symmetric with respect to $j = 1$ and $j = 2$. So we only consider (i). Observe also that a priori there is no help from a large σ_j. Let σ, δ be two positive constants to be determined later but such that $1 - \sigma > \delta$.

<u>Subcase 1:</u> Assume $N_2, N_4, N_5 < N^{1-\sigma}$, $N_3 \lesssim N^{\delta}$ and $N_1 \sim N \sim N_6$ in (4.35). Then we have the support condition on w_{N_1} and $\overline{w_{N_6}}$. Let us denote by $\sum_*$ the sum over the set of $N_j \leq N, 1 \leq j \leq 6$ such that $N_1, N_6 \sim N$, $N_j < N^{1-\sigma}$ for $j = 2, 4, 5$ and $N_3 \lesssim N^{\delta}$. By Cauchy-Schwarz, (2.5), Lemma 4.9 and Remark 4.12 we then have that (4.35) is less than or equal to

$$\begin{aligned}
&\sum_* N_3\, N_6 \;\; \max(N_2, N_3)^{\frac{1}{6}} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \times \\
&\qquad\qquad \times \max(N_4, N_5)^{\frac{1}{6}} N_6^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right) \\
\lesssim \;\; &\sum_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} \max(N_2, N_3)^{\frac{1}{6}} N_2^{-\frac{1}{2}+} N^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right)
\end{aligned}$$

since $N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4, N_5)^{\frac{1}{6}}$ is bounded. On the other hand the latter expression is worst possible when $\max(N_2, N_3) \sim N_3$; hence if $\delta < \frac{1}{2}$ we conclude by Hölder as before with a decay of $N^{-\frac{1}{3}} N^{\frac{2}{3}\delta}$.

<u>Subcase 2:</u> Assume $N_2, N_4, N_5 < N^{1-\sigma}$, $N_3 \gtrsim N^{\delta}$ and $N_1 \sim N \sim N_6$ in (4.35). We further subdivide as follows:

Subcase 2a) Assume $N_2, N_4, N_5 \ll N^{\delta}$, $N_3 \gtrsim N^{\delta}$ and $N_1 \sim N \sim N_6$ in (4.35). Then from (4.32) there exists $\sigma_j \gtrsim N^{1+\delta}$. Denote by $\sum_*$ the sum over the set of $N_j \leq N, 1 \leq j \leq 6$ such that $N_1, N_6 \sim N$, $N_j < N^{\delta}$ for $j = 2, 4, 5$ and $N_3 \geq N^{\delta}$.

• Suppose $j = 2, 4$ or 5; $j = 2$ or 4 are symmetric. So we treat first $j = 2$ and then $j = 5$. By Plancherel we have that (4.35) is less than or equal to

$$\begin{aligned}
&\sum_* \int_{\mathbb{R}} \int_{\mathbb{T}} N_3 N_6\, \sigma_2^{-\frac{1}{2}+}\, w_{N_1}\, \sigma_2^{\frac{1}{2}-} w_{N_2} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5} w_{N_6}}\, dxdt \\
\lesssim \;\; &\sum_* N_3^{\frac{1}{2}+} N_6^{\frac{1}{2}+} N^{-\frac{1}{2}-\frac{\delta}{2}} N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_4^{0+}\, N_5^{0+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right)
\end{aligned}$$

by Cauchy Schwarz placing $w_{N_1} \overline{w_{N_3} w_{N_6}}$ in L^2, $\sigma_2^{\frac{1}{2}} w_{N_2}$ in L^2 and $w_{N_4} \overline{w_{N_5}}$ in L^{∞}. From (2.5) and Lemma 4.9 we obtain the desired estimate with decay $N^{-\frac{\delta}{2}}$ so long as $\delta > 0$.

If $j = 5$ we proceed as above with same grouping in L^2 but exchanging the roles of w_{N_2} and $\overline{w_{N_5}}$ for the other L^2 and one of the L^{∞} bounds.

• Suppose $j = 3, 6$ or 1; $j = 3$ or 6 are symmetric. So we treat first $j = 3$ and then $j = 1$. Proceeding as above from (4.35) we now have

$$\begin{aligned}
&\sum_{*} \int_{\mathbb{R}} \int_{\mathbb{T}} N_3 N_6 \, \sigma_3^{-\frac{1}{2}+} \, w_{N_1} \, w_{N_2} \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5} w_{N_6}} \, dxdt \\
\lesssim \; & \sum_{*} N_3^{\frac{1}{2}+} N_6^{\frac{1}{2}+} N^{-\frac{1}{2}-\frac{\delta}{2}} N_1^{-\frac{1}{2}+} N_2^{0+} \, N_4^{-\frac{1}{2}+} N_5^{0+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right)
\end{aligned}$$

by Cauchy Schwarz placing $w_{N_1} w_{N_4} \overline{w_{N_6}}$ in L^2, $\sigma_3^{\frac{1}{2}-} \overline{w_{N_3}}$ in L^2 and $w_{N_2} \overline{w_{N_5}}$ in L^∞. We thus obtain the desired estimate as before with decay $N^{-\frac{\delta}{2}}$ so long as $\delta > 0$.

If $j = 1$ then we group $\overline{w_{N_3}} w_{N_4} \overline{w_{N_6}}$ in L^2, $\sigma_1^{\frac{1}{2}-} w_{N_1}$ in L^2 and the other two on L^∞ to reach the same estimate.

Subcase 2b) Suppose there exists $i \in \{2, 4, 5\}$ such that $N_i \gtrsim N^\delta$ and $N_j \ll N^\delta$ for $j \neq i$, and $i, j \in \{2, 4, 5\}$ while still $N_3 \gtrsim N^\delta$ and $N_1 \sim N \sim N_6$ in (4.35).

• Suppose $i = 2$ first. Then we further split the sum over this set into three sums, S_1, S_2 and S_3 according to whether $N^\delta \lesssim N_2 \ll N_3$; $N_2 \sim N_3$ or $N_2 \gg N_3$ respectively. When considering the sums over S_1 or over S_3 we have that from (4.32) there exists $\sigma_j \gtrsim N^{1+\delta}$ and hence the estimates for S_1 and S_3 follow exactly as those in *Subcase 2a)*.

We treat then S_2. Since $N_2 \sim N_3$ and $N_2 < N^{1-\sigma}$, we also have $N_3 < N^{1-\sigma}$; while $N_4, N_5 \lesssim N^\delta$. Thus we have the support condition in w_{N_1} and $\overline{w_{N_6}}$. Then from (4.35) by Cauchy-Schwarz, (2.5), Lemma 4.9 and Remark 4.12, grouping $w_{N_1} \, w_{N_2} \overline{w_{N_3}}$ in L^2 and $w_{N_4} \overline{w_{N_5} w_{N_6}}$ and (4.36) we have

$$\begin{aligned}
&\sum_{S_2} N_3 N_6 \max(N_2, N_3)^{\frac{1}{6}} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \; \times \\
&\qquad\qquad \times \max(N_4, N_5)^{\frac{1}{6}} N_6^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right) \\
\lesssim \; & \sum_{S_2} N_6^{\frac{1}{3}+} \max(N_2, N_3)^{\frac{1}{6}} N_1^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right)
\end{aligned}$$

since $N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4, N_5)^{\frac{1}{6}}$ is bounded and $N_2 \sim N_3$. Summing as usual, we get the desired estimate with decay $N^{-\frac{1}{6}+}$ regardless of $\sigma > 0$.

• Suppose $i = 4$. Again, we further split the sum over this set into three sums, S_1, S_2 and S_3 according now to whether $N^\delta \lesssim N_4 \ll N_3$; $N_4 \sim N_3$ or $N_4 \gg N_3$ respectively. For the sums over S_1 or over S_3, from (4.32) we have a $\sigma_j \gtrsim N^{1+\delta}$ and hence the estimates for S_1 and S_3 follow exactly as those in *Subcase 2a)*.

We treat then S_2. Since $N_4 \sim N_3$, $N_3 < N^{1-\sigma}$ while $N_2, N_5 \lesssim N^{\delta}$; so once again we have a support condition in w_{N_1} and $\overline{w_{N_6}}$. Proceeding as before we have

$$\begin{aligned}
&\sum_{S_2} N_3 N_6 \max(N_2, N_3)^{\frac{1}{6}} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \times \\
&\qquad\qquad \times \max(N_4, N_5)^{\frac{1}{6}} N_6^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right) \\
\lesssim \quad & \sum_{S_2} N_3^{\frac{1}{2}+} N_6 N_3^{\frac{1}{6}} N^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+\frac{1}{6}+} N_5^{-\frac{1}{2}+} N^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right) \\
\lesssim \quad & \sum_{S_2} N_3^{\frac{1}{3}+} N\, N^{-\frac{4}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right).
\end{aligned}$$

Since $N_4 \sim N_3$ and $N_3 < N^{1-\sigma}$ summing as before we have the desired estimate with decay $N^{-\frac{\sigma}{3}}$ so long as $\sigma > 0$.

- Suppose $i = 5$. We now split the sum over this set into three sums, S_1, S_2 and S_3 according to whether $N^{\delta} \lesssim N_5 \ll N_3$; $N_5 \sim N_3$ or $N_5 \gg N_3$ respectively. Again for the sums over S_1 or over S_3, from (4.32) we have a $\sigma_j \gtrsim N^{1+\delta}$ and hence the estimates for S_1 and S_3 follow exactly as those in *Subcase 2a)*.

We treat then S_2. Since $N_5 \sim N_3$, $N_3 < N^{1-\sigma}$ while $N_2, N_4 \lesssim N^{\delta}$; we have a support condition in w_{N_1} and $\overline{w_{N_6}}$. Proceeding as before we have

$$\begin{aligned}
&\sum_{S_2} N_3 N_6 \max(N_2, N_3)^{\frac{1}{6}} N_1^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \times \\
&\qquad\qquad \times \max(N_4, N_5)^{\frac{1}{6}} N_6^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right) \\
\lesssim \quad & \sum_{S_2} N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} N_3^{\frac{1}{6}} N^{-\frac{2}{3}+} N_2^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+\frac{1}{6}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right) \\
\lesssim \quad & \sum_{S_2} N_3^{\frac{1}{3}+} N^{-\frac{1}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}} \right)
\end{aligned}$$

which summing over S_2 gives the desired estimate with the same $N^{-\frac{\sigma}{3}}$ decay as in the previous case so long as $\sigma > 0$.

Subcase 2c) Suppose that there exist at least $i, j \in \{2, 4, 5\}$ $(i \neq j)$ such that $N_i, N_j \gtrsim N^{\delta}$ while $N_3 \gtrsim N^{\delta}$ and $N_1 \sim N \sim N_6$ in (4.35). Note that $N_4, N_5 < N^{1-\sigma}$ which ensures a support condition on $\overline{w_{N_6}}$.

- Suppose $(i,j)=(4,5)$. Proceeding as above and using similar arguments we have

$$\begin{aligned}
&\sum_{*} N_3\, N_6\, N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4,N_5)^{\frac{1}{6}} N_6^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)\\
=\ &\sum_{*} N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} N^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4,N_5)^{\frac{1}{6}} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)
\end{aligned}$$

from where using that $N_4, N_5 \gtrsim N^{\delta}$ and $N_3 \gtrsim N^{\delta}$ we get the desired bound with decay $N^{\frac{1}{3}-\frac{5}{6}\delta}$ so long as $\delta > \frac{2}{5}$.

- Suppose $(i,j)=(2,5)$. Once again proceeding as before and using similar arguments we have

$$\begin{aligned}
&\sum_{*} N_3\, N_6\, N_1^{-\frac{1}{2}+} N_2^{-\frac{1}{2}+} N_3^{-\frac{1}{2}+} N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4,N_5)^{\frac{1}{6}} N_6^{-\frac{2}{3}+} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)\\
\lesssim\ &\sum_{*} N_3^{\frac{1}{2}+} N_6^{\frac{1}{3}+} N^{-\frac{1}{2}+} N^{-\frac{\delta}{2}} N^{-\frac{\delta}{2}+\frac{\delta}{6}} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)
\end{aligned}$$

using that $N_2 \gtrsim N^{\delta}$ and that $N_4^{-\frac{1}{2}+} N_5^{-\frac{1}{2}+} \max(N_4,N_5)^{\frac{1}{6}}$ is worse possible when $N_4 \ll N_5$ but $N_5 \gtrsim N^{\delta}$. Hence we once again obtain the desired estimate with decay $N^{\frac{1}{3}-\frac{5}{6}\delta}$ so long as $\delta > \frac{2}{5}$.

- Suppose $(i,j)=(2,4)$. This is exactly as in the previous case by exchanging the roles of 4 and 5.

<u>Subcase 3:</u> Assume there exists at least one $i \in \{2,4,5\}$ such that $N_i \gtrsim N^{1-\sigma}$, $N_2, N_4, N_5 \ll N$ while $N_3 \ll N$ and $N_1 \sim N \sim N_6$ in (4.35). This case follows from Lemma 4.11 with $0<\sigma<\frac{1}{6}$ as in its statement.

All in all, for Case [IIIA] we need $\frac{2}{5} < \delta < \frac{1}{2}$ and $0 < \sigma < \frac{1}{6}$.

Remark 4.13. In the proof of the remaining cases, in order to keep the notation lighter, we will ignore the $+\epsilon$ appearing in the exponent of the N_i's in (4.36). For example we simply write $N_i^{-\frac{1}{2}}$ instead of $N_i^{-\frac{1}{2}+}$.

Case [IA]. Assume $N_3 \sim N \sim N_6$ while $N_1, N_2, N_4, N_5 \ll N$ in (4.35) and denote as before by $\sum_*$ the sum over this set. Observe that from (4.29)-(4.33) there exists $\sigma_j \gtrsim N^2$.

<u>Subcase 1:</u> Assume in addition $N_1, N_2 < N^{\delta}$ for some $\delta > 0$. We then have the support condition on $\overline{w_{N_3}}$.

- Suppose $j=3$ or 6; say $j=3$ ($j=6$ is similar). Then we rewrite (4.35) as follows:

$$\begin{aligned}
&\sum_{*} \int_{\mathbb{R}} \int_{\mathbb{T}} N^2\, \sigma_3^{-\frac{1}{2}+}\, w_{N_1}\, w_{N_2} \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5} w_{N_6}}\, dxdt\\
\lesssim\ &\sum_{*} N^2 N^{-1} N_1^{0+} N_2^{0+} \max(N_1,N_2)^{\frac{1}{6}} N_3^{-\frac{2}{3}} N_4^{-\frac{1}{2}} N_5^{-\frac{1}{2}} N_6^{-\frac{1}{2}} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)
\end{aligned}$$

by placing $\sigma_3^{\frac{1}{2}-}\overline{w_{N_3}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}w_{N_6}}$ in L^2_{xt}, $w_{N_1}w_{N_2}$ in L^∞_{xt} and using the support condition on $\overline{w_{N_3}}$. By Hölder's inequality, summing as above, we get the desired estimate with decay $N^{\frac{\delta}{6}-\frac{1}{6}}$ so long as $\delta<1$.

• Suppose $j=1,2,4$ or 5. By symmetry (relative to conjugates) $j=1,2,4$ are similar; so suppose $j=1$. We rewrite (4.35) as

$$\begin{aligned}&\sum_*\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_1^{-\frac{1}{2}+}\,w_{N_1}\sigma_1^{\frac{1}{2}-}\,w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dxdt\\ \lesssim\ &\sum_* N^2N^{-1}N_1^{-\frac{1}{2}}N_2^{0+}\max(N_1,N_2)^{\frac{1}{6}}N_3^{-\frac{2}{3}}N_4^{-\frac{1}{2}}N_5^{0+}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^6\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)\end{aligned}$$

by placing $\sigma_1^{\frac{1}{2}-}w_{N_1}$ in L^2_{xt}, $\overline{w_{N_3}}w_{N_4}\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt} and using the support condition on $\overline{w_{N_3}}$. Once again, by Hölder's inequality, summing as before we get the desired estimate with decay $N^{\frac{\delta}{6}-\frac{1}{6}}$ so long as $\delta<1$.

If $j=5$

$$\begin{aligned}&\sum_*\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_5^{-\frac{1}{2}+}\,w_{N_1}\,w_{N_2}\overline{w_{N_3}}w_{N_4}\sigma_5^{\frac{1}{2}-}\overline{w_{N_5}w_{N_6}}\,dxdt\\ \lesssim\ &\sum_* N^2N^{-1}N_1^{0+}N_2^{0+}\max(N_1,N_2)^{\frac{1}{6}}N_3^{-\frac{2}{3}}N_4^{-\frac{1}{2}}N_5^{-\frac{1}{2}}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^6\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)\end{aligned}$$

by placing $\sigma_5^{\frac{1}{2}-}\overline{w_{N_5}}$ in L^2_{xt}, $\overline{w_{N_3}}w_{N_4}\overline{w_{N_6}}$ in L^2_{xt}, $w_{N_1}w_{N_2}$ in L^∞_{xt} and using the support condition on $\overline{w_{N_3}}$. Once again, Hölder's inequality, summing as before we get the desired estimate with decay $N^{\frac{\delta}{6}-\frac{1}{6}}$ so long as $0<\delta<1$.

<u>Subcase 2:</u> Assume either N_1 or $N_2>N^\delta$. Suppose $N_1>N^\delta$; otherwise exchange the roles of w_{N_1} and w_{N_2} below. We no longer rely on the support condition but on the lower bound on N_1 as follows.

• Suppose $j=3$ or 6; say $j=3$ ($j=6$ is similar). Then proceeding as before we rewrite (4.35) as

$$\begin{aligned}&\sum_*\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_3^{-\frac{1}{2}+}\,w_{N_1}\,w_{N_2}\sigma_3^{\frac{1}{2}-}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dxdt\\ \lesssim\ &\sum_* N^2N^{-1}N_1^{-\frac{1}{2}}N_2^{0+}N_3^{-\frac{1}{2}}N_4^{0+}N_5^{-\frac{1}{2}}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^6\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)\end{aligned}$$

by placing $\sigma_3^{\frac{1}{2}-}\overline{w_{N_3}}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_5}w_{N_6}}$ in L^2_{xt}, $w_{N_2}w_{N_4}$ in L^∞_{xt} . By Hölder's inequality, summing as above, we get the desired estimate with decay $N^{-\frac{\delta}{2}}$ so long as $\delta>0$.

• Suppose $j=1$ or 2; say $j=1$ ($j=2$ is similar). We now write

$$\begin{aligned}&\sum_*\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_1^{-\frac{1}{2}+}\,w_{N_1}\sigma_1^{\frac{1}{2}-}\,w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dxdt\\ \lesssim\ &\sum_* N^2N^{-1}N_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}N_4^{0+}N_5^{0+}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^6\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)\end{aligned}$$

by placing $\sigma_1^{\frac{1}{2}-} w_{N_1}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}$ in L^∞_{xt} . Once again, by Hölder's inequality and summing as above, we get the desired estimate with decay $N^{-\frac{\delta}{2}}$ so long as $\delta > 0$.

• Suppose $j = 4$ then proceed as above but place $\sigma_4^{\frac{1}{2}-} w_{N_4}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt}, and $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt}.

• Suppose $j = 5$ then once again we proceed as above but now place $\sigma_5^{\frac{1}{2}-}\overline{w_{N_5}}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt}, and $w_{N_2}w_{N_4}$ in L^∞_{xt}.

Remark 4.14. Matching Subcases 1 and 2 above means $-\frac{\delta}{2} = \frac{\delta}{6} - \frac{1}{6}$ which requires $\delta = \frac{1}{4}$ and yields a decay of $N^{-\frac{1}{8}+}$.

Case [IIA]. Part (i) will follow similarly to Case [IA] while part (ii) to Case [IIIA].

Part (i) We are under the assumptions $N_3 \sim N \sim N_5$ while $N_1, N_2, N_4, N_6 \ll N$. It follows from (4.33), there exists $\sigma_j \gtrsim N^2$. We proceed exactly as in [IA] exchanging in each instance the roles of $\overline{w_{N_6}}$ and $\overline{w_{N_5}}$

Part (ii) We are under the assumptions $N_3 \sim N \sim N_4$ while $N_1, N_2, N_5, N_6 \ll N$. We have *a priori* no help from a large σ_j at our disposal. We proceed then as in [IIIA] above with the roles of $(N_3; \overline{w_{N_3}})$ switched with that of $(N_6; \overline{w_{N_6}})$ and $(N_1; w_{N_1})$ with $(N_4; w_{N_4})$. Hence for $\sigma, \delta > 0$ -to be determined- in Subcase 1 we are under the assumption $N_1, N_2, N_5 < N^{1-\sigma}$, $N_6 \lesssim N^\delta$ and $N_3 \sim N \sim N_4$. While in Subcase 2 we assume $N_1, N_2, N_5 < N^{1-\sigma}$ while $N_6 \gtrsim N^\delta$ and $N_3 \sim N \sim N_4$, and further subdivide just as before into *Subcase 2a)*: $N_1, N_2, N_5 \ll N^\delta$ while $N_6 \gtrsim N^\delta$ which implies from (4.31) the existence of a $\sigma_j \gtrsim N^{1+\delta}$; *Subcase 2b)*: there exists $i \in \{1, 2, 5\}$ such that $N_i \gg N^\delta$ and $N_j \lesssim N^\delta$ for $j \neq i$ and $i, j \in \{1, 2, 5\}$ while still $N_6 \gtrsim N^\delta$ and $N_3 \sim N \sim N_4$ in (4.35) and *Subcase 2c)*: that there exist at least $i, j \in \{1, 2, 5\}$ $(i \neq j)$ such that $N_i, N_j \gg N^\delta$ while $N_6 \gtrsim N^\delta$ and $N_3 \sim N \sim N_4$ in (4.35). Note that $N_1, N_2 < N^{1-\sigma}$ which ensures a support condition on $\overline{w_{N_3}}$. Subcase 3: Assume there exists at least one $i \in \{1, 2, 5\}$ such that $N_i \gtrsim N^{1-\sigma}$ $N_2, N_1, N_5 \ll N$ while $N_6 \ll N$ and $N_3 \sim N \sim N_4$ in (4.35). This case follows from Lemma 4.11 with $0 < \sigma < \frac{1}{6}$ as in its statement.

Proceeding then just as in [IIIA] we conclude the desired estimate with the same decay in N as in [IIIA] as long as $\frac{2}{5} < \delta < \frac{1}{2}$ and $0 < \sigma < \frac{1}{6}$ as before.

Case [IB]. We first note that parts (ii), (iii) and (iv) are all symmetric relative to conjugation; so we only consider (i) and (ii).

Part (i) We are under the assumptions $N_3 \sim N_5 \sim N_6 \sim N$ while $N_1, N_2, N_4 \ll N$. It follows from (4.33), there exists $\sigma_j \gtrsim N^2$.

• Suppose $j = 1, 2$ or 4. By symmetry is enough to consider $j = 1$ and $j = 4$. To obtain decay we need to use the support condition. This we further subdivide into two cases.

Subcase 1: Assume in addition $N_1, N_2 < N^\delta$ for some $\delta > 0$. We then have the support condition on $\overline{w_{N_3}}$. For $j = 1$ we have:

$$\sum_{*}\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_1^{-\frac{1}{2}+}\,w_{N_1}\sigma_1^{\frac{1}{2}-}\,w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dxdt$$
$$\lesssim \sum_{*} N^2N^{-1}N_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{-\frac{2}{3}}N^{\frac{\delta}{6}}N_4^{0+}N_5^{0+}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)$$

by placing $\sigma_1^{\frac{1}{2}-}w_{N_1}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}$ in L^∞_{xt}. By Hölder's inequality, summing as above, we get the desired estimate with decay $N^{-\frac{1}{6}+\frac{\delta}{6}}$ so long as $0<\delta<1$.

For $j=4$, we place $\sigma_4^{\frac{1}{2}}w_{N_4}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt} and $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt} and proceed similarly.

Subcase 2: Assume either N_1 or $N_2>N^\delta$. By symmetry suppose $N_1>N^\delta$; otherwise exchange the roles of w_{N_1} and w_{N_2} below. We use then the lower bound on N_1 as follows. For $j=1$:

$$\sum_{*}\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_1^{-\frac{1}{2}+}\,w_{N_1}\sigma_1^{\frac{1}{2}-}\,w_{N_2}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dxdt$$
$$\lesssim \sum_{*} N^2N^{-1}N_1^{-\frac{1}{2}}N_2^{-\frac{1}{2}}N_3^{-\frac{1}{2}}N_4^{0+}N_5^{0+}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)$$

by placing $\sigma_1^{\frac{1}{2}-}w_{N_1}$ in L^2_{xt}, $w_{N_2}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}}$ in L^∞_{xt}. Hence, by Hölder's inequality and summing as usual we get the desired estimate with decay $N^{-\frac{\delta}{2}}$ so long as $\delta>0$.

For $j=4$, we place $\sigma_4^{\frac{1}{2}-}w_{N_4}$ in L^2_{xt}, $w_{N_1}\overline{w_{N_3}w_{N_6}}$ in L^2_{xt} and $w_{N_2}\overline{w_{N_5}}$ in L^∞_{xt} and proceed similarly.

Remark 4.15. Note that once again, matching Subcases 1 and 2 above means $-\frac{\delta}{2}=\frac{\delta}{6}-\frac{1}{6}$ which requires $\delta=\frac{1}{4}$ and yields a decay of $N^{-\frac{1}{8}+}$.

• Suppose $j=3,6$ or 5. By symmetry relative to conjugation it's enough to consider -say- $j=3$. We have

$$\sum_{*}\int_{\mathbb{R}}\int_{\mathbb{T}} N^2\,\sigma_3^{-\frac{1}{2}+}\,w_{N_1}w_{N_2}\,\sigma_3^{\frac{1}{2}-}\overline{w_{N_3}}w_{N_4}\overline{w_{N_5}w_{N_6}}\,dxdt$$
$$\lesssim \sum_{*} N^2N^{-1}N_1^{0+}N_2^{0+}N_3^{-\frac{1}{2}}N_4^{-\frac{1}{2}}N_5^{-\frac{1}{2}}N_6^{-\frac{1}{2}}\left(\prod_{i=1}^{6}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\right)$$

by placing $\sigma_3^{\frac{1}{2}-}\overline{w_{N_3}}$ in L^2_{xt}, $w_{N_4}\overline{w_{N_5}w_{N_6}}$ in L^2_{xt}, $w_{N_1}w_{N_2}$ in L^∞_{xt}. Hence, by Hölder's inequality, summing as usual we get the desired estimate with decay $N^{-\frac{1}{2}+}$.

<u>Part (ii)</u> We are under the assumptions $N_3\sim N_4\sim N_6\sim N$ while $N_1,N_2,N_5\ll N$. It follows from (4.31), there exists $\sigma_j\gtrsim N^2$.

• Suppose $j = 1, 2$ or 5. Suppose $j = 1$ then

$$\begin{aligned}
&\sum_{*} \int_{\mathbb{R}} \int_{\mathbb{T}} N^2\, \sigma_1^{-\frac{1}{2}+}\, \sigma_1^{\frac{1}{2}-} w_{N_1} w_{N_2}\, \overline{w_{N_3}} w_{N_4} \overline{w_{N_5} w_{N_6}}\, dxdt \\
\lesssim\ & \sum_{*} N^2 N^{-1} N_1^{-\frac{1}{2}} N_2^{0+} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{0+} N_6^{-\frac{1}{2}} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right)
\end{aligned}$$

by placing $\sigma_1^{\frac{1}{2}-} w_{N_1}$ in L^2_{xt}, $w_{N_4} \overline{w_{N_3} w_{N_6}}$ in L^2_{xt}, $w_{N_2} \overline{w_{N_5}}$ in L^∞_{xt}. Hence, by Hölder's inequality and summing as usual we get the desired estimate with decay $N^{-\frac{1}{2}+}$.

If $j = 2, 5$ we proceed similarly; keeping $w_{N_4} \overline{w_{N_3} w_{N_6}}$ in L^2_{xt} and exchanging the roles of either w_{N_2} or $\overline{w_{N_5}}$ with that of w_{N_1} above.

• Suppose $j = 3, 6$ or 4. Suppose $j = 3$ then

$$\begin{aligned}
&\sum_{*} \int_{\mathbb{R}} \int_{\mathbb{T}} N^2\, \sigma_3^{-\frac{1}{2}+}\, w_{N_1} w_{N_2}\, \sigma_3^{\frac{1}{2}-} \overline{w_{N_3}} w_{N_4} \overline{w_{N_5} w_{N_6}}\, dxdt \\
\lesssim\ & \sum_{*} N^2 N^{-1} N_1^{0+} N_2^{-\frac{1}{2}} N_3^{-\frac{1}{2}} N_4^{-\frac{1}{2}} N_5^{0+} N_6^{-\frac{1}{2}} \left(\prod_{i=1}^{6} \|w_{N_i}\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}} \right)
\end{aligned}$$

by placing $\sigma_3^{\frac{1}{2}-} \overline{w_{N_3}}$ in L^2_{xt}, $w_{N_2} w_{N_4} \overline{w_{N_6}}$ in L^2_{xt}, $w_{N_1} \overline{w_{N_5}}$ in L^∞_{xt}. Hence, by Hölder's and summing as usual we get the desired estimate with decay $N^{-\frac{1}{2}+}$.

If $j = 6$ we proceed similarly exchanging the roles of $\overline{w_{N_3}}$ and $\overline{w_{N_6}}$ above.

If $j = 4$ we place $\sigma_4^{\frac{1}{2}-} w_{N_4}$ in L^2_{xt} and group $w_{N_2} \overline{w_{N_3} w_{N_6}}$ in L^2_{xt} to derive the same conclusion.

We now remove the assumption we made at the beginning of the proof. Suppose that there is at least a $\sigma_j > N^7$. It follows from (4.31) and (4.32) that there are two indices $1 \le i_1 \ne i_2 \le 6$ such that $\sigma_{i_1}, \sigma_{i_2} \gtrsim N^7$. Then, by (2.6) and (4.37), we have

$$\begin{aligned}
|I_1| \quad &\lesssim \sum_{N \le |n| \le 3N} \sum_{N_i \le N;\, i=1,\dots 6} \int_\tau \Bigg(\int_{\tau = \tau_1+\tau_2+\tau_3} \sum_{n = n_1+n_2+n_3} |\widehat{w_{N_1}}|\, |\widehat{w_{N_2}}|\, |n_3|\, |\widehat{\overline{w_{N_3}}}|\, d\tau_1 d\tau_2 \Bigg) \times \\
&\qquad \Bigg(\int_{-\tau = \tau_4+\tau_5+\tau_6} \sum_{-n = n_4+n_5+n_6} |\widehat{w_{N_4}}|\, |\widehat{\overline{w_{N_5}}}|\, |n_6|\, |\widehat{\overline{w_{N_6}}}|\, d\tau_4 d\tau_5 \Bigg) d\tau. \qquad (4.60) \\
&\lesssim \sum_{N \le |n| \le 3N} \sum_{N_i \le N;\, i=1,\dots 6} N^2 \|w_{N_1} w_{N_2} \overline{w_{N_3}}\|_{L^2_{xt}} \|w_{N_4} \overline{w_{N_5} w_{N_6}}\|_{L^2_{xt}} \\
&\lesssim \sum_{N \le |n| \le 3N} \sum_{N_i \le N;\, i=1,\dots 6} N^{-\frac{1}{3}+} \|w_{N_{i_1}}\|_{X^{\frac{1}{2}-, \frac{1}{2}-}} \|w_{N_{i_2}}\|_{X^{\frac{1}{2}-, \frac{1}{2}-}} \prod_{j \ne i_1, i_2} \|w_j\|_{X^{\frac{1}{2}-, \frac{1}{3}+}} \\
&\lesssim N^{-\frac{1}{3}+} \prod_{j=1}^{6} \|w_j\|_{X_3^{\frac{2}{3}-, \frac{1}{2}-}}.
\end{aligned}$$

To treat the remaining terms in (4.18) we first note that these are either higher order with no derivatives or same order as the first but with only one derivative term. We again start by assuming that $\sigma_j \lesssim N^9$ for all j. Under this assumption their estimate follow from the following lemma.

Lemma 4.16 (Remaining Terms)**.** *There exists $\beta > 0$ such that following estimates hold:*

$$\sum_{N\le|n|\le 3N}\ \sum_{N_i\le N;\, i=1,\dots 6}\int_\tau\Big(\int_{\tau=\tau_1+\tau_2+\tau_3}\ \sum_{n=n_1+n_2+n_3}|\widehat{w_{N_1}}||\widehat{w_{N_2}}|\,|\widehat{\overline{w_{N_3}}}|\Big)\quad\times \tag{4.61}$$
$$\Big(\int_{-\tau=\tau_4+\tau_5+\tau_6}\ \sum_{-n=n_4+n_5+n_6}|\widehat{w_{N_4}}|\,|\widehat{\overline{w_{N_5}}}|\,|m(n_6)|\,|\widehat{\overline{w_{N_6}}}|\Big)\,d\tau\lesssim N^{-\beta}\prod_{i=1}^{6}\|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}$$

$$\sum_{N\le|n|\le 3N}\ \sum_{N_i\le N;\, i=1,\dots 8}\int_\tau\Big(\int_{\tau=\sum_{i=1}^5\tau_i}\ \sum_{n=\sum_{i=1}^5 n_i}|\widehat{w_{N_1}}||\widehat{w_{N_2}}|\,|\widehat{\overline{w_{N_3}}}|\widehat{w_{N_4}}|\widehat{\overline{w_{N_5}}}|\Big)\quad\times \tag{4.62}$$
$$\Big(\int_{-\tau=\tau_6+\tau_7+\tau_8}\ \sum_{-n=n_6+n_7+n_8}|\widehat{w_{N_6}}|\,|\widehat{\overline{w_{N_7}}}|\,|m(n_8)|\,|\widehat{\overline{w_{N_8}}}|\Big)\,d\tau\lesssim N^{-\beta}\prod_{i=1}^{8}\|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}$$

where the multiplier m satisfies: $|m(\xi)|\le\langle\xi\rangle$.

Proof. Here we will only prove (4.62) since (4.61) is similar but simpler. Without loss of generality we can assume that $N_1\sim N\sim N_8$. Fix any $0<\sigma<1$ and consider the following cases.

Case 1: Assume that $N_i\lesssim N^\sigma$, $i\neq 1,8$. Then we have the support condition on w_{N_1} and $\overline{w_{N_8}}$. By Plancherel (4.62) is less than or equal to

$$\begin{aligned}
&\sum_{N_1,N_8\sim N;N_i\le N^\sigma,i\neq 1,8}\int_{\mathbb{R}}\int_{\mathbb{T}}\ N_8\ \ w_{N_1}\,w_{N_2}\,\overline{w_{N_3}}\,w_{N_4}\,\overline{w_{N_5}}\,w_{N_6}\overline{w_{N_7}w_{N_8}}\ \,dx\,dt \qquad (4.63)\\
\lesssim\ &\sum_{N_1,N_8\sim N;N_i\le N^\sigma,i\neq 1,8}N\|w_{N_1}\,w_{N_2}\,\overline{w_{N_3}}\|_{L^2_{x,t}}\|w_{N_4}\,\overline{w_{N_5}}\|_{L^\infty_{x,t}}\,\|w_{N_6}\overline{w_{N_7}w_{N_8}}\|_{L^2_{x,t}}\\
\lesssim\ &\sum_{N_1,N_8\sim N;N_i\le N^\sigma,i\neq 1,8}NN_1^{-\frac{2}{3}}\max(N_2,N_3,N_4,N_5)^{\frac{1}{6}}N_4^{0+}N_5^{0+}N_6^{-\frac{1}{2}}N_7^{-\frac{1}{2}}N_8^{-\frac{2}{3}}\\
\times\ &\max(N_6,N_7)^{\frac{1}{6}}\Big(\prod_{i=1}^{8}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)\lesssim N^{-\frac{1}{3}+\frac{\sigma}{6}+}\Big(\prod_{i=1}^{8}\|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big).
\end{aligned}$$

Case 2: Assume there exists $k\neq 1,8$ such that $N_k>N^\sigma$. Without loss of generality -say- $k=4$. Then we bound (4.63) as follows:

$$\begin{aligned}
&\sum_{N_1,N_8\sim N;N_4>N^\sigma,N_i\le N;i\neq 1,4,8}N\|w_{N_1}\,w_{N_4}\,\overline{w_{N_3}}\|_{L^2_{x,t}}\|w_{N_2}\,\overline{w_{N_5}}\|_{L^\infty_{x,t}}\,\|w_{N_6}\overline{w_{N_7}w_{N_8}}\|_{L^2_{x,t}}\\
\lesssim\ &\sum_{N_1,N_8\sim N;N_4>N^\sigma,N_i\le N;i\neq 1,4,8}NN_1^{-\frac{1}{2}}N_3^{-\frac{1}{2}}N_4^{-\frac{1}{2}}N_2^{0+}N_5^{0+}N_6^{-\frac{1}{2}}N_7^{-\frac{1}{2}}N_8^{-\frac{1}{2}}\Big(\prod_{i=1}^{8}\|w_{N_i}\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big)\\
\lesssim\ &N^{-\frac{\sigma}{2}+}\Big(\prod_{i=1}^{8}\|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\Big).
\end{aligned}$$

□

We now remove the assumption we made before the lemma above. Suppose that there is at least a $\sigma_j>N^9$. The term with six factors follows just in (4.60). To estimate the term with eight factors we first observe that as before there are at least two indices

$1 \le i_1 \ne i_2 \le 8$ such that $\sigma_i, \sigma_j \gtrsim N^9$. Next we use Hölder inequality to bound the left hand side of (4.62) by

(4.64)
$$\sum_{N\le|n|\le 3N} \sum_{N_i\le N;\, i=1,\dots 8} N \prod_{i=1}^{8} \|w_{N_i}\|_{L^8_{tx}} \lesssim \sum_{N\le|n|\le 3N} \sum_{N_i\le N;\, i=1,\dots 8} N \prod_{i=1}^{8} \|w_{N_i}\|_{X_3^{\frac{13}{24}+,\frac{3}{8}+}}$$

by (4.38). Using $\sigma_{i_1}, \sigma_{i_2} > N^9$ we conclude that

$$\begin{aligned}(4.64) \quad &\lesssim \sum_{N\le|n|\le 3N} \sum_{N_i\le N;\, i=1,\dots 8} N^{-\frac{1}{8}+} \|w_{Ni_1}\|_{X_3^{\frac{13}{24}+,\frac{1}{2}-}} \|w_{N_{i_2}}\|_{X_3^{\frac{13}{24}+,\frac{1}{2}-}} \prod_{i\ne i_1,i_2} \|w_{N_i}\|_{X_3^{\frac{13}{24}+,\frac{3}{8}+}} \\ &\lesssim N^{-\frac{1}{8}+} \prod_{i=1}^{8} \|w_i\|_{X_3^{\frac{2}{3}-,\frac{1}{2}-}}\end{aligned}$$

5. Construction of Weighted Wiener Measures

In this section we construct weighted Wiener measures and associated probability spaces on which we establish well-posedness. To construct these measures we make use of the conserved quantities $\mathcal{E}(v)$ (given in (2.16)) and the L^2-norm. As a motivation we recall a well known fact in finite dimensional spaces. Suppose we have a well-posed ODE $y_t = F(y)$, where F is a divergence-free vector field. Assume $G(y)$ is a constant of motion such that for reasonable f, $f(G(y)) \in L^1(dy)$. Then by Liouville's Theorem, $d\mu(y) = Z^{-1} f(G(y))dy$ is, for suitable normalization constant Z, an invariant probability measure for flow map for the ODE.

To construct the measures on infinite dimensional spaces we will consider conserved quantities of the form $\exp(-\frac{\beta}{2}\mathcal{E}(v))$. But there is a priori little hope of constructing a finite measure using this quantity since (a) the nonlinear part of $\mathcal{E}(v)$ is not bounded below and (b) the linear part is only non-negative but not positive definite. To resolve this we use the conservation of L^2-norm and consider instead the conserved quantity

$$\chi_{\{\|v\|_{L^2}\le B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx} \tag{5.1}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy, i.e.

$$\begin{aligned}\mathcal{N}(v) \;&=\; -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2\overline{v v}_x\,dx - \frac{1}{4\pi}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)\Big(\int_{\mathbb{T}}|v|^4\,dx\Big) + \\ &+\; \frac{1}{\pi}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)\Big(\mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x\,dx\Big) + \frac{1}{4\pi^2}\Big(\int_{\mathbb{T}}|v|^2\,dx\Big)^3.\end{aligned} \tag{5.2}$$

and B is a (suitably small) constant.

By analogy with the finite dimensional case we would like to construct the measure (with $v(x) = u(x) + iw(x)$)

$$\text{“ } d\mu_\beta = Z^{-1}\chi_{\{\|v\|_{L^2}\le B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)} e^{-\frac{\beta}{2}\int(|v|^2+|v_x|^2)dx} \prod_{x\in\mathbb{T}} du(x)dw(x) \text{ ”}. \tag{5.3}$$

This is a purely formal, although suggestive, expression since it is impossible to define the Lebesgue measure on an infinite-dimensional space as a countably additive measure. Moreover, it will turn out that $\int |u_x|^2 = \infty$, μ almost surely.

One uses instead a Gaussian measure as reference measure and the measure μ is constructed in two steps. First one constructs a Gaussian measure ρ as the limit of the

finite-dimensional measures on $\mathbb{R}^{4N+2}$ given by

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{\beta}{2} \sum_{|n|\le N} (1+|n|^2)|\widehat{v}_n|^2 \Big) \prod_{|n|\le N} da_n db_n \tag{5.4}$$

where $\widehat{v}_n = a_n + ib_n$. The construction of such Gaussian measures is a classical subject, see e.g. Gross [20] and Kuo [29]. For our purpose we will need to realize this measure as a measure supported on a suitable Banach space. Once this measure ρ has been constructed one constructs the measure μ as a measure which is absolutely continuous with respect to ρ and whose Radon-Nikodym derivative is

$$\frac{d\mu}{d\rho} = \tilde{Z}^{-1} \chi_{\{\|v\|_L^2 \le B\}} e^{-\frac{\beta}{2}\mathcal{N}(v)}.$$

For this measure to be normalizable it turns out that one needs B to be sufficiently small. Also the constant β in the measure does not play any role in the analysis (although the cutoff B depends on β) and thus in the sequel we will set $\beta = 1$. But note that the measures for different β are all invariant and they are all mutually singular [20, 29].

First let us recall some facts on Gaussian measures in Hilbert spaces and Banach spaces. For details see Zhidkov [48], Gross [20] and Kuo [29]. Let $n \in \mathbb{N}$ and $\mathcal{T}$ be a symmetric positive $n \times n$ matrix with real entries. The Borel measure ρ in $\mathbb{R}^n$ given by

$$d\rho(x) = \frac{1}{\sqrt{(2\pi)^n \det(\mathcal{T})}} \exp\big(-\tfrac{1}{2}\langle \mathcal{T}^{-1}x, x\rangle_{\mathbb{R}^n} \big)\, dx$$

is called a (nondegenerate centered) Gaussian measure in $\mathbb{R}^n$. Note that $\rho(\mathbb{R}^n) = 1$.

Now, we consider the analogous definition of the infinite dimensional (centered) Gaussian measures. Let H be a real separable Hilbert space and $\mathcal{T} : H \to H$ be a linear positive self-adjoint operator (generally unbounded) with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ and the corresponding eigenvectors $\{e_n\}_{n\in\mathbb{N}}$ forming an orthonormal basis of H. We call a set $M \subset H$ cylindrical if there exists an integer $n \ge 1$ and a Borel set $F \subset \mathbb{R}^n$ such that

$$M = \big\{ x \in H : (\langle x, e_1\rangle_H, \cdots, \langle x, e_n\rangle_H) \in F \big\}. \tag{5.5}$$

Given the operator $\mathcal{T}$, we denote by $\mathcal{A}$ the set of all cylindrical subsets of H and one can easily verify that $\mathcal{A}$ is a field. The centered Gaussian measure in H with correlation operator $\mathcal{T}$ is defined as the additive (but not countably additive in general) measure ρ defined on the field $\mathcal{A}$ via

$$\rho(M) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} \lambda_j^{-\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j^{-1} x_j^2} dx_1 \cdots dx_n, \text{ for } M \in \mathcal{A} \text{ as in (5.5).} \tag{5.6}$$

The following proposition tells us when this Gaussian measure ρ is countably additive.

Proposition 5.1. *The Gaussian measure ρ defined in* (5.6) *is countably additive on the field $\mathcal{A}$ if and only if $\mathcal{T}$ is an operator of trace class, i.e., $\sum_{n=1}^\infty \lambda_n < \infty$. If the latter holds, then the minimal σ-field $\mathcal{M}$ containing the field $\mathcal{A}$ of all cylindrical sets is the Borel σ-field on H.*

Consider a sequence of the finite dimensional Gaussian measures $\{\rho_n\}_{n\in\mathbb{N}}$ as follows. For fixed $n \in \mathbb{N}$, let $\mathcal{M}_n$ be the set of all cylindrical sets in H of the form (5.5) with this fixed n and arbitrary Borel sets $F \subset \mathbb{R}^n$. Clearly, $\mathcal{M}_n$ is a σ-field, and setting

$$\rho_n(M) = (2\pi)^{-\frac{n}{2}} \prod_{j=1}^{n} \lambda_j^{-\frac{1}{2}} \int_F e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j^{-1} x_j^2} dx_1 \cdots dx_n$$

for $M \in \mathcal{M}_n$, we obtain a countably additive measure ρ_n defined on $\mathcal{M}_n$. Then, one can extend the measure ρ_n onto the whole Borel σ-field $\mathcal{M}$ of H by setting $\rho_n(A) := \rho_n(A \cap \text{span}\{e_1, \cdots, e_n\})$ for $A \in \mathcal{M}$.[6] Then, we have

Proposition 5.2. *Let ρ in (5.6) be countably additive. Then, $\{\rho_n\}_{n\in\mathbb{N}}$ constructed above converges weakly to ρ as $n \to \infty$.*

For our problem then we consider the Gaussian measure ρ which is the weak limit of the finite dimensional Gaussian measures

$$d\rho_N = Z_{0,N}^{-1} \exp\Big(-\frac{1}{2} \sum_{|n|\le N} (1+|n|^2)|\widehat{v}_n|^2 \Big) \prod_{|n|\le N} da_n db_n \,. \tag{5.7}$$

Let $J_s := (1-\Delta)^{s-1}$, then we have

$$\sum_n (1+|n|^2)\,|\widehat{v}_n|^2 \;=\; \langle v, v\rangle_{H^1} \;=\; \langle J_s^{-1} v, v\rangle_{H_s} \,.$$

The operator $J_s : H_s \to H_s$ has the set of eigenvalues $\{(1+|n|^2)^{(s-1)}\}_{n\in\mathbb{Z}}$ and the corresponding eigenvectors $\{(1+|n|^2)^{-s/2} e^{inx}\}_{n\in\mathbb{Z}}$ form an orthonormal basis of H^s. Since J_s is of trace class if and only if $s < \frac{1}{2}$, by Proposition 5.1, ρ is a countably additive measure on H^s for any $s < 1/2$ (but not for $s \ge 1/2$.)

Unfortunately, (2.8) is locally well-posed in $H^s(\mathbb{T})$ only for $s \ge \frac{1}{2}$ [26]. Instead, we propose to work in the Fourier-Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T})$ defined in (2.2) in view of the local well-posedness result by Grünrock-Herr [22]. Since $\mathcal{F}L^{s,r}$ is not a Hilbert space, we need to construct ρ as a measure supported on a Banach space.

5.1. **General Banach space setting.** Let us recall the basic theory of abstract Wiener spaces [29]. Given a real separable Hilbert space H with norm $\|\cdot\|$, let $\mathcal{F}$ denote the set of finite dimensional orthogonal projections $\mathbb{P}$ of H. Then, define a cylinder set E by $E = \{x \in H : \mathbb{P}x \in F\}$ where $\mathbb{P} \in \mathcal{F}$ and F is a Borel subset of $\mathbb{P}H$, and let $\mathcal{R}$ denote the collection of such cylinder sets. Note that $\mathcal{R}$ is a field but not a σ-field. The Gaussian measure ρ on H is defined by

$$\rho(E) = (2\pi)^{-\frac{n}{2}} \int_F e^{-\frac{\|x\|^2}{2}} dx$$

for $E \in \mathcal{R}$, where $n = \dim \mathbb{P}H$ and dx is the Lebesgue measure on $\mathbb{P}H$. It is known that ρ is finitely additive but not countably additive in $\mathcal{R}$.

Definition 5.3 (Gross [20])**.** *A seminorm $|||\cdot|||$ in H is called measurable if for every $\varepsilon > 0$, there exists $\mathbb{P}_\varepsilon \in \mathcal{F}$ such that*

$$\rho(|||\mathbb{P}x||| > \varepsilon) < \varepsilon$$

for $\mathbb{P} \in \mathcal{F}$ orthogonal to $\mathbb{P}_\varepsilon$.

Any measurable seminorm is weaker than the norm of H, and H is not complete with respect to $|||\cdot|||$ unless H is finite dimensional. Let $\mathcal{B}$ be the completion of H with respect to $|||\cdot|||$ and denote by i the inclusion map of H into $\mathcal{B}$. The triple $(i, H, \mathcal{B})$ is called an abstract Wiener space.

[6]Note a slight abuse of notation. We use ρ_n to denote a Gaussian measure on $\text{span}\{e_1, \cdots, e_n\}$ as well as its extension on H. A similar comment applies in the following.

Now, regarding $y \in \mathcal{B}^*$ as an element of $H^* \equiv H$ by restriction, we embed $\mathcal{B}^*$ in H. Define the extension of ρ onto $\mathcal{B}$ (which we still denote by ρ) as follows. For a Borel set $F \subset \mathbb{R}^n$, set

$$\rho(\{x \in \mathcal{B} : ((x, y_1), \cdots, (x, y_n)) \in F\}) := \rho(\{x \in H : (\langle x, y_1\rangle_H, \cdots, \langle x, y_n\rangle_H) \in F\}),$$

where y_j's are in $\mathcal{B}^*$ and $(\cdot, \cdot)$ denote the natural pairing between $\mathcal{B}$ and $\mathcal{B}^*$. Let $\mathcal{R}_{\mathcal{B}}$ denote the collection of cylinder sets $\{x \in \mathcal{B} : ((x, y_1), \cdots, (x, y_n)) \in F\}$ in $\mathcal{B}$.

Proposition 5.4 (Gross [20])**.** *ρ is countably additive in the σ-field generated by $\mathcal{R}_{\mathcal{B}}$.*

5.2. **Back to our setting.** In the present context, we will let $H = H^1(\mathbb{T})$ and $\mathcal{B} = \mathcal{F}L^{s,r}(\mathbb{T})$ with $2 \le r < \infty$ and $(s-1)r < -1$. First we prove the following result.

Proposition 5.5. *Let $2 \le r < \infty$ and assume $(s-1)r < -1$. Then the seminorm $\|\cdot\|_{\mathcal{F}L^{s,r}}$ is measurable. Moreover, we have the following exponential tail estimate: there exists $C > 0$ and $c > 0$ (which both depends on (s, r)) such that, for $K > 0$,*

$$\rho(\|v\|_{\mathcal{F}L^{s,r}} > K) \le Ce^{-cK^2}. \tag{5.8}$$

This shows that $(i, H, \mathcal{B}) = (i, H^1, \mathcal{F}L^{s,r})$ $(2 \le r < \infty)$ is an abstract Wiener space if $(s-1)r < -1$ and thus the Wiener measure ρ can be realized as a countably additive measure supported on $\mathcal{F}L^{s,r}$ for $(s-1)r < -1$. This is hardly surprising since this is equivalent to $\sigma \equiv s + \frac{1}{r} - \frac{1}{2} < \frac{1}{2}$ and $\mathcal{F}L^{s,r}$ scale as H^σ.

The second part of Proposition 5.5 is a consequence of Fernique's theorem [19] (c.f. Theorem 3.1 of Chapter III in [29]).

Remark 5.6. Proposition 5.5 was essentially proved in [35] in the context of white noise for the KdV equation. We include here a proof in our DNLS context for completeness[7].

It is useful to note that the measure ρ_N given in (5.7) can be regarded as the induced probability measure on $\mathbb{C}^{2N+1} \cong \mathbb{R}^{4N+2}$ under the map

$$\omega \longmapsto \left\{ \frac{g_n}{\sqrt{1 + |n|^2}} \right\}_{|n| \le N}, \tag{5.9}$$

where $g_n(\omega)$, $|n| \le N$, are independent standard complex Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ (i.e. $\widehat{v}_n = \frac{g_n}{\sqrt{1+|n|^2}}$). In a similar manner, we can view ρ as the induced probability measure under the map $\omega \mapsto \{\frac{g_n}{\sqrt{1+|n|^2}}\}_{n \in \mathbb{Z}}$, where $g_n(\omega)$ are independent standard complex Gaussian random variables.

For the proof of Proposition, 5.5 we first recall the following result.

Lemma 5.7 (Lemma 4.7 [36])**.** *Let $\{g_n\}$ be a sequence of independent standard complex-valued Gaussian random variables. Then, for M dyadic and $\delta < \frac{1}{2}$, we have*

$$\lim_{M \to \infty} M^{2\delta} \frac{\max_{|n| \sim M} |g_n|^2}{\sum_{|n| \sim M} |g_n|^2} = 0 \quad a.s.$$

Proof of Proposition 5.5. Let $2 \le r < \infty$ and $(s-1)r < -1$. In view of Definition 5.3, it suffices to show that for given $\varepsilon > 0$, there exists a large M_0 such that

$$\rho\big(\|P_{M_0}^{\perp} v\|_{\mathcal{F}L^{s,r}} > \varepsilon\big) < \varepsilon, \tag{5.10}$$

[7] Proposition 5.5 also holds for $r < 2$ and $(s-1)r < -1$, albeit with a different proof (see [1] for details). For our purposes $2 \le r < \infty$ suffices and so we restrict ourselves to that case.

where $P_{M_0}^{\perp}$ is the projection onto the frequencies $|n| > M_0$. Note that if $\mathbb{P}$ is a finite dimensional projection such that $\mathbb{P} \perp P_{M_0}$ then $\|\mathbb{P}v\|_{\mathcal{F}L^{s,r}} \leq \|P_{M_0}^{\perp} v\|_{\mathcal{F}L^{s,r}}$.

In view of (5.9), we assume that v is of the form

$$v(x) = \sum_n \frac{g_n}{\sqrt{1+|n|^2}} e^{inx}, \tag{5.11}$$

where $\{g_n\}$ is as in (5.9).

Let $\delta < \frac{1}{2}$ to be chosen later. Then, by Lemma 5.7 and Egoroff's theorem, there exists a set E such that $\rho(E^c) < \frac{1}{2}\varepsilon$ and the convergence in Lemma 5.7 is uniform on E, i.e. we can choose dyadic M_0 large enough such that

$$\frac{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L^\infty_n}}{\|\{g_n(\omega)\}_{|n|\sim M}\|_{L^2_n}} \leq M^{-\delta}, \tag{5.12}$$

for all $\omega \in E$ and dyadic $M > M_0$. In the following, we will work only on E and drop '$\cap E$' for notational simplicity. However, it should be understood that all the events are under the intersection with E so that (5.12) holds.

Let $\{\sigma_j\}_{j\geq 1}$ be a sequence of positive numbers such that $\sum \sigma_j = 1$, and let $M_j = M_0 2^j$ dyadic. Note that $\sigma_j = C2^{-\lambda j} = CM_0^{\lambda} M_j^{-\lambda}$ for some small $\lambda > 0$ (to be determined later.) Then, from (5.11), we have

$$\rho\big(\|P_{M_0}^{\perp} v(\omega)\|_{\mathcal{F}L^{s,r}} > \varepsilon\big) \leq \sum_{j=1}^{\infty} \rho\big(\|\{\langle n\rangle^{s-1} g_n(\omega)\}_{|n|\sim M_j}\|_{L^r_n} > \sigma_j \varepsilon\big). \tag{5.13}$$

By interpolation and (5.12), we have

$$\begin{aligned}\|\{\langle n\rangle^{s-1} g_n\}_{|n|\sim M_j}\|_{L^r_n} &\sim M_j^{s-1} \|\{g_n\}_{|n|\sim M_j}\|_{L^r_n} \leq M_j^{s-1} \|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}^{\frac{2}{r}} \|\{g_n\}_{|n|\sim M_j}\|_{L^\infty_n}^{\frac{r-2}{r}} \\ &\leq M_j^{s-1} \|\{g_n\}_{|n|\sim M}\|_{L^2_n} \left(\frac{\|\{g_n\}_{|n|\sim M_j}\|_{L^\infty_n}}{\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}} \right)^{\frac{r-2}{r}} \leq M_j^{s-1-\delta\frac{r-2}{r}} \|\{g_n\}_{|n|\sim M_j}\|_{L^2_n}.\end{aligned}$$

Thus, if we have $\|\{\langle n\rangle^{s-1} g_n\}_{|n|\sim M_j}\|_{L^r_n} > \sigma_j \varepsilon$, then we have $\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j$ where $R_j := \sigma_j \varepsilon M_j^{\alpha}$ with $\alpha := -s + 1 + \delta\frac{r-2}{r}$. With $r = 2+\theta$, we have $\alpha = \frac{-(s-1)r+\delta\theta}{2+\theta} > \frac{1}{2}$ by taking δ sufficiently close to $\frac{1}{2}$ since $-(s-1)r > 1$. Then, by taking $\lambda > 0$ sufficiently small, $R_j = \sigma_j \varepsilon M_j^{\alpha} = C\varepsilon M_0^{\lambda} M_j^{\alpha-\lambda} \gtrsim C\varepsilon M_0^{\lambda} M_j^{\frac{1}{2}+}$. By a direct computation in polar coordinates, we have

$$\rho\big(\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j\big) \sim \int_{B^c(0,R_j)} e^{-\frac{1}{2}|g_n|^2} \prod_{|n|\sim M_j} dg_n \lesssim \int_{R_j}^{\infty} e^{-\frac{1}{2}s^2} s^{2\#\{|n|\sim M_j\}-1} ds.$$

Note that, in the inequality, we have dropped the implicit constant $\sigma(S^{2\#\{|n|\sim M_j\}-1})$, a surface measure of the $2\#\{|n|\sim M_j\}-1$ dimensional unit sphere, since $\sigma(S^n) = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2}) \lesssim 1$. By the change of variable $t = M_j^{-\frac{1}{2}} s$, we have $s^{2\#\{|n|\sim M_j\}-2} \lesssim s^{4M_j} \sim M_j^{2M_j} t^{4M_j}$. Since $t > M_j^{-\frac{1}{2}} R_j = C\varepsilon M_0^{\lambda} M_j^{0+}$, we have $M_j^{2M_j} = e^{2M_j \ln M_j} < e^{\frac{1}{8}M_j t^2}$ and $t^{4M_j} < e^{\frac{1}{8}M_j t^2}$ for M_0 sufficiently large. Thus, we have $s^{2\#\{|n|\sim M_j\}-2} < e^{\frac{1}{4}M_j t^2} = e^{\frac{1}{4}s^2}$ for $s > R_j$. Hence, we have

$$\rho\big(\|\{g_n\}_{|n|\sim M_j}\|_{L^2_n} \gtrsim R_j\big) \leq C \int_{R_j}^{\infty} e^{-\frac{1}{4}s^2} s ds \leq e^{-cR_j^2} = e^{-cC^2 M_0^{2\lambda} M_j^{1+\varepsilon^2}}. \tag{5.14}$$

From (5.13) and (5.14), we have

$$\rho\big(\|P_{M_0}^{\perp} v\|_{\mathcal{F}L^{s,r}} > \varepsilon\big) \leq \sum_{j=1}^{\infty} e^{-cC^2 M_0^{1+2\lambda+}(2^j)^{1+}\varepsilon^2} \leq \tfrac{1}{2}\varepsilon$$

by choosing M_0 sufficiently large as long as $(s-1)r < -1$. Hence, the seminorm $\|\cdot\|_{\mathcal{F}L^{s,r}}$ is measurable for $(s-1)r < -1$.

The tail estimate (5.8) is a direct consequence of Fernique's theorem [29, Theorem 3.1]. □

To construct the weighted Wiener measure μ let us define

$$R(v) := \chi_{\{\|v\|_{L^2} \leq B\}} e^{-\frac{1}{2}\mathcal{N}(v)}, \qquad R_N(v) := R(v^N) \tag{5.15}$$

where $\mathcal{N}(v)$ is the nonlinear part of the energy defined in (5.2) and at this stage and for the remainder of this section $v^N = P_N(v)$ for some generic function v. In the next section v^N will denote the solution to the (FGDNLS) (3.1) as in Section 3. We write

$$\mathcal{N}_N(v) := \mathcal{N}(v^N) = F_N(v) + G_N(v) + K_N(v), \tag{5.16}$$

where

$$F_N(v) = -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} (v^N)^2 \overline{v^N v_x^N}\, dx, \tag{5.17}$$

$$G_N(v) = -\frac{1}{4\pi}\bigg(\int_{\mathbb{T}} |v^N|^2\, dx\bigg)\bigg(\int_{\mathbb{T}} |v^N|^4\, dx\bigg), \tag{5.18}$$

$$K_N(v) = \frac{1}{\pi}\bigg(\int_{\mathbb{T}} |v^N|^2\, dx\bigg)\bigg(\mathrm{Im}\int_{\mathbb{T}} v^N(\overline{v}_x^N) dx\bigg) + \frac{1}{4\pi^2}\bigg(\int_{\mathbb{T}} |v^N|^2\, dx\bigg)^3. \tag{5.19}$$

We will construct the measure

$$d\mu = Z^{-1} R(v) d\rho\,, \tag{5.20}$$

for sufficiently small B, as the weak limit of the finite dimensional weighted Wiener measures μ_N on $\mathbb{R}^{4N+2}$ given by

$$\begin{aligned} d\mu_N &= Z_N^{-1} R_N(v) d\rho_N \\ &= Z_N^{-1} \chi_{\{\|v^N\|_{L^2} \leq B\}} e^{-\frac{1}{2}\mathcal{N}(v^N)} d\rho_N \end{aligned} \tag{5.21}$$

for suitable normalization Z_N.

Lemma 5.8. (a) *The sequence F_N converges in $L^2(d\rho)$ to*

$$F(v) = -\frac{1}{2}\mathrm{Im}\int_{\mathbb{T}} v^2 \overline{v v_x}\, dx.$$

Moreover, for $\alpha < \frac{3}{4}$, there exist $C, \delta > 0$ such that for all $M \geq N \geq 1$ and $\lambda > 0$, we have

$$\rho(|F_M(v) - F_N(v)| > \lambda) \leq C e^{-\delta N^{\alpha} \lambda^{\frac{1}{2}}} \tag{5.22}$$

(b) *Let $p \in [2, \infty)$. Then, there exist α, C such that for all $M \geq N \geq 1$ and $\lambda > 0$, we have*

$$\rho(\|P_N v\|_{L^p(\mathbb{T})} > \lambda) < C e^{-c\lambda^2} \tag{5.23}$$

$$\rho(\|P_M v - P_N v\|_{L^p(\mathbb{T})} > \lambda) < C e^{-cN^{2\alpha}\lambda^2} \tag{5.24}$$

Proof. Part (a) was proved by Thomann and Tzvetkov in [42] Proposition 3.1. using Proposition 5.10 below. Note that their proof only uses the fact that v is in the support of the measure and is independent of the function space v is in.

To prove part (b) we first note that for any $2 \le p < \infty$, and $N \le M$,

$$\|P_N v\|_{L^p(\mathbb{T})} \quad \le C\|P_N v\|_{\mathcal{F}L^{\frac{2}{3}-,3}(\mathbb{T})} \tag{5.25}$$

$$\|P_N v - P_M v\|_{L^p(\mathbb{T})} \quad \le C\tfrac{1}{N^\alpha}\|P_M v\|_{\mathcal{F}L^{\frac{2}{3}-,3}(\mathbb{T})}, \tag{5.26}$$

where $\alpha = \dfrac{1}{p}-$. Then use (5.25) and (5.26) in conjunction with (5.8) to conclude the proof. □

Lemma 5.9. *$K_N(v)$ is Cauchy in measure; i.e. for every $\gamma > 0$ and $N \le M$*

$$\lim_{N,M\to\infty} \rho(|K_M(v) - K_N(v)| > 2\gamma) = 0,$$

and hence K_N converges in measure to

$$K(v) = \frac{1}{\pi}\left(\int_{\mathbb{T}} |v|^2\,dx\right)\left(\mathrm{Im}\int_{\mathbb{T}} v\overline{v}_x\right) + \frac{1}{4\pi^2}\left(\int_{\mathbb{T}} |v|^2\,dx\right)^3.$$

Before the proof we need the following Proposition 5.10 (see Thomann and Tzevtkov [42] for a proof) and Lemma 5.11 which we prove below.

Proposition 5.10. *Let $d \ge 1$ and $c(n_1,\dots,n_k) \in \mathbb{C}$. Let $\{g_n\}_{1\le n\le d} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex L^2 normalized independent Gaussians. For $k \ge 1$ denote by $A(k,d) := \{(n_1,\dots,n_k) \in \{1,\dots,d\}^k,\ n_1 \le \dots \le n_k\}$ and*

$$S_k(\omega) = \sum_{A(k,d)} c(n_1,\dots,n_k) g_{n_1}(\omega)\dots g_{n_k}(\omega). \tag{5.27}$$

Then for all $d \ge 1$ and $p \ge 2$

$$\|S_k\|_{L^p(\Omega)} \le \sqrt{k+1}(p-1)^{\frac{k}{2}}\|S_k\|_{L^2(\Omega)}.$$

Let $X_N(v) = \int_{\mathbb{T}} v^N \overline{v}_x^N$.

Lemma 5.11. *For any $N \le M$ and $\varepsilon > 0$ we have*

$$|X_N(v)| \quad \lesssim \quad N^{2\varepsilon}\|v^N\|^2_{\mathcal{F}L^{\frac{2}{3}-\varepsilon,3}} \tag{5.28}$$

$$\|X_M(v) - X_N(v)\|_{L^4} \quad \lesssim \quad \frac{1}{N^{\frac{1}{2}}}. \tag{5.29}$$

$$\textit{Moreover,} \qquad \|X_M(v) - X_N(v)\|_{L^q} \quad \lesssim \quad c(q-1)\frac{1}{N^{\frac{1}{2}}} \qquad \textit{for any } q \ge 2. \tag{5.30}$$

Proof. To prove (5.28) we use Plancherel and Hölder inequality to obtain

$$\begin{aligned} |X_N(v)| \quad &\le \quad \sum_{|n|\le N} |n|\,|\widehat{v^N}(n)|^2 \\ &\le \quad \left(\sum_{|n|\le N} |n|^{-1+6\varepsilon}\right)^{\frac{1}{3}} \left(\sum_{|n|\le N} (|n|^{\frac{2}{3}-\varepsilon}|\widehat{v^N}(n)|)^3\right)^{\frac{2}{3}} \le N^{2\varepsilon}\|v^N\|^2_{\mathcal{F}L^{\frac{2}{3}-\varepsilon,3}}. \end{aligned}$$

To prove (5.29) we start by recalling that $v^N(\omega,x) := \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle} e^{inx}$. Then by Plancherel

$$X_N(v) = -i \sum_{|n|\le N} n \frac{|g_n(\omega)|^2}{\langle n\rangle^2} \quad \text{and} \quad X_M(v) - X_N(v) = -i \sum_{N\le |n|<M} n \frac{|g_n(\omega)|^2}{\langle n\rangle^2},$$

and

$$|X_M(v) - X_N(v)|^2 = \sum_{N\le |n_1|,|n_2|<M} n_1 n_2 \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1\rangle^2 \langle n_2\rangle^2} =: Y^1_{N,M} + Y^2_{N,M} + Y^3_{N,M}, \tag{5.31}$$

where

$$Y^1_{N,M} := \sum_{N\le |n_2|,|n_1|<M} n_1 n_2 \frac{(|g_{n_1}(\omega)|^2 - 1)(|g_{n_2}(\omega)|^2 - 1)}{\langle n_1\rangle^2 \langle n_2\rangle^2} \tag{5.32}$$

$$Y^2_{N,M} := \sum_{N\le |n_2|,|n_1|<M} n_1 n_2 \frac{(|g_{n_1}(\omega)|^2 - 1) + (|g_{n_2}(\omega)|^2 - 1)}{\langle n_1\rangle^2 \langle n_2\rangle^2} \tag{5.33}$$

$$Y^3_{N,M} := \sum_{N\le |n_2|,|n_1|<M} \frac{n_1 n_2}{\langle n_1\rangle^2 \langle n_2\rangle^2}.$$

By symmetry $Y^3_{N,M} = 0$. We now observe that

$$\|X_M(v) - X_N(v)\|^4_{L^4} \lesssim \|Y^1_{N,M}\|^2_{L^2} + \|Y^2_{N,M}\|^2_{L^2}. \tag{5.34}$$

We now proceed as in [42], denote by $G_n(\omega) := |g_n(\omega)|^2 - 1$ and note that by the independence of $g_n(\omega)$ (c.f. (5.9)),

$$\mathbb{E}[G_n(\omega) G_m(\omega)] = 0 \quad \text{for } n \ne m. \tag{5.35}$$

Since

$$|Y^1_{N,M}|^2 = \sum_{N\le |n_1|,|n_2|,|n_3|,|n_4|<M} n_1 n_2 n_3 n_4 \frac{G_{n_1} G_{n_2} G_{n_3} G_{n_4}}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle n_3\rangle^2 \langle n_4\rangle^2}.$$

We compute $\mathbb{E}[|Y^1_{N,M}|^2]$ and by (5.35) the only contributions come from ($n_1 = n_3$ and $n_2 = n_4$), ($n_1 = n_2$ and $n_3 = n_4$) or ($n_2 = n_3$ and $n_1 = n_4$) . Hence by symmetry and using that the fourth moment of the Gaussian $g_n(\omega)$ are bounded we have

$$\|Y^1_{N,M}\|^2_{L^2} = E[|Y^1_{N,M}|^2] \le C \sum_{N\le |n_1|,|n_2|<M} \frac{n_1^2 n_2^2}{\langle n_1\rangle^4 \langle n_2\rangle^4} \lesssim \frac{1}{N^2}. \tag{5.36}$$

On the other hand, since

$$|Y^2_{N,M}|^2 = \sum_{N\le |n_1|,|n_2|,|n_3|,|n_4|<M} n_1 n_2 n_3 n_4 \frac{(G_{n_1} + G_{n_2})(G_{n_3} + G_{n_4})}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle n_3\rangle^2 \langle n_4\rangle^2},$$

by symmetry it is enough to consider a single term of the form

$$\sum_{N\le |n_1|,|n_2|,|n_3|,|n_4|<M} n_1 n_2 n_3 n_4 \frac{G_{n_j} G_{n_k}}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle n_3\rangle^2 \langle n_4\rangle^2},$$

with $1 \leq j \neq k \leq 4$, which we set without any loss of generality to be $j = 1$, $k = 3$. We then have

$$\|Y^2_{N,M}\|^2_{L^2} = E[|Y^2_{N,M}|^2] \leq C \sum_{N \leq |n_1|,|n_2|,|n_4| \leq M} \frac{n_1^2 n_2 n_4}{\langle n_1\rangle^4 \langle n_2\rangle^2 \langle n_4\rangle^2} = 0$$

by symmetry. From (5.34) and (5.36) we obtain (5.29) as desired.

To prove (5.30) we use (5.31) to define

$$S_{M,N}(v) := |X_M(v) - X_N(v)|^2 = \sum_{N \leq |n_1|,|n_2| < M} n_1 n_2 \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1\rangle^2 \langle n_2\rangle^2} \tag{5.37}$$

which fits the framework of (5.27) in Proposition 5.10 with $k = 4$. Then it follows that for any $p \geq 2$

$$\|S_{M,N}(v)\|_{L^p} \lesssim (p-1)^2 \|S_{M,N}(v)\|_{L^2} = (p-1)^2 \|X_M(v) - X_N(v\|^2_{L^4} \lesssim (p-1)^2 \frac{1}{N}. \tag{5.38}$$

On the other if we set $q = 2p$, then by (5.38) we have that

$$\|X_M(v) - X_N(v)\|_{L^q} = \|S_{M,N}(v)\|^{\frac{1}{2}}_{L^p} \lesssim (q-1)\frac{1}{N^{\frac{1}{2}}},$$

hence (5.30) for $q \geq 4$. Finally, Hölder's inequality gives the (5.30) for $2 \leq q \leq 4$.

□

Proof of Lemma 5.9. Let us denote $M_N(v) := \int_{\mathbb{T}} |v_N|^2\, dx$. Up to absolute constants we write

$$\begin{aligned} \rho(|K_M(v) - K_N(v)| > 2\gamma) &\leq \rho(|X_M(v)M_M(v) - X_N(v)M_N(v)| > \gamma) \\ &+ \rho(|M_M(v)^3 - M_N(v)^3| > \gamma). \end{aligned} \tag{5.39}$$

Then

$$\begin{aligned} \rho(|X_M(v)M_M(v) - X_N(v)M_N(v)| > \gamma) &\leq \rho(|X_M(v) - X_N(v)|\, M_M(v) > \frac{\gamma}{2}) \\ &+ \rho(|M_M(v) - M_N(v)|\, |X_N(v)| > \frac{\gamma}{2}) = I_1 + I_2. \end{aligned}$$

Let $\lambda > 0$ to be determined. Then by (5.28), (5.8) and (5.26) with $p = 2$, $\alpha = \frac{1}{2}-$, we have that

$$\begin{aligned} I_2 &\leq \rho(|X_N(v)| > \lambda) + \rho(|M_M(v) - M_N(v)| > \frac{\gamma}{2}\lambda^{-1}) \\ &\leq e^{-c\lambda N^{-2\varepsilon}} + \rho(\|v^N - v^M\|_{L^2} > \frac{\gamma}{4B}\lambda^{-1}) \\ &\leq e^{-c\lambda N^{-2\varepsilon}} + e^{-c_{\gamma,B} N^{1-}\lambda^{-2}}. \end{aligned}$$

By setting $\lambda = N^{\frac{1}{3}+\frac{2\varepsilon}{3}-}$ we have that $I_2 \lesssim e^{-c_{\gamma,B} N^{\frac{1}{3}-\frac{4\varepsilon}{3}-}}$.

To estimate I_1 we first note that

$$M_M(v) \leq \|v\|^2_{L^2} \leq B^2. \tag{5.40}$$

Then by (5.30) and Tchebishev's inequality[8] we have that

$$I_1 \leq \rho(|X_M(v) - X_N(v)| > \frac{\gamma}{2B^2}) \lesssim e^{-C_B N^{\frac{1}{2}}\gamma}. \tag{5.41}$$

[8]C.f. Lemma 4.5 in [46].

To estimate the second term of (5.39), we use (5.40) to obtain

$$\rho(|M_M(v)^3 - M_N(v)^3| > \gamma) \leq \rho(|M_M(v) - M_N(v)| > c_B\,\gamma) \leq e^{-C_B\gamma^2 N^{1-}}$$

by arguing as in the estimate for I_2 above.

□

Lemma 5.12. *$R_N(v)$ converges in measure to $R(v)$.*

Proof. If $\|P_N v\|_{L^2} \leq B$ for all $N \in \mathbb{N}$, then we have $\|v\|_{L^2} \leq B$. Then, by continuity from above, we have, for $\delta \in (0,1)$,

$$\begin{aligned}
&\lim_{N\to\infty} \rho\Big(\big\{v; |\chi_{\{\|v^N\|_{L^2}\leq B\}} - \chi_{\{\|v\|_{L^2}\leq B\}}| > \delta\big\}\Big) \\
&\quad = \lim_{N\to\infty} \rho(\|v^N\|_{L^2} \leq B) - \rho(\|v\|_{L^2} \leq B) \\
&\quad = \rho\Big(\bigcap_{N=1}^{\infty} \big\{\|v^N\|_{L^2} \leq B\big\}\Big) - \rho(\|v\|_{L^2} \leq B) = 0.
\end{aligned}$$

Thus, $\chi_{\{\|v^N\|_{L^2}\leq B\}}$ converges to $\chi_{\{\|v\|_{L^2}\leq B\}}$ in measure. By Lemma 5.8 (a), F_N converges to F in measure and by Lemma 5.9 K_N converges to K in measure.

Lastly, we consider $G_N(v)$ and show it is Cauchy in measure provided $\|v\|_{L^2} \leq B$. Assume $N \leq M$ then,

$$\begin{aligned}
4\pi G_N(v) - 4\pi G_M(v) &= \left(\int_{\mathbb{T}} |v^M|^2 - |v^N|^2\,dx\right)\left(\int_{\mathbb{T}} |v^M|^4\,dx\right) + \left(\int_{\mathbb{T}} |v^N|^2\,dx\right)\left(\int_{\mathbb{T}} |v^M|^4 - |v^N|^4\,dx\right) \\
&\leq c_B\,\|v^M - v^N\|_{L^2}\|v^M\|_{L^4}^4 + \|v^N\|_{L^2}^2\,\big|\|v^M\|_{L^4}^4 - \|v^N\|_{L^4}^4\big| \\
&\leq C_B\Big[\|v^M - v^N\|_{L^2}\|v^M\|_{L^4}^4 + 3(\|v^M\|_{L^4}^3 + \|v^N\|_{L^4}^3)\,\|v^M - v^N\|_{L^4}\Big]. \qquad (5.42)
\end{aligned}$$

Fix any $\gamma > 0$; then

$$\begin{aligned}
\rho(|4\pi G_M(v) - 4\pi G_N(v)| > \gamma) \;&\leq\; \rho(\|v^M - v^N\|_{L^2}^2\|v^M\|_{L^4}^4 > \frac{\gamma}{2C_B}) \\
&+\; \rho\Big((\|v^M\|_{L^4}^3 + \|v^N\|_{L^4}^3)\|v^M - v^N\|_{L^4} > \frac{\gamma}{6C_B}\Big).
\end{aligned}$$

To treat the first term we write

$$\rho(\|v^M - v^N\|_{L^2}\|v^M\|_{L^4}^4 > \frac{\gamma}{2C_B}) \leq \rho(\|v^M - v^N\|_{L^2} > \lambda^{-1}\frac{\gamma}{2C_B}) + \rho(\|v^M\|_{L^4}^4 > \lambda)$$

for some $\lambda > 0$ to be determined. We use (5.24) with $\alpha = \frac{1}{2}-$ corresponding to $p = 2$ and (5.23) to get that

$$\rho(\|v^M - v^N\|_{L^2} > c_B\gamma\,\lambda^{-1}) \leq e^{-c_B'\gamma^2 N^{1-}\lambda^{-2}}$$

and

$$\rho(\|v^M\|_{L^4} > \lambda^{\frac{1}{4}}) \leq e^{-c\lambda^{\frac{1}{2}}}.$$

A decay of $e^{-C_B N^{\frac{1}{5}-}\gamma^{\frac{2}{5}}}$ follows by setting $\lambda = N^{\frac{2}{5}-}\gamma^{\frac{4}{5}}$.

For the second term write

$$\begin{aligned}
&\rho\bigg(\|v^M - v^N\|_{L^4}(\|v^M\|_{L^4}^3 + \|v^N\|_{L^4}^3) > \frac{\gamma}{6C_B}\bigg) \\
\leq\ & \rho(\|v^M - v^N\|_{L^4} > c_B\gamma\lambda^{-1}) + \rho(\|v^M\|_{L^4} > c_1\,\lambda^{\frac{1}{3}}) + \rho(\|v^N\|_{L^4} > c_2\lambda^{\frac{1}{3}}) \\
\leq\ & e^{-c_B'\gamma^2 N^{\frac{1}{2}-}\lambda^{-2}} + 2e^{-c\lambda^{\frac{2}{3}}},
\end{aligned}$$

since $\alpha = \frac{1}{4}-$ when $p = 4$ in (5.24). A decay of $e^{-C_B N^{\frac{1}{8}-}\gamma^{\frac{1}{2}}}$ follows by setting $\lambda = N^{\frac{3}{16}-}\gamma^{\frac{3}{4}}$.

Thus, $G_N(v)$ converges to $G(v)$ in measure and hence, by composition and multiplication of continuous functions, $R_N(v)$ converges to $R(v)$ in measure. □

The following proposition shows that the weight $R(v)$ is indeed integrable with respect to the Wiener measure ρ.

Proposition 5.13. (a) *For sufficiently small $B > 0$, we have $R(v) \in L^2(d\rho)$. In particular, the weighted Wiener measure μ is a probability measure, absolutely continuous with respect to the Wiener measure ρ.*
(b) *We have the following tail estimate. Let $2 \leq r < \infty$ and $(s-1)r < -1$; then there exists a constant c such that*

$$\mu(\|v\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-cK^2} \tag{5.43}$$

for sufficiently large $K > 0$.
(c) *The finite dimensional weighted Wiener measure μ_N in* (5.21) *converges weakly to μ.*

Proof. (a) By Hölder inequality, we have

$$\begin{aligned}
\int R_N^2(v)d\rho(v) \ &\leq\ C_B\bigg(\int \chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-3\mathrm{Im}\int (v^N)^2\overline{v^N v_x^N}\,dx} d\rho(v)\bigg)^{\frac{1}{3}} \\
&\times\ \bigg(\int \chi_{\{\|v^N\|_{L^2}\leq B\}} e^{\frac{3B^2}{2\pi}(\int |v^N|^4\,dx)} d\rho(v)\bigg)^{\frac{1}{3}} \\
&\times\ \bigg(\int \chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-\frac{6}{\pi}M_N(v)\mathrm{Im}\int (v^N\overline{v_x^N})\,dx} d\rho(v)\bigg)^{\frac{1}{3}}.
\end{aligned}$$

It follows from Lemma 3.10 in [3] (see also [30]) that the second factor is finite for any $B > 0$, whereas it was shown in [42, Proposition 4.2] that the first factor is finite for sufficiently small $B > 0$. For the third factor we proceed as in the proof of [42] Proposition 4.2. In what follows we always implicitly assume that $\|v_N\|_{L^2} \leq B$. If we define

$$A_{\gamma,N} = \{\chi_{\{\|v^N\|_{L^2}\leq B\}} e^{-\frac{6}{\pi}M_N(v)\mathrm{Im}\int (v^N\overline{v_x^N})\,dx} > \gamma\},$$

then we need to show that

$$\int_0^\infty \gamma^2 \rho(A_{\gamma,N})\,d\gamma, \tag{5.44}$$

is convergent uniformly with respect to N for $B > 0$ small enough. Let $N_0 = \ln\gamma$ and assume first that $N \leq N_0 \leq \frac{C}{B^2}\ln\gamma$, for B small enough. We first observe that

$$\bigg|M_N(v)\mathrm{Im}\int (v^N\overline{v_x^N})\,dx\bigg| \leq CB^2\|\partial_x(v^N)^2\|_{L^\infty(\mathbb{T})}.$$

We also note that

$$\rho(A_{\gamma,N}) \leq \rho\left(\left|M_N(v)\mathrm{Im}\int (v^N\overline{v_x^N})\,dx\right| > C\ln\gamma\right) \tag{5.45}$$

and combining (5.45) and (5.44) with Proposition 4.1 in [42], we can continue with

$$\rho(A_{\gamma,N}) \leq \rho(\|\partial_x(v^N)^2\|_{L^\infty(\mathbb{T})} > CB^{-2}\ln\gamma) \lesssim e^{-\frac{C}{B^2}\ln\gamma} = \gamma^{-\frac{C}{B^2}},$$

and the convergence of (5.44) follows from taking B small enough.

Assume now that $N > N_0 = \ln\gamma$ then we observe that $A_{\gamma,N} \subset B_{\gamma,N} \cup C_{\gamma,N}$ where

$$B_{\gamma,N} := \{|X_{N_0}(v)| > \frac{\pi}{12B^2}\ln\gamma\},$$

$$C_{\gamma,N} := \{|X_N - X_{N_0}(v)| > \frac{\pi}{12B^2}\ln\gamma\}.$$

We first observe that from the argument above

$$\rho(B_{\gamma,N}) \leq \rho(\|\partial_x(v^{N_0})^2\|_{L^\infty(\mathbb{T})} > CB^{-2}\ln\gamma) \lesssim \gamma^{-\frac{C}{B^2}}.$$

On the other hand from (5.41) and the fact that $N > \ln\gamma$ we have that

$$\rho(C_{\gamma,N}) \lesssim e^{-C_B N^{\frac{1}{2}}\ln\gamma} \leq e^{-C_B(\ln\gamma)^{(1+\frac{1}{2})}} \leq C_{B,L}\gamma^L,$$

for any $L \geq 1$ and an appropriate constant $C_{B,L}$ depending on B and L. From here again the convergence of (5.44) follows.

Hence we have that $R_N(v) \in L^2(d\rho)$ for sufficiently small $B > 0$, independent of N. Then, by Lemma 5.12 and Fatou's lemma, we obtain $R(v) \in L^2(d\rho)$.

(b) By Cauchy-Schwarz inequality, we have

$$\int \chi_{\{\|v\|_{\mathcal{F}L^{s,r}} > K\}} d\mu \leq \|R(v)\|_{L^2(d\rho)}\big\{\rho(\|v\|_{\mathcal{F}L^{s,r}} > K)\big\}^{\frac{1}{2}}.$$

Then, (5.43) follows from (5.8).

(c) Let us define

$$\mathcal{H} := \bigcup_M \{F\,; F = G(\widehat{v}_{-M}, \cdots, \widehat{v}_M),\ G \text{ bounded and continuous}\}\ . \tag{5.46}$$

Note this is a dense set in $L^1(\mathcal{F}L^{s,r}, \mu)$ with $2 \leq r < \infty$ and $(s-1)r < -1$. Fix $F \in \mathcal{H}$, then F depends on M finitely many modes, for some M. Fix $\varepsilon > 0$. Then, for $N > M$, we have

$$\begin{aligned}
\left|\int F(v)d\mu_N - \int F(v)d\mu\right| &= \left|\int F(v)\big(R_N(v) - R(v)\big)d\rho\right| \\
&\leq \left|\int_{\{|R_N(v)-R(v)|<\varepsilon\}} F(v)\big(R_N(v) - R(v)\big)d\rho\right| \\
&\qquad + \left|\int_{\{|R_N(v)-R(v)|\geq\varepsilon\}} F(v)\big(R_N(v) - R(v)\big)d\rho\right| \\
&\leq \varepsilon\sup|F| + \sup|F|\,\|R_N(v) - R(v)\|_{L^2(d\rho)}\big\{\rho(|R_N(v) - R(v)| \geq \varepsilon)\big\}^{\frac{1}{2}}.
\end{aligned}$$

From the proof of Proposition 5.13, we have $\|R_N(v) - R(v)\|_{L^2(d\rho)} \leq \|R_N(v)\|_{L^2(d\rho)} + \|R(v)\|_{L^2(d\rho)} < C < \infty$ for all N. By Lemma 5.12, $\rho(|R_N(v) - R(v)| \geq \varepsilon) \to 0$ as $n \to \infty$.

Now, let F be a general bounded continuous function on $\mathcal{F}L^{s,r}$ with $2 \leq r < \infty$ and $(s-1)r < -1$. Let F_M denote its restriction on E_M, i.e. $F_M(v) = F(v^M)$ where $v^M = P_M v$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big| \int F(v)d\mu - \int F_M(v)d\mu \Big| &= \Big| \int \big(F(v) - F(v^M)\big) R(v) d\rho \Big| \\ &\leq \|R(v)\|_{L^2(d\rho)} \bigg(\int |F(v) - F(v^M)|^2 d\rho \bigg)^{\frac{1}{2}}. \end{aligned} \tag{5.47}$$

By continuity of F, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|P_M^\perp v\|_{\mathcal{F}L^{s,r}} = \|v - v^M\|_{\mathcal{F}L^{s,r}} < \delta \quad \Longrightarrow \quad |F(v) - F(v^M)| < \varepsilon.$$

Then, the contribution in (5.47) from $\{ v ; \|P_M^\perp v\|_{\mathcal{F}L^{s,r}} < \delta\}$ is at most $\varepsilon \, \|R(v)\|_{L^2(d\rho)}$. Without loss of generality, assume $\delta \leq \varepsilon^2$. By the measurability of the $\mathcal{F}L^{s,r}$-norm (see Definition 5.3), the contribution in (5.47) from $\{ v ; \|P_M^\perp v\|_{\mathcal{F}L^{s,r}} \geq \delta\}$ is at most

$$\begin{aligned} &2 \sup |F| \cdot \|R(v)\|_{L^2(d\rho)} \big\{ \rho(\|P_M^\perp v\|_{\mathcal{F}L^{s,r}} \geq \delta) \big\}^{\frac{1}{2}} \\ &\qquad \leq 2 \sup |F| \cdot \|R(v)\|_{L^2(d\rho)} \, \delta^{\frac{1}{2}} \leq 2 \sup |F| \cdot \|R(v)\|_{L^2(d\rho)} \, \varepsilon \end{aligned}$$

for sufficiently large M. A similar argument can be used to show $|\int F(v)d\mu_N - \int F_M(v)d\mu_N| \leq C(f,R)\varepsilon$, independent of N. Hence, μ_N converges weakly to μ. □

Remark 5.14. A tail estimate similar to (5.43) holds for the finite dimensional weighted Wiener measure μ_N; i.e. we have

$$\mu_N(\|v^N\|_{\mathcal{F}L^{s,r}} > K) \leq e^{-cK^2}, \tag{5.48}$$

where the constant is independent of N.

Remark 5.15. The measure ρ_N is not absolutely continuous with respect to μ_N but its restriction on $\{\|v^N\|_{L^2} \leq B\}$, i.e., $\widetilde{\rho}_N = \widehat{Z}_N^{-1} \chi_{\{\|v^N\|_{L^2} \leq B\}} \rho_N$ is absolutely continuous with respect to μ_N and from (5.21), we have that

$$\frac{d\widetilde{\rho}_N}{d\mu_N} := \tilde{R}_N = \tilde{Z}_N^{-1} \chi_{\{\|v^N\|_{L^2} \leq B\}} e^{\frac{1}{2}\mathcal{N}(v^N)} \tag{5.49}$$

for suitable renormalization $\tilde{Z}_N$. Since $\mathcal{N}(v^N)$ does not have a definite sign Lemma 5.8 Lemma 5.12 and part (a) of Proposition 5.13 hold for $\tilde{R}_N$ and its corresponding limit $\tilde{R}$. In particular, for sufficiently small B, $\tilde{R}_N \in L^2(d\rho)$ for all N with bound independent of N. The latter fact will be used in the proof of Proposition 6.2 in Section 6.

Remark 5.16. Given any $p < \infty$, one can prove $R(v) \in L^p(d\rho)$ for sufficiently small $B \leq B(p)$. However, $B(p) \to 0$ as $p \to \infty$. i.e. there is no uniform lower bound on the size of the L^2-cutoff. For our purpose, the integrability with $p = 2$ suffices.

6. Almost sure well-posedness of FGDNLS and invariance of the measure

In order to prove the global well-posedness of μ-almost all solution of (FGDNLS) (3.1) we fix once again $s = \frac{2}{3}-$ and $r = 3$ so that we have at our disposal the local well posedness result in $\mathcal{F}L^{s,r}$, that the measure is supported on $\mathcal{F}L^{s,r}$ and also the energy growth estimates in Theorem 4.2 as explained in Remark 4.3.

We first use the almost invariance of the finite-dimensional measure μ_N under the flow of the truncated equation (3.1) to control the growth of solutions.

Lemma 6.1. *For any given $T > 0$ and $\varepsilon > 0$ there exists an integer $N_0 = N_0(T, \varepsilon)$ and sets $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon, T) \subset \mathbb{R}^{4N+2}$ such that for $N > N_0$*

(a) $\mu_N\left(\widetilde{\Omega}_N\right) \geq 1 - \varepsilon$.

(b) *For any initial condition $v_0^N \in \widetilde{\Omega}_N$, (FGDNLS)* (3.1) *is well-posed on $[-T, T]$ and its solution $v^N(t)$ satisfies the bound*

$$\sup_{|t|\leq T} \|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

Proof. It is enough to consider $t \in [0, T]$, the argument is similar for $t \in [-T, 0]$. We set

$$C_N(K, B) := \left\{ w^N \in \mathbb{R}^{4N+2} \,:\, \|w^N\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \leq K \,,\, \|w^N\|_{L^2} \leq B \right\}.$$

If the initial condition $v_0^N \in C_N(K, B)$ then (FGDNLS) (3.1) is locally well-posed on the time interval of length $\delta \sim K^{-\gamma}$ by Theorem 3.2 , where $\gamma > 0$ is independent of N. Furthermore, if μ_N is given by (5.21), then for sufficiently large K we have that $\mu_N(C_N(K, B)^c) \leq e^{-cK^2}$ for some constant c which is independent of N by (5.48).

Let $\Phi_N(t)$ the flow map of (3.1). We define $\widetilde{\Omega}_N$ by

$$\widetilde{\Omega}_N := \left\{ v_0^N \,:\, \Phi_N(j\delta)(v_0^N) \in C_N(K, B) \,,\, j = 0, 1, \cdots \left[\frac{T}{\delta}\right] \right\}.$$

Note that $\widetilde{\Omega}_N^c = \bigcup_{k=0}^{[\frac{T}{\delta}]} D_k$, where

$$\begin{aligned} D_k &= \left\{ v_0^N \,;\, k = \min\left\{ j \,:\, \Phi_N(j\delta)(v_0^N) \in C_N(K, B)^c \right\} \right\}, \\ &= \left[\bigcap_{j=0}^{k-1} \Phi_N(-j\delta) C_N(K, B)) \right] \cap \Phi_N(-k\delta)(C_N(B, K)^c). \end{aligned} \tag{6.1}$$

One verifies easily that the sets D_k satisfy

$$D_0 = C_N(K, B)^c, \quad D_k = C_N(K, B) \cap \Phi_N(-\delta)(D_{k-1}). \tag{6.2}$$

By Lemma 4.1, the Lebesgue measure $d\mu_N^0 \equiv \prod_{|n|\leq N} da_n db_n$ is invariant under the flow $\Phi_N(t)$ (i.e. for any $f \in L^1(d\mu_N^0)$ we have $\int f \circ \Phi_N(t) d\mu_N^0 = \int f d\mu_N^0$).

Using the energy growth estimate[9] in Theorem 4.2 and the invariance of the L^2 norm $m(v) = \frac{1}{2\pi}\|v\|_{L^2}$ under $\Phi_N(t)$ (i.e. $m \circ \Phi_N(t) = m$ for all t; see Remark 3.1) we have for

[9] Without loss of generality we assume $\max(K^6, K^8) = K^8$ in Theorem 4.2.

any set $A \subset \mathbb{R}^{4N+2}$

$$\begin{aligned}
\mu_N(C_N(K,B)\cap A) &= Z_N^{-1}\int \chi_{\{C_N(K,B)\cap A\}}\chi_{\{m\le 2\pi B^2\}}e^{-\frac{1}{2}\mathcal{E}-\pi m}d\mu_N^0 \\
&= Z_N^{-1}\int \chi_{\{C_N(K,B)\cap A\}}\circ\Phi_N(-\delta)\chi_{\{m\le 2\pi B^2\}}e^{-\mathcal{E}\circ\Phi_N(-\delta)-\pi m}d\mu_N^0 \\
&= \int \chi_{\{\Phi_N(\delta)(C_N(K,B)\cap A)\}}e^{-\frac{1}{2}(\mathcal{E}\circ\Phi_N(-\delta)-\mathcal{E})}d\mu_N \qquad (6.3)\\
&\le e^{c(\delta)N^{-\beta}K^8}\mu_N\left(\Phi_N(\delta)(C_N(K,B)\cap A)\right) \\
&\le e^{c(\delta)N^{-\beta}K^8}\mu_N\left(\Phi_N(\delta)(A)\right).
\end{aligned}$$

Applying (6.3) to (6.2) with $A = \Phi_N(-\delta)(D_{k-1})$ and iterating in $k \in \{0,\dots[\frac{T}{\delta}]\}$, we obtain

$$\mu_N(D_K) \le e^{c(\delta)N^{-\beta}K^8}\mu_N(D_{K-1}) \le e^{k\,c(\delta)N^{-\beta}K^8}e^{-cK^2}$$

and thus

$$\mu_N(\widetilde{\Omega}_N^c) \le \sum_{k=0}^{[\frac{T}{\delta}]} e^{k\,c(\delta)N^{-\beta}K^8}e^{-cK^2} \lesssim \left[\frac{T}{\delta}\right]e^{-cK^2} \sim TK^{\gamma}e^{-cK^2}, \qquad (6.4)$$

for $N \ge N_0(T,K)$. By choosing $K \sim \left(\log\frac{T}{\varepsilon}\right)^{\frac{1}{2}}$, we have $\mu_N(\widetilde{\Omega}_N^c) < \varepsilon$ as desired.

Finally, by construction, we have $\|v^N(j\delta)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \le K$ for $j = 0,\cdots,[\frac{T}{\delta}]$ and by the local theory, we have

$$\sup_{0\le t\le T}\|v^N(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \le 2K \sim \left(\log\frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

□

Combining Lemma 6.1 with the approximation Lemma 3.3 we can now prove a similar result for the solution of the initial value problem (GDNLS) (2.8).

Proposition 6.2. *For any given $T > 0$ and $\varepsilon > 0$ there exists a set $\Omega(\varepsilon,T)$ such that*
(a) $\mu\left(\Omega(\varepsilon,T)\right) \ge 1-\varepsilon$.
(b) *For any initial condition $v_0 \in \Omega(\varepsilon,T)$ the initial value problem (GDNLS)* (2.8) *is well-posed on $[-T,T]$ with the bound*

$$\sup_{|t|\le T}\|v(t)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \lesssim \left(\log\frac{T}{\varepsilon}\right)^{\frac{1}{2}}.$$

Proof. Let $\widetilde{\Omega}_N = \widetilde{\Omega}_N(\varepsilon,T)$ be the set given in Lemma 6.1 for $N \ge N_0(\varepsilon,T)$. This set is defined in terms of $K \sim \left(\log\frac{T}{\varepsilon}\right)^{1/2}$ and for that same K we define the set

$$\Omega_N := \Omega_N(\varepsilon,T) := \left\{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3} \,:\, \|v_0\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \le K\,,\ \ P_N v_0 \in \widetilde{\Omega}_N\right\}$$

If $v_0 \in \Omega_N$ then by Lemma 6.1 we have

$$\sup_{t\le T}\|\Phi_N(t)(P_N v_0)\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \le 2K\,. \qquad (6.5)$$

On the other hand for $v_0 \in \Omega_N$ the local well posedness theorem in [22] gives a $\delta > 0$ and a solution $v(t)$ of (GDNLS) (2.8) for $|t| \le \delta$.

By (3.9) in the proof of the Lemma 3.3, with K in place of A, we obtain that for every $s_1 < \frac{2}{3}-$

$$\|v(\delta) - v^N(\delta)\|_{\mathcal{F}L^{s_1,3}} \lesssim KN^{s_1-\frac{2}{3}+}.$$

By choosing a larger N_0 if necessary, so that $\left[\frac{T}{\delta}\right] KN^{s_1-\frac{2}{3}+} \ll 1$ for $N > N_0$ we can repeat this argument $\left[\frac{T}{\delta}\right]$ times over the intervals $[j\delta, (j+1)\delta], j = 0, 1, \ldots, \left[\frac{T}{\delta}\right] - 1$ and obtain

$$\|v(j\delta) - v^N(j\delta)\|_{\mathcal{F}L^{s_1,3}} < 1. \tag{6.6}$$

Then from (6.5) and (6.6) we conclude

$$\|v(t)\|_{\mathcal{F}L^{s_1,3}} \lesssim (2K+1) \sim \left(\log \frac{T}{\varepsilon}\right)^{\frac{1}{2}},$$

and since the right hand side is independent of $s_1 < \frac{2}{3}-$ we obtained the desired estimate.

To estimate $\mu(\Omega_N)$ note first that

$$\Omega_N^c \subset \left\{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3} : \|v_0\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \geq K\right\} \cup \left\{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3} : P_N v_0 \in \tilde{\Omega}_N^c\right\} \tag{6.7}$$

The first set on the right hand side of (6.7) has μ measure less than ε by the tail bound in Proposition 5.13. The set $F_N \equiv \left\{v_0 \in \mathcal{F}L^{\frac{2}{3}-,3}; P_N v_0 \in \tilde{\Omega}_N^c\right\}$ is a cylinder set and we have $F_N \cap E_N = \widetilde{\Omega}_N^c$ (recall $E_N = \text{span}\{e^{inx}\}_{|n|\leq N}$). Thus $\rho(F_N) = \rho_N(F_N) = \rho_N(\widetilde{\Omega}_N^c)$. On the other hand, recall that $\mu \ll \rho$ and that, $\widetilde{\rho}_N$ the restriction of ρ_N to the ball $\{\|v^N\|_{L^2} \leq B\}$ is absolutely continuous with respect to μ_N (see Remark 5.15). Then using Cauchy-Schwarz repeatedly we obtain

$$\begin{aligned}
\mu(F_N) &\leq \left(\int R^2 d\rho\right)^{\frac{1}{2}} \left(\int_{\widetilde{\Omega}_N^c} \chi_{\{\|v^N\|_{L^2}\leq B\}}\, d\rho_N\right)^{\frac{1}{2}} \\
&\leq \left(\int R^2 d\rho\right)^{\frac{1}{2}} \left(\int \tilde{R}_N^2 d\mu_N\right)^{\frac{1}{4}} \mu_N(\widetilde{\Omega}_N^c)^{\frac{1}{4}} \\
&\leq \left(\int R^2 d\rho\right)^{\frac{1}{2}} \left(\int \tilde{R}_N d\rho_N\right)^{\frac{1}{4}} \mu_N(\widetilde{\Omega}_N^c)^{\frac{1}{4}} \qquad (6.8)
\end{aligned}$$

where $\tilde{R}_N$ is as defined in Remark 5.15 and where in the last inequality we have used that by definition $\tilde{R}_N^2 R_N = \tilde{R}_N$.

By relying on Lemma 5.12, Proposition 5.13 and Remark 5.15 we can bound the first two terms in (6.8) by a constant independent of N. This combined with Lemma 6.1 allows us to conclude that there exist a constant $d > 0$ and $N_1(\varepsilon, T)$ such that $\mu(F_N) \leq d\,\varepsilon$ for $N \geq N_1$. So for $N \geq \max(N_0, N_1)$, any set $\Omega(\varepsilon, T) := \Omega_N(\varepsilon, T)$ satisfies the desired hypothesis. □

Theorem 6.3 (Almost sure global well-posedness)**.** *There exists a subset Ω of the space $\mathcal{F}L^{\frac{2}{3}-,3}$ with $\mu(\Omega^c) = 0$ such that for every $v_0 \in \Omega$ the initial value problem (GDNLS)* (2.8) *with initial data v_0 is globally well-posed.*

Proof. Fix an arbitrary T and let $\varepsilon = 2^{-i}$. Using the sets given in Proposition 6.2 we set

$$\Omega(T) := \bigcup_i \Omega(2^{-i}, T). \tag{6.9}$$

If $v_0 \in \Omega(T)$ then the initial value problem (GDNLS) (2.8) is well-posed up to time T. Since $\mu(\Omega(T)) \geq 1 - 2^{-i}$ for any $i \in \mathbb{N}$, the set $\Omega(T)$ has full measure.

Finally by taking $T := 2^j$ the set

$$\Omega = \bigcap_j \Omega(2^j) \tag{6.10}$$

has also full measure and if $v_0 \in \Omega$ then the initial value problem (GDNLS) (2.8) is globally well-posed. □

Remark 6.4. We note that by slightly modifying the proof of Theorem 6.3 above we could also derive a logarithmic bound in time on solutions similar to the one in [3] and [12].

Now that we have a well-defined flow on the measure space $(\mathcal{F}L^{\frac{2}{3}-,3}, \mu)$ we show that μ is invariant under the flow $\Phi(t)$, following the argument in [38].

Theorem 6.5. *The measure μ is invariant under the flow $\Phi(t)$.*

Proof. Let us consider the measure space $(\mathcal{F}L^{\frac{2}{3}-,3}, \mu)$. We need to show that for any measurable A we have $\mu(A) = \mu(\Phi(-t)(A))$ for all $t \in \mathbb{R}$. Note that by the group property of the flow without loss of generality we can assume that $|t| \leq \delta$. An equivalent characterization of invariance is that for all $F \in L^1(\mathcal{F}L^{\frac{2}{3}-,3}, \mu)$ we have

$$\int F(\Phi(t)(v))d\mu \, = \, \int F(v)d\mu \, . \tag{6.11}$$

By an elementary approximation argument it is enough to show (6.11) for F in a dense set in $L^1(\mathcal{F}L^{\frac{2}{3}-,3}, \mu)$ which we choose as in (5.46) to be

$$\mathcal{H} \, := \bigcup_M \{F \, : F = G(\widehat{v}_{-M}, \cdots, \widehat{v}_M), \; G \text{ bounded and continuous}\} \, .$$

For $F \in \mathcal{H}$ let us choose an arbitrary $\epsilon > 0$ and assume $N \geq M$. By Proposition 5.13 μ_N converges weakly to μ and thus

$$\left| \int F d\mu - \int F d\mu_N \right| + \left| \int F \circ \Phi(t) d\mu - \int F \circ \Phi(t) d\mu_N \right| \leq \epsilon \, . \tag{6.12}$$

Let $\Phi_N(t)$ be the flow map for FGDNLS (3.1). For $s_1 < \frac{2}{3}-$, by the Lemma 3.3, we have that $\|\Phi(t)(v) - \Phi_N(t)(v)\|_{\mathcal{F}L^{s_1,3}}$ converges to 0 uniformly on $\{v \, ; \; \|v\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \leq K\}$. Using the tail estimate $\mu_N(\|v^N\|_{\mathcal{F}L^{\frac{2}{3}-,3}} > K) \leq e^{-cK^2}$ (uniformly in N) and the continuity of F in $\mathcal{F}L^{s_1,3}$ we obtain

$$\left| \int F \circ \Phi(t) d\mu_N - \int F \circ \Phi_N(t) d\mu_N \right| \leq 2\|F\|_\infty e^{-cK^2} + \epsilon \leq 3\epsilon \tag{6.13}$$

for large enough K and N.

Finally using again the tail estimate for μ_N, the invariance of Lebesgue measure under $\Phi_N(t)$ and the energy estimate given in Theorem 4.2 we obtain

$$\begin{aligned} &\left| \int F \circ \Phi_N(t) \, d\mu_N - \int F \, d\mu_N \right| \\ &\leq \, 2\|F\|_{L^\infty} e^{-cK^2} + \left| \int_{\{\|v\|_{\mathcal{F}L^{\frac{2}{3}-,3}} \leq K\}} F \left[e^{-\frac{1}{2}(\mathcal{E} \circ \Phi_N(-t) - \mathcal{E})} - 1 \right] d\mu_N \right| \\ &\leq \, 2\epsilon + \|F\|_{L^\infty} \left(e^{c(\delta) N^{-\beta} K^8} - 1 \right) \leq 3\epsilon \, , \end{aligned} \tag{6.14}$$

for sufficiently large N. By combining (6.12), (6.13), and (6.14) we obtain invariance. □

7. The ungauged DNLS equation

Recall that if $u(t,x)$ is a solution of DNLS (2.1) then $w(t,x) = G(u(t,x))$ where $G(f)(x) = \exp(-iJ(f))\, f(x)$ (see (2.7)) is a solution of

$$w_t - iw_{xx} - 2m(w)w_x = -w^2\overline{w}_x + \frac{i}{2}|w|^4 w - i\psi(w)w - im(w)|w|^2 w \tag{7.1}$$

with initial data $w(0) = G(u(0))$. Furthermore $v(t,x) = w(t, x - 2tm(w))$ is a solution of (2.8) with initial condition $v(0) = w(0)$. If $\Phi(t)$ denotes the flow map for GDNLS (2.8), let $\widetilde{\Phi}(t)$ denote the flow map of (7.1) and let $\Psi(t)$ denote the flow map of (2.1).

Clearly we have the relation

$$\Psi(t) = G^{-1} \circ \widetilde{\Phi}(t) \circ G\,. \tag{7.2}$$

To elucidate the relation between $\Phi(t)$ and $\widetilde{\Phi}(t)$ let $\tau_\alpha(s)$ denote the action of the group of spatial translations on functions, i.e., $(\tau_\alpha(s)w)(x) := w(x - \alpha s)$. We define a state dependent translation

$$(\Gamma(s)w)(x) := (\tau_{2m(w)}(s)w)(x) = w(x - 2\,s\,m(w))\,.$$

Note the H^s, L^p and $\mathcal{F}L^{s,r}$ norms are all invariant under this transformation. Furthermore we have

$$v(t,x) := (\Gamma(t)w)(t,x)\,.$$

Since m is preserved under G, $\Gamma(s)$ and both flows $\Psi(t)$ and $\tilde{\Phi}(t)$ we have the relation

$$\Phi(t) = \Gamma(t)\widetilde{\Phi}(t) = \widetilde{\Phi}(t)\Gamma(t)\,, \tag{7.3}$$

in particular $\widetilde{\Phi}(t)$ and $\Gamma(t)$ commute.

Finally if μ is a measure on Ω as in Theorem 6.3 and $\varphi : \Omega \to \Omega$ is a measurable map then we define the measure $\nu = \mu \circ \varphi^{-1}$ by

$$\nu(A) := \mu(\varphi^{-1}(A)) = \mu(\{x\,;\, \varphi(x) \in A\})\,.$$

for all measurable sets A or equivalently by

$$\int F d\nu = \int F \circ \varphi\, d\mu$$

for integrable F.

Consider the measure defined by

$$\nu := \mu \circ G. \tag{7.4}$$

Since the measure μ constructed in Proposition 5.13 is invariant under the flow $\Phi(t)$ we show that the flow $\Psi(t)$ for DNLS is well defined ν almost surely and that ν is invariant under the flow $\Psi(t)$.

Theorem 7.1 (Almost sure global well-posedness for DNLS)**.** *There exists a subset Σ of the space $\mathcal{F}L^{\frac{2}{3}-,3}$ with $\nu(\Sigma^c) = 0$ such that for every $u_0 \in \Sigma$ the IVP (DNLS)* (2.1) *with initial data u_0 is globally well-posed.*

Proof. Let Ω be the set of full μ measure given in Theorem 6.3 and let $\Sigma = G^{-1}(\Omega)$. Note that Σ is a set of full ν-measure by (7.4). For $v_0 \in \Omega$ the IVP (GDNLS) (2.8) with initial data v_0 is globally well-posed. Hence since the map $\mathcal{G} : C([-T,T]; \mathcal{F}L^{s,r}) \to C([-T,T]; \mathcal{F}L^{s,r})$ is a homeomorphism if $s > \frac{1}{2} - \frac{1}{r}$ when $2 < r < \infty$ the IVP (DNLS) (2.1) with initial data $u_0 = G^{-1}(v_0)$ is also globally well-posed.

□

Finally we show that the measure ν is invariant under the flow map of DNLS (2.1).

Theorem 7.2. *The measure $\nu = \mu \circ G$ is invariant under the flow $\Psi(t)$.*

Proof. First we note that the measure μ is invariant under $\Gamma(t)$. The density of μ with respect to ρ is $R(v)$, see (5.15), and it is verified easily that $R \circ \Gamma(t) = R$. Furthermore one also verifies easily that the finite-dimensional measures ρ_N are also invariant under $\Gamma(t)$. As a consequence since μ is invariant under $\Phi(t)$ by Theorem 6.5 then μ is also invariant under $\tilde{\Phi}(t)$ because of (7.3). Finally ν is invariant under $\Psi(t)$ since by (7.2)

$$\int F \circ \Psi(t)\, d\nu \;=\; \int F \circ G^{-1} \circ \tilde{\Phi}(t) \circ G\, d\mu \circ G \;=\; \int F \circ G^{-1} \circ \tilde{\Phi}(t) d\mu \;=\; \int F \circ G^{-1} d\mu \;=\; \int F d\nu.$$

□

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